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ON COEFFICIENT PROBLEMS FOR STARLIKE FUNCTIONS RELATED TO VERTICAL STRIP DOMAINS

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Abstract. In the present paper, we find the sharp bound for the fourth coefficient of starlike functions f which are normalized by $f(0) = 0 = f'(0) - 1$ and satisfy the following two-sided inequality:

$$1 + \frac{\gamma - \pi}{2\sin\gamma} < \Re\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\gamma}{2\sin\gamma}, \quad z \in \mathbb{D},$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk and γ is a real number such that $\pi/2 \leq \gamma < \pi$. Moreover, the sharp bound for the fifth coefficient of f defined above with γ in a subset of $[\pi/2, \pi)$ also will be found.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass of f normalized by $f(0) = 0 = f'(0) - 1$. That is, f has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1}$$

Let the parameters α and β be real numbers such that $0 \leq \alpha < 1 < \beta$. A function $p \in \mathcal{H}$ is said to belong to the class $\mathcal{P}(\alpha, \beta)$ if p satisfies $p(0) = 1$ and the inequality $\alpha < \Re\{p(z)\} < \beta$ for all $z \in \mathbb{D}$. In [4], the authors introduced the class $\mathcal{S}(\alpha, \beta)$ of functions $f \in \mathcal{A}$ for which satisfies $zf'(z)/f(z) \in \mathcal{P}(\alpha, \beta)$ for all $z \in \mathbb{D}$. They obtained for functions $f \in \mathcal{S}(\alpha, \beta)$ which have the form given by (1), the following inequality holds.

$$|a_n| \leq \prod_{k=2}^{n} \frac{k - 2 + \frac{2(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}}{(n-1)!}, \quad n \in \mathbb{N} \setminus \{1\}. \tag{2}$$

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Moreover, they obtained the sharp results for the second and third coefficients for functions in $\mathcal{S}(\alpha,\beta)$ as follows (See [4, 5]):

$$|a_2| \leq \frac{2(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha} \tag{3}$$

and

$$|a_3| \leq \frac{(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha} \left[\cos\frac{\pi(1-\alpha)}{\beta-\alpha} + \frac{2(\beta-\alpha)}{\pi} \sin\frac{\pi(1-\alpha)}{\beta-\alpha}\right]. \tag{4}$$

Given $\gamma \in [\pi/2, \pi)$, we consider a class denoted by $\mathcal{M}(\gamma)$ of functions $f \in \mathcal{A}$ such that

$$1 + \frac{\gamma-\pi}{2\sin\gamma} < \Re\left\{\frac{zf'(z)}{f(z)}\right\} < 1 + \frac{\gamma}{2\sin\gamma}, \quad z \in \mathbb{D}.$$

The class $\mathcal{M}(\gamma)$ was introduced and investigated by Kargar *et al.* [3]. And this class can be reduced by the class $\mathcal{S}(\alpha,\beta)$ by putting

$$\alpha = 1 + \frac{\gamma-\pi}{2\sin\gamma} \quad \text{and} \quad \beta = 1 + \frac{\gamma}{2\sin\gamma}, \quad \gamma \in [\pi/2, \pi).$$

Therefore, for functions $f \in \mathcal{M}(\gamma)$ with the form given by (1), we obtain the inequalities $|a_n| \leq 1$ for $n \in \mathbb{N} \setminus \{1\}$ by (2). Furthermore, from (3) and (4), we get the sharp inequalities $|a_2| \leq 1$ and $|a_3| \leq (1-\cos\gamma)/2$.

The purpose of this paper is to obtain the sharp bound for the fourth and fifth coefficients of function in $\mathcal{M}(\gamma)$. For this, the following notions and results on them are required.

For analytic functions f and g, we say that f is subordinate to g, denoted by $f \prec g$, if there is an analytic function $w : \mathbb{D} \to \mathbb{D}$ with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. Further, if g is univalent, then the definition of subordination $f \prec g$ simplifies to the conditions $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

By using the notion of subordination, we can obtain the following equivalent condition for $f \in \mathcal{M}(\gamma)$:

$$\frac{zf'(z)}{f(z)} \prec B_\gamma(z) := 1 + \frac{1}{2\mathrm{i}\sin\gamma} \log\left(\frac{1+z\mathrm{e}^{\mathrm{i}\gamma}}{1+z\mathrm{e}^{-\mathrm{i}\gamma}}\right), \quad z \in \mathbb{D}, \tag{5}$$

where $\pi/2 \leq \gamma < \pi$. We note that the function B_γ defined by (5) can be represented by

$$B_\gamma(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad z \in \mathbb{D},$$

where

$$A_n = \frac{(-1)^{n-1}\sin n\gamma}{n\sin\gamma}, \quad n \in \mathbb{N}.$$

Throughout this paper, for given $\gamma \in [\pi/2, \pi)$, we will denote

$$\tau = \tau(\gamma) = \cos\gamma \in (-1, 0],$$

for the sake of our convenient notation. We note that, using the notation τ, several initial coefficients of B_γ can be represented by

$$A_1 = 1, \quad A_2 = -\tau, \quad A_3 = \frac{1}{3}(4\tau^2 - 1) \quad \text{and} \quad A_4 = \tau - 2\tau^3.$$

