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UNIQUENESS OF HOMOGENEOUS DIFFERENTIAL POLYNOMIALS CONCERNING WEAKLY WEIGHTED-SHARING

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Abstract. In 2006, S. Lin and W. Lin introduced the definition of weakly weighted-sharing of meromorphic functions which is between "CM" and "IM". In this paper, using the notion of weakly weighted-sharing, we study the uniqueness of nonconstant homogeneous differential polynomials $P[f]$ and $P[g]$ generated by meromorphic functions f and g, respectively. Our results generalize the results due to S. Lin and W. Lin, and H.-Y. Xu and Y. Hu.

1. Introduction and main result

Let $\mathbb{C}$ denote the complex plane and let $f(z)$ be a nonconstant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notions used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see [1, 6, 7]). By $S(r,f)$ we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if either $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers. For $a \in \mathbb{C} \cup \{\infty\}$ the quantities

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T(r,f)}$$

and

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}.$$

are respectively called the deficiency and ramification index of a for the function f.

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For any two nonconstant meromorphic functions f and g, and $a \in S(f) \cap S(g)$, we say that f and g share a IM (CM) provided that $f-a$ and $g-a$ have the same zeros ignoring (counting) multiplicities. If $\frac{1}{f}$ and $\frac{1}{g}$ share 0 IM (CM), we say that f and g share ∞ IM (CM).

Definition 1.1. Let k be a nonnegative integer or infinity and $a(z) \in S(f)$. We denote by $E_k(a,f)$ the set of all zeros of $f-a$, where a zero of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that $f,\ g$ share the function $a(z)$ with weight k. We write f and g share (a,k) to mean that f and g share the function $a(z)$ with weight k. Since $E_k(a,f) = E_k(a,g)$ implies that $E_l(a,f) = E_l(a,g)$ for any integer $l\,(0 \leq l < k)$, if $f,\ g$ share (a,k), then $f,\ g$ share (a,l), $(0 \leq l < k)$. Moreover, we note that f and g share the function $a(z)$ IM or CM if and only if f and g share $(a,0)$ or (a,∞) respectively.

Definition 1.2 ([3])**.** Let $N_E(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ with the same multiplicities, and $N_0(r,a)$ be the counting function of all common zeros of $f-a$ and $g-a$ ignoring multiplicities. Denote by $\overline{N}_E(r,a)$ and $\overline{N}_0(r,a)$ the reduced counting functions of f and g corresponding to the counting functions $N_E(r,a)$ and $N_0(r,a)$ respectively. If

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_E(r,a) = S(r,f) + S(r,g),$$

then we say that f and g share a "CM". If

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a) = S(r,f) + S(r,g),$$

then we say that f and g share a "IM".

Let k be a positive integer, and let f be a meromorphic function and $a \in S(f)$.

(i) $\overline{N}_{k)}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}_{(k}(r,a;f)$ denotes the counting function of those a-points of f whose multiplicities are not less than k, where each a-point is counted only once.

(iii) $N_p(r,a;f)$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m counted m times if $m \leq p$ and p times if $m > p$.

We denote by $\delta_p(a,f)$ the quantity

$$\delta_p(a,f) = 1 - \limsup_{r\to\infty} \frac{N_p(r,a;f)}{T(r,f)},$$

where p is a positive integer. Clearly $\delta_p(a,f) \geq \delta(a,f)$.

Let f and g be two nonconstant meromorphic functions sharing a "IM" for $a \in S(f) \cap S(g)$, and a positive integer k or ∞.

(i) $\overline{N}^{E}_{k)}(r,a)$ denotes the counting function of those a-points of f whose multiplicities are equal to the corresponding a-points of g, both of their multiplicities are not greater than k, where each a-point is counted only once.

(ii) $\overline{N}^{0}_{(k}(r,a)$ denotes the reduced counting function of those a-points of f which are a-points of g, both of their multiplicities are not less than k, where each a-point is counted only once.

Definition 1.3 ([3])**.** For $a \in S(f) \cap S(g)$, if k is a positive integer or ∞, and

$$\overline{N}_{k)}(r,a;f) + \overline{N}_{k)}(r,a;g) - 2\overline{N}^{E}_{k)}(r,a) = S(r,f) + S(r,g),$$

$$\overline{N}_{(k+1}(r,a;f) + \overline{N}_{(k+1}(r,a;g) - 2\overline{N}^{0}_{(k+1}(r,a) = S(r,f) + S(r,g)$$

or if $k = 0$ and

$$\overline{N}(r,a;f) + \overline{N}(r,a;g) - 2\overline{N}_0(r,a) = S(r,f) + S(r,g),$$

then we say f and g weakly share a with weight k. Here, we write f, g share "(a,k)" to mean that f, g weakly share a with weight k.

