

STUDY OF THE ANNIHILATOR IDEAL GRAPH OF A SEMICOMMUTATIVE RING

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ABSTRACT. Let R be an associative ring with nonzero identity. The annihilator ideal graph of R , denoted by $\Gamma_{\text{Ann}}(R)$, is a graph whose vertices are all nonzero proper left ideals and all nonzero proper right ideals of R , and two distinct vertices I and J are adjacent if $I \cap (\ell_R(J) \cup r_R(J)) \neq 0$ or $J \cap (\ell_R(I) \cup r_R(I)) \neq 0$, where $\ell_R(K) = \{b \in R \mid bK = 0\}$ is the left annihilator of a nonempty subset $K \subseteq R$, and $r_R(K) = \{b \in R \mid Kb = 0\}$ is the right annihilator of a nonempty subset $K \subseteq R$. In this paper, we assume that R is a semicommutative ring. We study the structure of $\Gamma_{\text{Ann}}(R)$. Also, we investigate the relations between the ring-theoretic properties of R and graph-theoretic properties of $\Gamma_{\text{Ann}}(R)$. Moreover, some combinatorial properties of $\Gamma_{\text{Ann}}(R)$, such as domination number and clique number, are studied.

1. Introduction

In recent years, assigning graphs to algebraic structures has played an important role in the study of algebraic structures, for instance, see [1], [2] and [3].

Let $G = (V, E)$ be a simple graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. We say G is *empty* if $V = \emptyset$. By $|G|$, $\text{diam}(G)$, $\text{gr}(G)$, $\gamma(G)$, $\alpha(G)$ and $\omega(G)$, we mean the number of vertices, the diameter, the girth, the domination number, the independence number and the clique number of G , respectively. Also, for a vertex $v \in V$, the degree of v , denoted by $\deg(v)$, is the number of incident edges. For two distinct vertices u and v in G , the notation $u - v$ means that u and v are adjacent, or neighbors. The set of neighbors of a vertex v in G is denoted by $N(v)$, that is, $N(v) := \{u \in V \setminus \{v\} \mid \{u, v\} \in E\}$. For any undefined notation or terminology in graph theory, we refer the reader to [14].

Let R be an associative ring with nonzero identity. A ring R is called *semi-commutative* [11] if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Bell [4] and Shin

Received May 4, 2018; Revised July 19, 2018; Accepted July 24, 2018.

2010 *Mathematics Subject Classification.* 16U99, 05C69.

Key words and phrases. domination number, annihilator ideal graph, reversible ring, semicommutative ring.

[13] used the terms IFP and SI for semicommutative, respectively. According to Cohn [5], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Clearly, *reduced* rings (i.e., rings with no nonzero nilpotent elements) and commutative rings are reversible. For a nonempty subset $X \subseteq R$, let $\ell_R(X) = \{b \in R \mid bX = 0\}$ be the left annihilator of X , and $r_R(X) = \{b \in R \mid Xb = 0\}$ be the right annihilator of X . Note that if R is a reversible ring and $X \subseteq R$, then $\ell_R(X) = r_R(X)$, and we denote it by $\text{Ann}(X)$. We write $Z_\ell(R)$, $Z_r(R)$, $Z(R)$ and $J(R)$ for the set of all left zero-divisors of R , the set of all right zero-divisors of R , the set $Z_\ell(R) \cup Z_r(R)$ and the Jacobson radical of R , respectively. By \mathbb{Z} and \mathbb{Z}_n , we mean the integers and the integers modulo n , respectively. Also, the nonzero elements of $X \subseteq R$ will be denoted by X^* . A *regular element* in a ring R is any element $a \in R \setminus Z(R)$. A prime ideal P of R is called *completely prime*, if $ab \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$. We denote the number of elements of a set S by $|S|$.

According to [1], the *annihilator ideal graph* of a commutative ring R , denoted by $\Gamma_{\text{Ann}}(R)$, is a graph whose vertices are all nontrivial ideals of R (i.e., distinct from 0 and R) and two distinct vertices I and J are adjacent if $I \cap \text{Ann}(J) \neq 0$ or $J \cap \text{Ann}(I) \neq 0$. In this paper, we extend this concept to any arbitrary ring R with nonzero identity as follows:

Definition. Let R be an associative ring with nonzero identity. We associate a simple graph $\Gamma_{\text{Ann}}(R)$ to R whose vertices are all nonzero proper left ideals and all nonzero proper right ideals of R , and two distinct vertices I and J are adjacent if $I \cap (\ell_R(J) \cup r_R(J)) \neq 0$ or $J \cap (\ell_R(I) \cup r_R(I)) \neq 0$.