Let $\mathcal{P}$ be the class of Carathéodory functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{6}$$

having a positive real part in $\mathbb{D}$. For given $p \in \mathcal{P}$ define a function $w : \mathbb{D} \to \mathbb{C}$ by $w(z) = (p(z) - 1)/(p(z) + 1)$. Then the property $w(\mathbb{D}) \subset \mathbb{D}$ holds and therefore $|w(z)| \le |z|$ for all $z \in \mathbb{D}$.

Now, let us recall several results for the class $\mathcal{P}$ will be used in further considerations.

Lemma 1.1 ([2, p. 41]). *If $p \in \mathcal{P}$ is of the form* (6), *then*

$$|c_n| \le 2, \quad n \in \mathbb{N}. \tag{7}$$

The inequality (7) *is sharp and the equality holds for the function L defined by*

$$L(z) = \frac{1+z}{1-z} = 1 + 2\sum_{n=1}^{\infty} z^n, \quad z \in \mathbb{D}.$$

Lemma 1.2 ([8, 9, Libera and Zlotkiewicz]). *If $p \in \mathcal{P}$ is of the form* (6) *with $c_1 \ge 0$, then*

$$2c_2 = c_1^2 + \zeta(4 - c_1^2) \tag{8}$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)\zeta - c_1(4 - c_1^2)\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{9}$$

for some ζ and η such that $|\zeta| \le 1$ and $|\eta| \le 1$.

Lemma 1.3 ([6, Kwon, Lecko and Sim]). *Let $p \in \mathcal{P}$ be of the form given by* (6) *and the formula* (8) *with $c_1 \in [0, 2)$ and $\zeta \in \mathbb{T}$ holds. Then p must be of the form*

$$p(z) = \frac{1 + \mu(1+\zeta)z + \zeta z^2}{1 - \mu(1-\zeta)z - \zeta z^2}, \quad z \in \mathbb{D},$$

where $\mu \in [0, 1)$. Here, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

We remark here that a special case Lemma 1.3 with $\zeta = -1 \in \mathbb{T}$ implies the result in [7, Lemma 2.3].

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [1] (see also [10]). Define

$$Y(a, b, c) = \max_{z \in \overline{\mathbb{D}}} \left(|a + bz + cz^2| + 1 - |z|^2 \right), \quad a, b, c \in \mathbb{R},$$

where $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$.

Lemma 1.4 ([1, Choi, Kim and Sugawa]). *If* $ac \geq 0$, *then*

$$Y(a,b,c) = \begin{cases} |a|+|b|+|c|, & |b| \geq 2(1-|c|), \\ 1+|a|+\dfrac{b^2}{4(1-|c|)}, & |b| < 2(1-|c|). \end{cases}$$

If $ac < 0$, *then*

$$Y(a,b,c) = \begin{cases} 1-|a|+\dfrac{b^2}{4(1-|c|)}, & -4ac(c^{-2}-1) \leq b^2 \wedge |b| < 2(1-|c|), \\ 1+|a|+\dfrac{b^2}{4(1+|c|)}, & b^2 < \min\left\{4(1+|c|)^2, -4ac(c^{-2}-1)\right\}, \\ R(a,b,c), & \textit{otherwise}, \end{cases}$$

where

$$R(a,b,c) = \begin{cases} |a|+|b|-|c|, & |c|(|b|+4|a|) \leq |ab|, \\ -|a|+|b|+|c|, & |ab| \leq |c|(|b|-4|a|), \\ (|c|+|a|)\sqrt{1-\dfrac{b^2}{4ac}}, & \textit{otherwise.} \end{cases}$$

2. On the fourth coefficients of functions in $\mathcal{M}(\gamma)$

Let $x_1 \approx -0.468$, $x_2 \approx -0.454$, $x_3 \approx -0.141$ and $x_4 \approx -0.072$ be the unique zeros in $[-1,0]$ of the polynomials q_1, q_2, q_3 and q_4 defined by

$$q_1(x) = -11+83x+72x^2-272x^3+128x^4, \tag{10}$$

$$q_2(x)=1001-9658x+10533x^2+29820x^3-75792x^4+66624x^5-26624x^6+4096x^7, \tag{11}$$