Obviously if f and g share "(a,k)", then f and g share "(a,p)" for any p $(0 \le p \le k)$. Also, we note that f and g share a "IM" or "CM" if and only if f and g share "$(a,0)$" or "(a,∞)", respectively.

Suppose F and G share 1 "IM". By $\overline{N}_L(r,1;F)$ we denotes the counting function of the 1-points of F whose multiplicities are greater than 1-points of G. $\overline{N}_L(r,1;G)$ is defined similarly.

Definition 1.4. Let f be a nonconstant meromorphic function. An expression of the form

$$P[f] = \sum_{k=1}^{n} a_k \prod_{j=0}^{p} \left(f^{(j)}\right)^{l_{kj}}, \tag{1.1}$$

where $a_k \in S(f)$ for $k = 1,2,\ldots,n$ and l_{kj} are nonnegative integers for $k = 1,2,\ldots,n$; $j = 0,1,2,\ldots,p$ and $d = \sum_{j=0}^{p} l_{kj}$ for $k = 1,2,\ldots,n$, is called a homogeneous differential polynomial of degree d generated by f. Also we denote by Q the quantity $Q = \max_{1\le k\le n} \sum_{j=0}^{p} j.l_{kj}$.

In 2006 S. Lin and W. Lin [3] first defined and used the concept of weakly-weighted sharing of functions to prove the uniqueness of a meromorphic function and its derivative and proved the following theorems:

Theorem 1.1. *Let $n \ge 1$ and $2 \le k \le \infty$, let f be a nonconstant meromorphic function, $a \in S(f)$ and $a \not\equiv 0,\infty$. If f and $f^{(n)}$ share "(a,k)" and*

$$4\Theta(\infty,f) + 2\delta_{2+n}(0,f) > 5,$$

then $f \equiv f^{(n)}$.

Theorem 1.2. *Let $n \ge 1$ and let f be a nonconstant meromorphic function, $a \in S(f)$ and $a \not\equiv 0,\infty$. If f and $f^{(n)}$ share "$(a,1)$" and*

$$\left(\frac{n+9}{2}\right)\Theta(\infty,f) + \frac{5}{2}\delta_{2+n}(0,f) > \frac{n}{2} + 6,$$

then $f \equiv f^{(n)}$.

Theorem 1.3. *Let $n \geq 1$ and let f be a nonconstant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. If f and $f^{(n)}$ share "$(a,0)$" and*

$$(7+2n)\Theta(\infty, f) + 5\delta_{2+n}(0,f) > 2n + 11,$$

then $f \equiv f^{(n)}$.

Later in 2011, H.-Y. Xu and Y. Hu [4] generalize Theorems 1.1–1.3 by proving the following theorems:

Theorem 1.4. *Let $n \geq 1$ and $2 \leq k \leq \infty$, let f be a nonconstant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose $L(f) = f^{(n)} + a_{n-1}f^{(n-1)} + \cdots + a_0 f$. If f and $L(f)$ share "(a,k)" and*

$$4\Theta(\infty, f) + 2\delta_{2+n}(0,f) > 5,$$

then $f \equiv L(f)$.

Theorem 1.5. *Let $n \geq 1$, let f be a nonconstant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose $L(f)$ be defined as in Theorem 1.4. If f and $L(f)$ share "$(a,1)$" and*

$$\left(\frac{7}{2} + n\right)\Theta(\infty, f) + \frac{3}{2}\delta_2(0,f) + \delta_{n+2}(0,f) > n + 5,$$

then $f \equiv L(f)$.

Theorem 1.6. *Let $n \geq 1$, let f be a nonconstant meromorphic function, $a \in S(f)$ and $a \not\equiv 0, \infty$. Suppose $L(f)$ be defined as in Theorem 1.4. If f and $L(f)$ share "$(a,0)$" and*

$$(6+2n)\Theta(\infty, f) + \delta_2(0,f) + 2\Theta(0,f) + 2\delta_{2+n}(0,f) > 2n + 10,$$

then $f \equiv L(f)$.