Remark 1.1. Let R be an associative ring with nonzero identity. According to the commutative case, we assume that R is a ring such that every left (or right) annihilator over R is an ideal of R . Thus by [9, Lemma 1.1], R must be a semicommutative ring. Hence, we impose the semicommutativity condition on a ring R . Thus, we assume that throughout this paper, R is a semicommutative ring with nonzero identity.

This generalization, at least in our opinion, is comprehensive and natural enough. Since it is worth to mention that some of our results in this paper appear at first time for the case of non-commutative rings, and there do not exist the counterpart results for the case of commutative rings in the literature, for example, see Propositions 2.1, Corollary 2.2(1), Corollary 2.5, Proposition 3.15 and Lemma 3.21. Moreover, some results are stronger than those results given for the counterpart results on the previous annihilator ideal graph, for example, see Proposition 3.12 and [1, Proposition 21 and Theorem 22]. In addition, the definition of the edges has been chosen to get as many results which are analogous to the commutative case as possible (for example, assume that I and J are adjacent if $I \cap (\ell_R(J) \cap r_R(J)) \neq 0$ or $J \cap (\ell_R(I) \cap r_R(I)) \neq 0$ and see Example 1.2, Proposition 3.10 and [1, Theorem 10(i)]).

Example 1.2. Let $R = \mathbb{Z}_2\langle a, b \rangle / \langle a^2, ab, b^2 \rangle$, where $\mathbb{Z}_2\langle a, b \rangle$ is the free associative algebra, with 1, over \mathbb{Z}_2 generated by two indeterminates (as labeled above) and $\langle a^2, ab, b^2 \rangle$ is the ideal generated by a^2 , ab and b^2 . Then by [12, Page 3], R is a semicommutative ring, but R is not reversible. Moreover, it is easy to check that $\Gamma_{\text{Ann}}(R)$ is complete. Also, $I = \bar{a}R$ and $J = R\bar{b}$ are two distinct vertices of $\Gamma_{\text{Ann}}(R)$ such that $I \cap (\ell_R(J) \cap r_R(J)) = \bar{0}$ and $J \cap (\ell_R(I) \cap r_R(I)) = \bar{0}$.

In this paper, we study the structure of $\Gamma_{\text{Ann}}(R)$. Also, we investigate the relations between the ring-theoretic properties of R and graph-theoretic properties of $\Gamma_{\text{Ann}}(R)$. Moreover, we study some combinatorial properties of $\Gamma_{\text{Ann}}(R)$ such as domination number and clique number.

2. The structure of $\Gamma_{\text{Ann}}(R)$

In this section, we want to study the structure of $\Gamma_{\text{Ann}}(R)$. We start with the following proposition, that will be useful in the sequel.

Proposition 2.1. *Let R be a ring. If I and J are two non-adjacent vertices of $\Gamma_{\text{Ann}}(R)$, then $N(I) = N(J)$.*

Proof. Assume that I and J are two distinct vertices of $\Gamma_{\text{Ann}}(R)$ such that I is not adjacent to J . Also, assume that $K \in N(I) \setminus N(J)$. We consider the following cases:

Case 1: Suppose that I and J are left ideals. Since I is not adjacent to J , we have $I \cap \ell_R(J) = 0$, $I \cap r_R(J) = 0$, $J \cap \ell_R(I) = 0$ and $J \cap r_R(I) = 0$. Then one can easily see that $\ell_R(I) = \ell_R(J)$, $r_R(J) \subseteq \ell_R(I)$ and $r_R(I) \subseteq \ell_R(J)$. Moreover, since I is adjacent to K , we have $I \cap \ell_R(K) \neq 0$, $I \cap r_R(K) \neq 0$, $K \cap \ell_R(I) \neq 0$ or $K \cap r_R(I) \neq 0$. Now we consider two following subcases:

Subcase 1: Assume that K is a left ideal. Since K is not adjacent to J , we have $\ell_R(K) = \ell_R(J)$, $r_R(J) \subseteq \ell_R(K)$ and $r_R(K) \subseteq \ell_R(J)$. Now it is easy to see that $I \cap \ell_R(K) = 0$, $I \cap r_R(K) = 0$, $K \cap \ell_R(I) = 0$ and $K \cap r_R(I) = 0$, a contradiction.