$$q_3(x) = 71+420x-560x^2+192x^3 \tag{12}$$

and

$$q_4(x) = 11+140x-176x^2+64x^3,$$

respectively. And define the functions g, h, $k : [-1,0] \to \mathbb{R}$ by

$$g(x) = \begin{cases} \frac{32(479-1092x+784x^2-192x^3)}{3(7-4x)^3}, & x \in [-1, x_3]; \\ 16, & x \in [x_3, 0], \end{cases} \tag{13}$$

$$h(x) = \begin{cases} \frac{8\sqrt{6}(7-4x)^2(1-9x+8x^2)}{3-4x}\left(\frac{3-4x}{29-127x+124x^2-32x^3}\right)^{3/2}, & x \in [-1, x_2]; \\ \frac{16(5-4x)^{3/2}}{3\sqrt{16-21x+8x^2}}, & x \in [x_2, x_4]; \\ \frac{32(479-1092x+784x^2-192x^3)}{3(7-4x)^3}, & x \in [x_4, 0] \end{cases} \tag{14}$$

and

(15)
$$k(x) = \begin{cases} \frac{8}{3}(1-9x+8x^2), & x \in [-1, x_1]; \\ \frac{4(-7+39x+176x^2-336x^3+128x^4)}{57-48x^2}\sqrt{k_1(x)}\sqrt{k_2(x)}, & x \in [x_1, x_2]; \\ \frac{8\sqrt{6}(7-4x)(-7+67x-92x^2+32x^3)}{-29+127x-124x^2+32x^3}\left(\frac{3-4x}{29-127x+124x^2-32x^3}\right)^{1/2}, & x \in [x_2, 0] \end{cases}$$

with

$$k_1(x) := \frac{64-39x-224x^2+336x^3-128x^4}{95+171x-232x^2-144x^3+128x^4} \tag{16}$$

and

$$k_2(x) := \frac{266+912x-832x^2-768x^3+512x^4}{64-39x-224x^2+336x^3-128x^4}, \tag{17}$$

respectively. By comparing the functions g, h and k for fixed $x \in [-1, 0]$, we can obtain the following relation which will be used for the proof of our result (see figures below):

$$\max\{g(x), h(x), k(x)\} = \begin{cases} k(x), & \text{when } x \in [-1, x_2]; \\ h(x), & \text{when } x \in [x_2, x_3]; \\ g(x), & \text{when } x \in [x_3, 0]. \end{cases} \tag{18}$$

Now, we suggest the sharp bound for the fourth coefficient of $f \in \mathcal{M}(\gamma)$ for $\gamma \in [\pi/2, \pi)$.

Theorem 2.1. *Let $\gamma \in [\pi/2, \pi)$. If $f \in \mathcal{M}(\gamma)$ has the form given by* (1)*, then*

(19)
$$|a_4| \le \begin{cases} \frac{1}{3}, & \text{when } \pi/2 \le \gamma \le \gamma_3, \\ \frac{(5-4\tau)^{3/2}}{9\sqrt{16-21\tau+8\tau^2}}, & \text{when } \gamma_3 \le \gamma \le \gamma_2, \\ \frac{-7+39\tau+176\tau^2-336\tau^3+128\tau^4}{12(57-48\tau^2)}\sqrt{k_1(\tau)}\sqrt{k_2(\tau)}, & \text{when } \gamma_2 \le \gamma \le \gamma_1, \\ \frac{1}{18}(1-\tau)(1-8\tau), & \text{when } \gamma_1 \le \gamma < \pi \end{cases}$$

with k_1 and k_2 given by (16) *and* (17)*, respectively. Here, $\gamma_1 \approx 2.058$, $\gamma_2 \approx 2.042$ and $\gamma_3 \approx 1.712$ is the unique root in $[\pi/2, \pi]$ of the equation $q_1(\cos\gamma) = 0$, $q_2(\cos\gamma) = 0$ and $q_3(\cos\gamma) = 0$, where q_1, q_2 and q_3 is defined by* (10)*,* (11) *and* (12)*, respectively. This result is sharp.*

Proof. For given $\gamma \in [\pi/2, \pi)$, let $f \in \mathcal{M}(\gamma)$ be of the form (1). Then there exists $p \in \mathcal{P}$ with the form given by (6) such that

$$\frac{zf'(z)}{f(z)} = B_\gamma\left(\frac{p(z)-1}{p(z)+1}\right). \tag{20}$$

Putting the series (1) and (6) into (20) by equating the coefficient we get

$$48a_4 = 9c_3 - 2(4\tau+1)c_1c_2 + \frac{1}{3}(8\tau^2+3\tau-2)c_1^3. \tag{21}$$

Since the class $\mathcal{M}(\gamma)$ is invariant under the rotations, by Lemma 1.1, we may assume that $c_1 := t \in [0,2]$. By using (8) and (9) in Lemma 1.2 we have

$$\begin{aligned}48|a_4| = \Big|&\frac{1}{3}(8\tau^2 - 9\tau + 1)t^3 + (3-4\tau)t(4-t^2)\zeta - 2t(4-t^2)\zeta^2 \\ &+ 4(4-t^2)(1-|\zeta|^2)\eta\Big|,\end{aligned} \tag{22}$$

where $\zeta \in \overline{\mathbb{D}}$ and $\eta \in \overline{\mathbb{D}}$.