Motivated by such uniqueness investigation, it is natural to consider the problem in a more general setting: Let f and g be any two nonconstant meromorphic functions, $P[f]$ and $P[g]$ be nonconstant homogeneous differential polynomials of f and g respectively, and $a(z) \in S(f) \cap S(g)$, $a \not\equiv 0, \infty$. If $P[f]$ and $P[g]$ share "(a,k)", then what will be the relation between $P[f]$ and $P[g]$? In this paper we prove that under certain conditions either $P[f] \equiv P[g]$ or $P[f].P[g] \equiv a^2$.

Now, we state the main result of this paper.

Theorem 1.7. *Let f and g be two transcendental meromorphic functions, $a = a(z)$ $(a \not\equiv 0, \infty) \in S(f) \cap S(g)$. Suppose $P[f]$ and $P[g]$, defined by (1.1) are nonconstant. If $P[f]$ and $P[g]$ share "(a,k)" with one of the following conditions*:

(i) *$k \geq 2$ and*

$$\min\{(Q+4)\Theta(\infty,f) + 2\delta_{2+p}(0,f), (Q+4)\Theta(\infty,g) + 2\delta_{2+p}(0,g)\} > 6 + Q - d, \tag{1.2}$$

(ii) $k = 1$ *and*

$$\min\{(3Q+9)\Theta(\infty, f) + 5\delta_{2+p}(0, f), (3Q+9)\Theta(\infty, g) + 5\delta_{2+p}(0, g)\} > 3Q + 14 - 2d, \tag{1.3}$$

(iii) $k = 0$ *and*

$$\min\{(4Q+7)\Theta(\infty, f) + 5\delta_{2+p}(0, f), (4Q+7)\Theta(\infty, g) + 5\delta_{2+p}(0, g)\} > 4Q + 12 - d, \tag{1.4}$$

then either $P[f] \equiv P[g]$ *or* $P[f].P[g] \equiv a^2$.

2. Lemmas

In this section we present some lemmas which will needed in the sequel.

Lemma 2.1 ([2]). *Let* f *be a nonconstant meromorphic function and* $P[f]$ *be defined by* (1.1). *Then*

$$N(r, \infty; P) \le dN(r, \infty; f) + Q\overline{N}(r, \infty; f) + S(r, f).$$

Lemma 2.2. *Let* f *be a transcendental meromorphic function and* $P[f]$ *be same as in* (1.1). *If* $P[f] \not\equiv 0$, *then we have*
(i) $N_2(r, 0; P) \le N_{2+p}(r, 0; f) + Q\overline{N}(r, \infty; f) + S(r, f)$,
(ii) $N_2(r, 0; P) \le N_{2+p}(r, 0; f) + T(r, P) - dT(r, f) + S(r, f)$.

Proof.

$$\begin{aligned}
N_2(r, 0; P) &\le N(r, 0; P) - \sum_{k=3}^{\infty} \overline{N}(r, 0; P| \ge k) \\
&= T(r, P) - m(r, 0; P) - \sum_{k=3}^{\infty} \overline{N}(r, 0; P| \ge k) + O(1) \\
&\le T(r, P) - m(r, 0; f^d) + m(r, \infty; \frac{P}{f^d}) - \sum_{k=3}^{\infty} \overline{N}(r, 0; P| \ge k) + O(1) \\
&\le T(r, P) - dT(r, f) + N(r, 0; f^d) - \sum_{k=3}^{\infty} \overline{N}(r, 0; P| \ge k) + S(r, f) \\
&\le T(r, P) - dT(r, f) + N_{2+p}(r, 0; f^d) \\
&\quad + \sum_{k=3+p}^{\infty} \overline{N}(r, 0; f^d| \ge k) - \sum_{k=3}^{\infty} \overline{N}(r, 0; P| \ge k) + S(r, f) \\
&\le T(r, P) - dT(r, f) + N_{2+p}(r, 0; f) + S(r, f).
\end{aligned}$$

This proves (ii).

Now using Lemma 2.1 we have,

$$T(r, P) = N(r, \infty; P) + m(r, \infty; P)$$

$$
\begin{aligned}
&\le m(r,\infty;f^d) + m(r,\infty;\frac{P}{f^d}) + N(r,\infty;P)\\
&= dm(r,\infty;f) + N(r,\infty;P) + S(r,f)\\
&\le dm(r,\infty;f) + dN(r,\infty;f) + Q\overline{N}(r,\infty;f) + S(r,f)\\
&\le dT(r,f) + Q\overline{N}(r,\infty;f) + S(r,f).
\end{aligned}
$$

Therefore, $N_2(r,0;P) \le N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;f) + S(r,f)$. □