Subcase 2: Assume that K is a right ideal. Since K is not adjacent to J , we have $r_R(K) = \ell_R(J)$, $\ell_R(K) \subseteq \ell_R(J)$ and $r_R(J) \subseteq r_R(K)$. Then one can easily see that $I \cap \ell_R(K) = 0$, $I \cap r_R(K) = 0$, $K \cap \ell_R(I) = 0$ and $K \cap r_R(I) = 0$, a contradiction.

The other cases follow similarly. Therefore, we conclude that $N(I) = N(J)$. \square

Corollary 2.2. *Let R be a ring.*

- (1) *If $S \neq \emptyset$ is an induced subgraph of $\Gamma_{\text{Ann}}(R)$, then $\text{diam}(S) \in \{0, 1, 2, \infty\}$.*
- (2) *$\text{diam}(\Gamma_{\text{Ann}}(R)) \in \{0, 1, 2, \infty\}$.*
- (3) *$\Gamma_{\text{Ann}}(R)$ is disconnected or empty if and only if R is a domain.*

Proof. (1) If $|S| = 1$, then $\text{diam}(S) = 0$. Thus, we may assume that I and J are two distinct vertices of S such that I is not adjacent to J . Then $N_S(I) = N_S(J)$. If $N_S(I) = \emptyset$, then S is disconnected and hence $\text{diam}(S) = \infty$.

Otherwise, we can suppose that $Z \in N_S(I)$. Then $I - Z - J$ is a path of length two in S . Therefore, we conclude that $\text{diam}(S) \in \{0, 1, 2, \infty\}$.

(2) It follows from item (1).

(3) If $\Gamma_{\text{Ann}}(R)$ is empty, then it is easy to see that R is a division ring. Thus, we suppose that there exists no path between I and J , where I and J are two distinct vertices of $\Gamma_{\text{Ann}}(R)$. Then by Proposition 2.1, $N(I)$ and $N(J)$ are empty. We show that $\ell_R(I) \cup r_R(I) = \ell_R(J) \cup r_R(J) = 0$. To see this, let $\ell_R(I) \cup r_R(I) \neq 0$. Without loss of generality, we can suppose that $\ell_R(I) \neq 0$. Since I contains no neighbors in $\Gamma_{\text{Ann}}(R)$, we have $\ell_R(I) = I$. Also, $J \cap I = 0$. Thus, we have $IJ = 0$ or $JI = 0$. Hence, I is adjacent to J which is a contradiction. Thus $\ell_R(I) \cup r_R(I) = 0$. Similarly, $\ell_R(J) \cup r_R(J) = 0$. Now, if $\Gamma_{\text{Ann}}(R)$ contains a vertex, say K , such that $\ell_R(K) \cup r_R(K) \neq 0$, then one can see that $I - K - J$ is a path between I and J which is a contradiction. Therefore, we conclude that R is a domain.

The converse is clear. \square

In next two propositions, we study some relations between two distinct maximal independent sets.

Proposition 2.3. *Let R be a ring. If S_1 and S_2 are two distinct maximal independent sets of $\Gamma_{\text{Ann}}(R)$, then $S_1 \cap S_2 = \emptyset$.*

Proof. Assume to the contrary that $I \in S_1 \cap S_2$. Then by Proposition 2.1, we have $N(I) = N(J)$ for all vertices J of $S_1 \cup S_2$. Since S_1 and S_2 are distinct maximal independent sets, there exist two adjacent vertices A and B such that $A \in S_1$ and $B \in S_2$. Thus I is adjacent to A or B which is impossible. Therefore, $S_1 \cap S_2 = \emptyset$. \square

Proposition 2.4. *Let R be a ring. Let S_1 and S_2 be two distinct maximal independent sets of $\Gamma_{\text{Ann}}(R)$. If $I \in S_1$ and $J \in S_2$, then I is adjacent to J .*

Proof. Let $I \in S_1$ and $J \in S_2$, where S_1 and S_2 are two distinct maximal independent sets of $\Gamma_{\text{Ann}}(R)$. Also, assume to the contrary that I is not adjacent to J . Then by Proposition 2.1, we have $N(I) = N(J)$. Thus $S_1 \cup \{J\}$ is an independent set, a contradiction. Therefore, I is adjacent to J . \square

Now we have the following corollary.