Assume first that $t = 2$. Then

$$|a_4| = \frac{1}{18}(1-\tau)(1-8\tau). \tag{23}$$

On the other hand, for $t = 0$, we have

$$|a_4| = \frac{1}{3}(1-|\zeta|^2)|\eta| \leq \frac{1}{3}, \quad (\zeta,\eta) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}}. \tag{24}$$

Now let $t \in (0,2)$. Applying the triangle inequality to (22) we have

$$48|a_4| \leq 4(4-t^2)\Gamma(A,B,C), \tag{25}$$

where Γ is defined by

$$\Gamma(A,B,C) = |A + B\zeta + C\zeta^2| + 1 - |\zeta|^2, \quad \zeta \in \overline{\mathbb{D}}$$

with

$$A = \frac{(8\tau^2 - 9\tau + 1)t^3}{12(4-t^2)}, \quad B = \frac{1}{4}(3-4\tau)t, \quad \text{and} \quad C = -\frac{1}{2}t. \tag{26}$$

We will find $\max_{\zeta \in \overline{\mathbb{D}}} \Gamma(A,B,C)$ by using Lemma 1.4. For this, we note that $AC < 0$ since $\tau \in (-1,0]$ and $t \in (0,2)$. We also note that

$$-4AC(C^{-2} - 1) \leq B^2, \quad t \in (0,2),$$

since

$$4AC(C^{-2} - 1) + B^2 = \frac{1}{48}c^2(19 - 16\tau^2) \geq 0, \quad t \in (0,2).$$

Moreover, we note that

$$|C|(|B| + 4|A|) \geq |AB| \tag{27}$$

for all $t \in (0,2)$ and $\tau \in (-1,0]$. Indeed the inequality (27) is equivalent to $t^2 \leq \lambda_\tau$, where

$$\lambda_\tau = \frac{24(3-4\tau)}{-32\tau^3 - 4\tau^2 + 17\tau + 13}$$

and we can easily check that $\lambda_\tau > 4$ for given $\tau \in (-1,0]$.

Now, let

$$\tau^* = \sqrt{\frac{24(3-4\tau)}{29 - 127\tau + 124\tau^2 - 32\tau^3}}.$$

We note that $0 < 8/(7-4\tau) < \tau^* < 2$ for given $\tau \in (-1,0]$. And we consider subintervals I_1, I_2 and I_3 of $(0,2)$ defined by $I_1 = (0, 8/(7-4\tau)]$, $I_2 = [8/(7-4\tau), \tau^*]$ and $I_3 = [\tau^*, 2)$, respectively.

Let us consider the case $t \in I_1$. Then we have $|B| < 2(1-|C|)$, and by Lemma 1.4 we get

$$4(4-t^2)\Gamma(A,B,C) \le 4(4-t^2)\left[1+|A|+\frac{B^2}{4(1-|C|)}\right] = \Psi(t),$$

where $\Psi : I_1 \cup \{0\} \to \mathbb{R}$ is defined by

$$\Psi(x) = \frac{1}{24}(19-16\tau^2)x^3 - \frac{1}{4}(7-4\tau)(1+4\tau)x^2 + 16.$$

When $\tau \in (-1, -1/4]$, since

$$\Psi''(x) = \frac{1}{4}(19-16\tau^2)x - \frac{1}{2}(7-4\tau)(1+4\tau) \ge -\frac{1}{2}(1+4\tau)(7-4\tau) \ge 0$$

for $x \in (0,2)$, Ψ is convex on the interval I_1. Therefore, we have

$$\Psi(x) \le \max\left\{\Psi(0), \Psi\left(\frac{8}{7-4\tau}\right)\right\}, \quad x \in I_1. \tag{28}$$

When $\tau \in (-1/4, 0]$, we have $\Psi'(x) = 0$ occurs at $x = 0$ or $x = \mathring{\tau}$, where

$$\mathring{\tau} := \frac{4(7-4\tau)(1+4\tau)}{19-16\tau^2} > 0.$$

Since the leading coefficient of Ψ is positive for given $\tau \in (-1/4, 0]$, $\Psi(\mathring{\tau})$ is a local minimum on I_1 and therefore we have the inequality (28) again for the case $\tau \in (-1/4, 0]$. Furthermore, we can obtain $\Psi(8/(7-4\tau)) \ge 16$ when $\tau \in [-1, x_3]$ and $\Psi(8/(7-4\tau)) \le 16$ when $\tau \in [x_3, 0]$. Therefore, we can obtain

$$\Psi(x) \le \begin{cases} \Psi\left(\frac{8}{7-4\tau}\right), & \text{when } \tau \in (-1, x_3], \\ 16, & \text{when } \tau \in [x_3, 0], \end{cases} \quad x \in I_1. \tag{29}$$

And, it follows from (24), (25) and (29) that

$$48|a_4| \le g(\tau), \quad \text{when } t \in I_1 \cup \{0\} = [0, 8/(7-4\tau)],$$

where g is defined by (13).