Lemma 2.3 ([3]). *Let k be a nonnegative integer or infinity, F and G be nonconstant meromorphic functions, F and G share "$(1,k)$". Let*

$$H = \left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F-1}\right) - \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G-1}\right).$$

If $H \not\equiv 0$, then

(i) *If $2 \le k \le \infty$, then*

$$
\begin{aligned}
T(r,F) \le\ & N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)\\
&+ S(r,F) + S(r,G).
\end{aligned}
$$

(ii) *If $k = 1$, then*

$$
\begin{aligned}
T(r,F) \le\ & N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)\\
&+ \overline{N}_L(r,1;F) + S(r,F) + S(r,G).
\end{aligned}
$$

(iii) *If $k = 0$, then*

$$
\begin{aligned}
T(r,F) \le\ & N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)\\
&+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G).
\end{aligned}
$$

The same inequality holds for $T(r,G)$.

Lemma 2.4 ([4]). *Let F and G be nonconstant meromorphic functions such that F and G share "$(1,1)$". Then*

$$\overline{N}_L(r,1;F) \le \frac{1}{2}\overline{N}(r,0;F) + \frac{1}{2}\overline{N}(r,\infty;F) + S(r,F).$$

Lemma 2.5 ([4]). *Let F and G be nonconstant meromorphic functions such that F and G share "$(1,0)$". Then*

$$\overline{N}_L(r,1;F) \le \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + S(r,F).$$

Lemma 2.6 ([5]). *Let f be a nonconstant meromorphic function and let*

$$p(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0,$$

where $a_i \in S(f)$ for $i = 0,1,\dots,n$; $a_n \ne 0$ be a polynomial of degree n. Then $T(r,p(f)) = nT(r,f) + S(r,f)$.

3. Proof of the main theorem

Proof of Theorem 1.7. Let

$$F = \frac{P[f]}{a}, \ G = \frac{P[g]}{a}.$$

Since $P[f]$ and $P[g]$ share "(a, k)", it follows that F, G share "$(1, k)$" except at the zeros and poles of a.

Let H be same as in Lemma 2.3. Suppose that $H \not\equiv 0$.

Now we consider the following three cases:

Case 1: $2 \leq k \leq \infty$.

From (i) of Lemma 2.3

$$\begin{aligned} T(r, F) \leq\ & N_2(r, \infty; F) + N_2(r, \infty; G) + N_2(r, 0; F) + N_2(r, 0; G) \\ & + S(r, F) + S(r, G). \end{aligned}$$

Using Lemma 2.2

$$\begin{aligned} T(r, F) \leq\ & 2\overline{N}(r, \infty; F) + 2\overline{N}(r, \infty; G) + T(r, F) - dT(r, f) + N_{2+p}(r, 0; f) \\ & + Q\overline{N}(r, \infty; g) + N_{2+p}(r, 0; g) + S(r, F) + S(r, G) \end{aligned}$$

and so

$$\begin{aligned} dT(r, f) \leq\ & 2\overline{N}(r, \infty; f) + N_{2+p}(r, 0; f) + (2 + Q)\overline{N}(r, \infty; g) \\ & + N_{2+p}(r, 0; g) + S(r, f) + S(r, g). \end{aligned} \tag{3.1}$$

Similarly,

$$\begin{aligned} dT(r, g) \leq\ & 2\overline{N}(r, \infty; g) + N_{2+p}(r, 0; g) + (2 + Q)\overline{N}(r, \infty; f) \\ & + N_{2+p}(r, 0; f) + S(r, f) + S(r, g). \end{aligned} \tag{3.2}$$

Adding (3.1) and (3.2)

$$\begin{aligned} dT(r, f) + dT(r, g) \leq\ & 2N_{2+p}(r, 0; f) + (Q + 4)\overline{N}(r, \infty; f) + 2N_{2+p}(r, 0; g) \\ & + (Q + 4)\overline{N}(r, \infty; g) + S(r, f) + S(r, g) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & \{2\delta_{2+p}(0, f) + (Q + 4)\Theta(\infty, f) - (6 + Q - d)\}\, T(r, f) \\ & + \{2\delta_{2+p}(0, g) + (Q + 4)\Theta(\infty, g) - (6 + Q - d)\}\, T(r, g) \\ & \leq S(r, f) + S(r, g). \end{aligned}$$

Which contradict our hypothesis (1.2).