Next, we consider the case $t \in I_2$. In this case we have $|AB| \le |C|(|B| - 4|A|)$. Therefore, by Lemma 1.4,

$$4(4-t^2)\Gamma(A,B,C) \le 4(4-t^2)(-|A|+|B|+|C|) = \Lambda(t), \tag{30}$$

where $\Lambda : I_2 \to \mathbb{R}$ is defined by

$$\Lambda(x) = -\frac{1}{3}(8\tau^2 - 21\tau + 16)x^3 + (20-16\tau)x.$$

Moreover, $\Lambda'(x) = 0$ at $x = \tilde{\tau}$, where

$$\tilde{\tau} := \sqrt{\frac{20-16\tau}{16-21\tau+8\tau^2}}. \tag{31}$$

Comparing the values $8/(7-4\tau)$, τ^* and $\tilde{\tau}$, we have

$$\begin{cases} \frac{8}{7-4\tau} \le \tau^* \le \tilde{\tau}, & \text{when } \tau \in (-1, x_2]; \\ \frac{8}{7-4\tau} \le \tilde{\tau} \le \tau^*, & \text{when } \tau \in [x_2, x_4]; \\ \tilde{\tau} \le \frac{8}{7-4\tau} \le \tau^*, & \text{when } \tau \in [x_4, 0]. \end{cases}$$

Furthermore, Λ is increasing on I_2 when $\tau \in (-1, x_2]$, and Λ is decreasing on I_2 when $\tau \in [x_4, 0]$. On the other hand, when $\tau \in [x_2, x_4]$, Λ has its local maximum at $\tilde{\tau}$. Using these facts, we can obtain the maximum for Λ on I_2 as follows:

$$\Lambda(x) \le \begin{cases} \Lambda(\tau^*), & \text{when } \tau \in (-1, x_2]; \\ \Lambda(\tilde{\tau}), & \text{when } \tau \in [x_2, x_4]; \\ \Lambda\left(\frac{8}{7-4\tau}\right), & \text{when } \tau \in [x_4, 0], \end{cases} \quad x \in I_2. \tag{32}$$

Combining (25), (30) and (32), we have

$$48|a_4| \le h(\tau), \quad \text{when } t \in I_2,$$

where h is defined by (14).

Next, let us consider the case $t \in I_3$. In this case, from Lemma 1.4, we have

$$4(4-t^2)\Gamma(A, B, C) \le 4(4-t^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = \Phi(t), \tag{33}$$

where $\Phi : I_3 \cup \{2\} \to \mathbb{R}$ is defined by

$$\Phi(x) = \frac{1}{3}(24x + (8\tau^2 - 9\tau - 5)x^3)\sqrt{\phi(x)}$$

with

$$\phi(x) = 1 + \frac{3(3-4\tau)^2(4-x^2)}{8(8\tau^2 - 9\tau + 1)x^2}.$$

Differentiating Φ, we have

$$\Phi'(x) = \frac{\kappa_1 + \kappa_2 x^2}{2(1 - 9\tau + 8\tau^2)\sqrt{\phi(x)}}, \tag{34}$$

where

$$\kappa_1 = 8(-64 + 39\tau + 224\tau^2 - 336\tau^3 + 128\tau^4)$$

and

$$\kappa_2 = (5 + 9\tau - 8\tau^2)(19 - 16\tau^2).$$

We note that $\kappa_2 \ge 0$ for $\tau \in [\frac{9-\sqrt{241}}{16}, 0]$ and $\kappa_2 \le 0$ for $\tau \in (-1, \frac{9-\sqrt{241}}{16}]$.

When $\tau \in [\frac{9-\sqrt{241}}{16}, 0]$, we have

$$\kappa_1 + \kappa_2 x^2 \le \kappa_1 + 4\kappa_2 = 12(-11 + 83\tau + 72\tau^2 - 272\tau^2 + 128\tau^4) < 0, \quad x \in I_3.$$

Therefore, by (34), Φ is decreasing on I_3. Also, Φ is decreasing on I_3 for the case $\tau \in [x_2, \frac{9-\sqrt{241}}{16}]$, since

$$\kappa_1 + \kappa_2 x^2 \leq \kappa_1 + \kappa_2 (\tau^*)^2 = \frac{-8q_2(\tau)}{29 - 127\tau + 124\tau^2 - 32\tau^3} \leq 0, \quad x \in I_3.$$

For the case $\tau \in (-1, x_2]$, we note that $\kappa_1 \kappa_2 < 0$ and $\Phi'(x) = 0$ occurs at $x = \hat{\tau}$, where

$$\hat{\tau} := \sqrt{\frac{\kappa_1}{-\kappa_2}} = \sqrt{\frac{8(-64 + 39\tau + 224\tau^2 - 336\tau^3 + 128\tau^4)}{-(5 + 9\tau - 8\tau^2)(19 - 16\tau^2)}}. \tag{35}$$

Comparing the values 2, τ^* and $\hat{\tau}$, we have

$$\begin{cases} \tau^* \leq 2 \leq \hat{\tau}, & \text{when } \tau \in (-1, x_1]; \\ \tau^* \leq \hat{\tau} \leq 2, & \text{when } \tau \in [x_1, x_2]. \end{cases}$$

Furthermore, Φ is increasing on $[\tau^*, 2]$ when $\tau \in (-1, x_1]$, and Φ has the local maximum at $x = \hat{\tau}$ when $\tau \in [x_1, x_2]$. Using these facts, we can obtain the following maximum for Φ on I_3 for each cases:

$$\Phi(x) \leq \begin{cases} \Phi(2), & \text{when } \tau \in (-1, x_1]; \\ \Phi(\hat{\tau}), & \text{when } \tau \in [x_1, x_2]; \\ \Phi(\tau^*), & \text{when } \tau \in [x_2, 0], \end{cases} \quad x \in I_3. \tag{36}$$

Combining (23), (25), (33) and (36), we have

$$48|a_4| \leq k(\tau), \quad \text{when } t \in I_3 \cup \{2\} = [\tau^*, 2],$$

where k is defined by (15).

Consequently, we obtain

$$48|a_4| \leq \begin{cases} g(\tau), & \text{when } t \in [0, 8/(7 - 4\tau)]; \\ h(\tau), & \text{when } t \in [8/(7 - 4\tau), \tau^*]; \\ k(\tau), & \text{when } t \in [\tau^*, 2]. \end{cases} \tag{37}$$

We note that each conditions $\gamma \in [\pi/2, \gamma_3]$, $\gamma \in [\gamma_3, \gamma_2]$, $\gamma \in [\gamma_2, \gamma_1]$ and $\gamma \in [\gamma_1, \pi)$ are equivalent to $\tau \in [x_3, 0]$, $\tau \in [x_2, x_3]$, $\tau \in [x_1, x_2]$ and $\tau \in (-1, x_1]$, respectively. Thus, it follows from (37) and (18) that the inequality (19) holds.

From now, we will show that this result is sharp. The equality for the first case (i.e., $\gamma \in [\pi/2, \gamma_3]$) in (19) holds for the function $f_1 \in \mathcal{M}(\gamma)$ defined by

$$f_1(z) = z \exp\left[\int_0^z \frac{1}{\xi}\left(B_\gamma(\xi^3) - 1\right) \mathrm{d}\xi\right] = z + \frac{1}{3}z^4 + \cdots, \quad z \in \mathbb{D}.$$

And the equality for the fourth case (i.e., $\gamma \in [\gamma_1, \pi)$) in (19) holds for the function $f_2 \in \mathcal{M}(\gamma)$ defined by

$$\begin{aligned} f_2(z) &= z \exp\left[\int_0^z \frac{1}{\xi}\left(B_\gamma(\xi) - 1\right) \mathrm{d}\xi\right] \\ &= z + z^2 + \frac{1}{2}(1-\tau)z^3 + \frac{1}{18}(1 - 9\tau + 8\tau^2)z^4 \\ &\quad + \frac{1}{72}(-5 + 41\tau^2 - 36\tau^3)z^5 + \cdots, \quad z \in \mathbb{D}. \end{aligned} \tag{38}$$

To consider the sharpness for the second case in (19), let us fix $\gamma \in [\gamma_3, \gamma_2]$. We note that the equality in (19) holds for ζ which satisfies

$$4(4-t^2)(|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2) = 4(4-t^2)(-|A| + |B| + |C|) = \Lambda(\tilde{\tau}),$$

where $\tilde{\tau}$ is given by (31) and A, B, C are given by (26) with $t = \tilde{\tau}$. And we can easily check that this relation is satisfied for $\zeta = -1$ and $t = c_1 = \tilde{\tau}$. Therefore, by Lemma 1.2, we have

$$c_2 = \tilde{\tau}^2 - 2 \quad \text{and} \quad c_3 = \tilde{\tau}^3 - 3\tilde{\tau}.$$

We consider the function $f_3 : \mathbb{D} \to \mathbb{C}$ defined by

$$f_3(z) = z \exp\left[\int_0^z \frac{1}{\xi}\left(B_\gamma\left(\frac{p(\xi) - 1}{p(\xi) + 1}\right) - 1\right) \mathrm{d}\xi\right], \quad z \in \mathbb{D}, \tag{39}$$

where $p \in \mathcal{P}$ has the form given by

$$p(z) = 1 + \tilde{\tau}z + (\tilde{\tau}^2 - 2)z^2 + (\tilde{\tau}^3 - 3\tilde{\tau})z^3 + \cdots. \tag{40}$$