Thus $H \equiv 0$. That is

$$\left(\frac{F^{(2)}}{F^{(1)}} - 2\frac{F^{(1)}}{F - 1}\right) = \left(\frac{G^{(2)}}{G^{(1)}} - 2\frac{G^{(1)}}{G - 1}\right) \Rightarrow \frac{1}{G - 1} = \frac{A}{F - 1} + B,$$

where $A \neq 0$ and B are constants.

Thus

$$G = \frac{(B + 1)F + (A - B - 1)}{BF + (A - B)} \tag{3.3}$$

and

$$F = \frac{(B-A)G + (A-B-1)}{BG - (B+1)}. \tag{3.4}$$

Next we consider following three subcases:

Subcase 1. $B \neq 0, -1$. Then from (3.4) we have

$$\overline{N}(r, \frac{B+1}{B}; G) = \overline{N}(r, \infty; F).$$

By Nevanlinna second fundamental theorem and (ii) of Lemma 2.2 we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{B+1}{B}; G) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,\infty;F) + S(r,G) \\ &\leq \overline{N}(r,\infty;G) + T(r,G) - dT(r,g) + N_{2+p}(r,0;g) \\ &\quad + \overline{N}(r,\infty;F) + S(r,G), \end{aligned}$$

i.e.,

$$dT(r,g) \leq \overline{N}(r,\infty;f) + N_{2+p}(r,0;g) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \tag{3.5}$$

If $A - B - 1 \neq 0$, then it follows from (3.3) that

$$\overline{N}(r, \frac{-A+B+1}{B+1}; F) = \overline{N}(r, 0; G).$$

Again by Nevanlinna second fundamental theorem and Lemma 2.2

$$T(r,F) \leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r, \frac{-A+B+1}{B+1}; F) + S(r,F)$$

$$\begin{aligned} \Rightarrow \quad dT(r,f) &\leq \overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + Q\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) \\ &\quad + S(r,f) + S(r,g). \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6)

$$\begin{aligned} dT(r,f) + dT(r,g) \leq\ & N_{2+p}(r,0;f) + 2\overline{N}(r,\infty;f) + 2N_{2+p}(r,0;g) \\ & + (Q+1)\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \end{aligned}$$

which again contradict (1.2).

Hence $A - B - 1 = 0$. Then by (3.3)

$$\overline{N}(r, 0; F + \frac{1}{B}) = \overline{N}(r, \infty; G).$$

Again by Nevanlinna second fundamental theorem

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,0;F + \frac{1}{B}) + S(r,F) \\ &\leq \overline{N}(r,\infty;f) + T(r,F) - dT(r,f) + N_{2+p}(r,0;f) + \overline{N}(r,\infty;g) \\ &\quad + S(r,f) + S(r,g), \end{aligned}$$

i.e.,

$$dT(r,f) \le \overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + \overline{N}(r,\infty;g) + S(r,f) + S(r,g). \tag{3.7}$$

Combining (3.5) and (3.7)

$$\begin{aligned} dT(r,f) + dT(r,g) \le\ & N_{2+p}(r,0;f) + 2\overline{N}(r,\infty;f) + N_{2+p}(r,0;g) \\ & + 2\overline{N}(r,\infty;g) + S(r,f) + S(r,g), \end{aligned}$$

which violates our given assumption.

Subcase 2. $B = -1$. Then

$$G = \frac{A}{A+1-F}$$

and

$$F = \frac{(1+A)G - A}{G}.$$

If $A + 1 \neq 0$, then we obtain

$$\overline{N}(r, A+1; F) = \overline{N}(r,\infty;G),$$

$$\overline{N}(r, \frac{A}{A+1}; G) = \overline{N}(r,0;F).$$

By similar argument, we have a contradiction.

Therefore, $A + 1 = 0$, then

$$FG = 1 \;\Rightarrow\; P[f].P[g] \equiv a^2.$$

Subcase 3. $B = 0$. Then (3.3) and (3.4) gives $G = \frac{F+A-1}{A}$ and $F = AG + 1 - A$.

If $A - 1 \neq 0$, $N(r,0;A-1+F) = N(r,0;G)$ and $N(r,\frac{A-1}{A};G) = N(r,0;F)$. Proceeding similarly as in Subcase 1, we get a contradiction.

Therefore, $A - 1 = 0$, then $F \equiv G$, i.e.,

$$P[f] \equiv P[g].$$

This complete the proof of Case 1.