From (21), we can easily check that equality in (19) holds for this function f_3. Therefore, it is enough to construct a function $p \in \mathcal{P}$ with the form given by (40). Since $c_2 = c_1^2 - 2$, it follows from Lemma 1.3 with $\zeta = -1$ that the function p should be defined by

$$p(z) = \frac{1 - z^2}{1 - \tilde{\tau}z + z^2}, \quad z \in \mathbb{D}$$

and this guarantees the sharpness of the inequality (19) for the second case.

Finally, it is remained to show the sharpness of (19) for the third case. Fix $\gamma \in [\gamma_2, \gamma_1]$. We note that the equality in (19) holds for ζ which satisfies

$$4(4-t^2)(|A + B\zeta + C\zeta^2| + 1 - |\zeta|^2) = 4(4-t^2)(|C| + |A|)\sqrt{1 - \frac{B^2}{4AC}} = \Phi(\hat{\tau}),$$

where $\hat{\tau}$ is given by (35) and A, B, C are given by (26) with $t = \hat{\tau}$. Moreover, this relation holds for $\zeta = \mathrm{e}^{\mathrm{i}\theta}$, where $\theta \in [0, 2\pi)$ is defined so that

$$\cos\theta = -\frac{B(A+C)}{4AC} = \frac{(3-4\tau)(-163 - 105\tau + 368\tau^2 - 336\tau^3 + 128\tau^4)}{8(-64 + 39\tau + 224\tau^2 - 336\tau^3 + 128\tau^4)}.$$

Therefore, by Lemma 1.2, we have

$$c_2 = \frac{1}{2}[\hat{\tau}^2 + \zeta(4 - \hat{\tau}^2)] \quad \text{and} \quad c_3 = \frac{1}{4}\hat{\tau}[\hat{\tau}^2 + \zeta(2-\zeta)(4 - \hat{\tau}^2)].$$

Since the equality in (19) holds for the function f_3 defined by (39) with $p \in \mathcal{P}$, where

$$p(z) = 1 + \hat{\tau} z + \frac{1}{2}[\hat{\tau}^2 + \zeta(4 - \hat{\tau}^2)]z^2 + \frac{1}{4}\hat{\tau}[\hat{\tau}^2 + \zeta(2 - \zeta)(4 - \hat{\tau}^2)]z^3 + \cdots,$$

it is enough to construct a function $p \in \mathcal{P}$ with this representation. Similar methods with the second case and Lemma 1.3 lead us to get the desired function $p \in \mathcal{P}$ defined by

$$p(z) = \frac{2 + \hat{\tau}(1 + \zeta)z + 2\zeta z^2}{2 - \hat{\tau}(1 - \zeta)z - 2\zeta z^2}$$

and this completes the proof of Theorem 2.1. □

We give here 5 figures which present the graphs of g, h and k defined by (13), (14) and (15), respectively, to justify the equality (18).