Case 2: $k = 1$. From (ii) of Lemma 2.3

$$\begin{aligned} T(r,F) \le\ & N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G) + \overline{N}_L(r,1;F) \\ & + S(r,F) + S(r,G). \end{aligned}$$

Using Lemma 2.2

$$\begin{aligned} T(r,F) \le\ & 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) + N_{2+p}(r,0;f) \\ & + Q\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + \overline{N}_L(r,1;F) + S(r,F) + S(r,G). \end{aligned}$$

So by Lemma 2.2 and Lemma 2.4

$$\begin{aligned} dT(r,f) \le\ & 2\overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + (2+Q)\overline{N}(r,\infty;g) \\ & + N_{2+p}(r,0;g) + \frac{1}{2}\overline{N}(r,\infty;f) + \frac{1}{2}Q\overline{N}(r,\infty;f) + \frac{1}{2}N_{2+p}(r,0;f) \\ & + S(r,f) + S(r,g) \end{aligned}$$

$$\leq \frac{5+Q}{2}\overline{N}(r,\infty;f) + \frac{3}{2}N_{2+p}(r,0;f) + (2+Q)\overline{N}(r,\infty;g)$$
$$+ N_{2+p}(r,0;g) + S(r,f) + S(r,g),$$

i.e.,

$$dT(r,f) \leq \frac{5+Q}{2}\overline{N}(r,\infty;f) + \frac{3}{2}N_{2+p}(r,0;f) + (2+Q)\overline{N}(r,\infty;g)$$
$$+ N_{2+p}(r,0;g) + S(r,f) + S(r,g).$$

Similarly

$$dT(r,g) \leq \frac{5+Q}{2}\overline{N}(r,\infty;g) + \frac{3}{2}N_{2+p}(r,0;g) + (2+Q)\overline{N}(r,\infty;f)$$
$$+ N_{2+p}(r,0;f) + S(r,f) + S(r,g).$$

Adding the above two inequalities we get

$$dT(r,f) + dT(r,g) \leq \frac{3Q+9}{2}\overline{N}(r,\infty;f) + \frac{5}{2}N_{2+p}(r,0;f) + \frac{3Q+9}{2}\overline{N}(r,\infty;g)$$
$$+ \frac{5}{2}N_{2+p}(r,0;g) + S(r,f) + S(r,g),$$

which contradict our hypothesis (1.3).

Proceeding similarly as in Case 1, we get the result for this case.

Case 3: $k = 0$. From (iii) of Lemma 2.3

$$T(r,F) \leq N_2(r,\infty;F) + N_2(r,\infty;G) + N_2(r,0;F) + N_2(r,0;G)$$
$$+ 2\overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + S(r,F) + S(r,G).$$

Using Lemmas 2.2 and 2.5

$$T(r,F) \leq 2\overline{N}(r,\infty;F) + 2\overline{N}(r,\infty;G) + T(r,F) - dT(r,f) + N_{2+p}(r,0;f)$$
$$+ Q\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + 2\overline{N}(r,\infty;F) + 2\overline{N}(r,0;F)$$
$$+ \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + S(r,F) + S(r,G),$$

i.e.,

$$dT(r,f) \leq 4\overline{N}(r,\infty;f) + N_{2+p}(r,0;f) + (3+Q)\overline{N}(r,\infty;g)$$
$$+ N_{2+p}(r,0;g) + 2Q\overline{N}(r,\infty;f) + 2N_{2+p}(r,0;f)$$
$$+ Q\overline{N}(r,\infty;g) + N_{2+p}(r,0;g) + S(r,f) + S(r,g)$$
$$\leq (4+2Q)\overline{N}(r,\infty;f) + 3N_{2+p}(r,0;f)$$
$$+ (3+2Q)\overline{N}(r,\infty;g) + 2N_{2+p}(r,0;g) + S(r,f) + S(r,g).$$

Similarly,

$$dT(r,g) \leq (4+2Q)\overline{N}(r,\infty;g) + 3N_{2+p}(r,0;g)$$
$$+ (3+2Q)\overline{N}(r,\infty;f) + 2N_{2+p}(r,0;f) + S(r,f) + S(r,g).$$

Combining the above two inequalities we get,

$$\begin{aligned} dT(r,f) + dT(r,g) \leq{} & (4Q+7)\overline{N}(r,\infty;f) + 5N_{2+p}(r,0;f) \\ & + (4Q+7)\overline{N}(r,\infty;g) + 5N_{2+p}(r,0;g) \\ & + S(r,f) + S(r,g), \end{aligned}$$

which contradict our hypothesis (1.4).

Approaching similarly as in Case 1, we get the result for this case. □

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