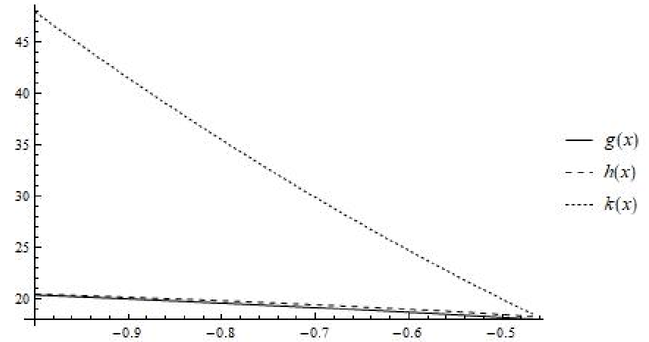

FIGURE 1. Graphs of g, h, k on the interval $[-1, x_1]$

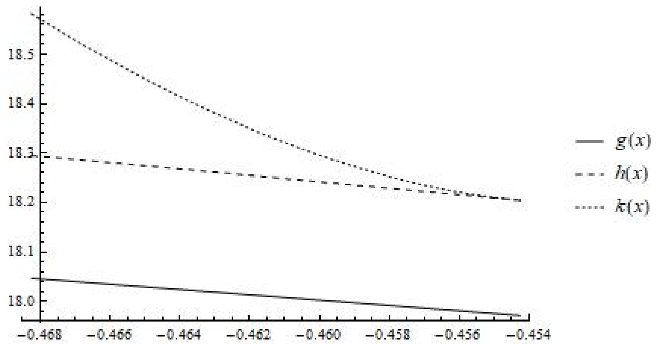

FIGURE 2. Graphs of g, h, k on the interval $[x_1, x_2]$

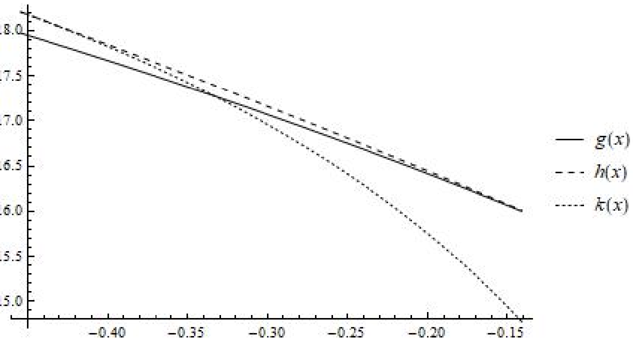

FIGURE 3. Graphs of g, h, k on the interval $[x_2, x_3]$

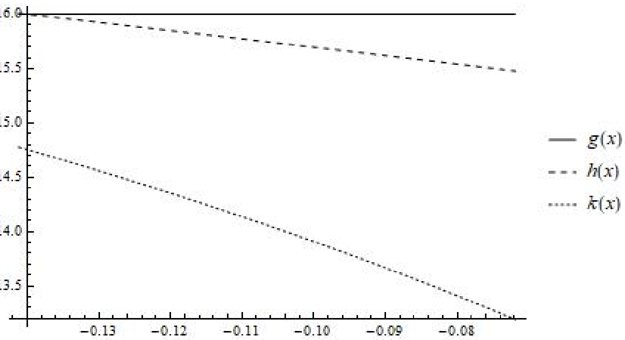

FIGURE 4. Graphs of g, h, k on the interval $[x_3, x_4]$

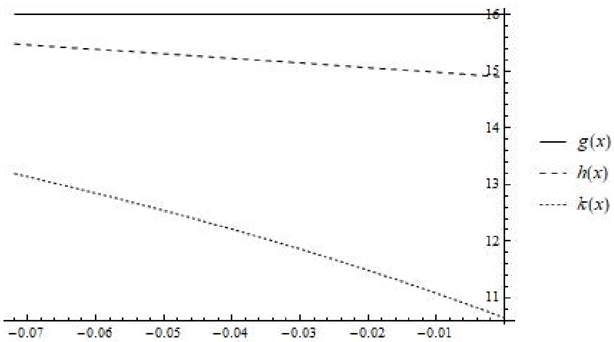

FIGURE 5. Graphs of g, h, k on the interval $[x_4, 0]$

3. On the fifth coefficients of functions in $\mathcal{M}(\gamma)$

In this section, we obtain the sharp bound for the fifth coefficient of $f \in \mathcal{M}(\gamma)$ for γ in a subset of $[\pi/2, \pi)$. For this, let $x_5 \approx -0.882$ be the zero in the interval $(-1,-1/2)$ of the polynomial r_1 defined by

$$r_1(x) = 10 + 12x - 31x^2 - 36x^3. \tag{41}$$

Theorem 3.1. *Let $\gamma_5 \approx 2.651$ be a root of the equation $r_1(\cos\gamma) = 0$ and let $\gamma \in [\gamma_5, \pi)$. If $f \in \mathcal{M}(\gamma)$ has the form given by (1), then*

$$|a_5| \leq \frac{1}{72}(-5 + 41\tau^2 - 36\tau^3). \tag{42}$$

This result is sharp.

Proof. By the same methods in the proof of Theorem 2.1, we have

$$64a_5 = \frac{1}{18}r_1(\tau)c_1^4 + \frac{2}{3}r_2(\tau)c_1^2c_2 + r_3(\tau)c_2^2 + \frac{8}{3}r_4(\tau)c_1c_3 + 8c_4, \tag{43}$$

where r_1 is defined by (41),

$$r_2(x) = -2 + 7x + 12x^2,$$

$$r_3(x) = -4x - 2$$

and

$$r_4(x) = -3x - 1.$$

And we can easily check that

$$r_i(x) \geq 0, \quad i = 1, 2, 3, 4$$

for $x \in [-1, x_5]$. Or, equivalently, $r_i(\tau) \geq 0$ $(i = 1, 2, 3, 4)$ hold when $\gamma \in [\gamma_5, \pi)$. Therefore, by applying Lemma 1.1 to the inequality which obtained by applying the triangle inequality to (43), we obtain

$$64|a_5| \leq \frac{8}{9}r_1(\tau) + \frac{16}{3}r_2(\tau) + 4r_3(\tau) + \frac{32}{3}r_4(\tau) + 16 = -\frac{8}{9}(5 - 41\tau^2 + 36\tau^3),$$

which implies the inequality (42). The equality in (42) holds for f_2 defined by (38) and this completes the proof of our result. □

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