Corollary 2.5. *Let R be a ring and $r \geq 2$ be a positive integer. If the vertex set of $\Gamma_{\text{Ann}}(R)$ can be partitioned into r maximal independent sets, then $\Gamma_{\text{Ann}}(R)$ is a complete r -partite graph.*

3. Some combinatorial properties of $\Gamma_{\text{Ann}}(R)$

In this section, we investigate the relations between the ring-theoretic properties of R and the graph-theoretic properties of $\Gamma_{\text{Ann}}(R)$. Then, we study some combinatorial properties of $\Gamma_{\text{Ann}}(R)$ such as the domination number and clique number. We start this section with the following proposition.

Proposition 3.1. *Let $R = R_1 \times R_2$, where R_i is a ring for $i = 1, 2$. If $\Gamma_{\text{Ann}}(R)$ is complete, then $\Gamma_{\text{Ann}}(R_i)$ is complete for $i = 1, 2$.*

Proof. Assume that $\Gamma_{\text{Ann}}(R)$ is complete. If $|\Gamma_{\text{Ann}}(R_i)| \in \{0, 1\}$ for $i = 1, 2$, then $\Gamma_{\text{Ann}}(R_i)$ is complete. Hence, we can suppose that I and J are two distinct vertices of $\Gamma_{\text{Ann}}(R_1)$. Then $I \times 0$ and $J \times 0$ are two distinct vertices of $\Gamma_{\text{Ann}}(R)$. Now since $I \times 0$ is adjacent to $J \times 0$ in $\Gamma_{\text{Ann}}(R)$, we conclude that I is adjacent to J in $\Gamma_{\text{Ann}}(R_1)$. Therefore, $\Gamma_{\text{Ann}}(R_1)$ is a complete graph. Similarly, one can see that $\Gamma_{\text{Ann}}(R_2)$ is a complete graph. \square

In [8, Theorem 81], it is proved that if R is a commutative ring, I_1, \dots, I_n are ideals of R and $S \subseteq R$ is a ring that is contained in the set theoretic union $I_1 \cup \dots \cup I_n$ and at least $n - 2$ of the I 's are prime, then S is contained in some I_j . By a similar method as used in the proof of [8, Theorem 81] we have the following proposition.

Proposition 3.2. *Let R be a ring and P_1, \dots, P_n a finite number of ideals of R and S a subring of R that is contained in the set theoretic union $P_1 \cup \dots \cup P_n$ and at least $n - 2$ of the P 's are completely prime. Then S is contained in some P_j .*

Recall that an *annihilator prime* for a left (or right) R -module M is any prime ideal P of R which equals the annihilator of some nonzero submodule of M . An *associated prime* of M is any annihilator prime ideal P which equals the annihilator of some nonzero submodule N of M such that P must equal the annihilator of each nonzero submodule of N . The set of all associated primes of M is denoted $\text{Ass}(M)$.

Lemma 3.3. *Let R be a ring which is not a domain.*

- (1) *If R is left Noetherian, then $Z_\ell(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and left annihilator of a nonzero element of $Z_r(R)$.*
- (2) *If R is right Noetherian, then $Z_r(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and right annihilator of a nonzero element of $Z_\ell(R)$.*
- (3) *If R is Noetherian, then $Z(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and left or right annihilator of a nonzero element of $Z(R)$.*

Proof. (1) Assume that R is a left Noetherian ring and $x \in Z_\ell(R)$. Then there exists a nonzero element $y \in Z_r(R)$ such that $xy = 0$. Put $\Sigma := \{\ell_R(a) \mid x \in \ell_R(a) \text{ and } a \in R^*\}$. Now since R is a left Noetherian ring, Σ has a maximal element, say $P = \ell_R(b)$. Thus we have $x \in P$. We show that P is completely prime. To see this, let $rs \in P$ for $r, s \in R$. Now if $sb \neq 0$, then since R is a semicommutative ring, one can see that $r \in P = \ell_R(sb) = \ell_R(b)$. Hence P is a completely prime ideal. Also by [6, Proposition 2.12], we have $P \in \text{Ass}(R)$ (R is viewed as a left module over itself). Now by [6, Exercise 2J], we have

$|\text{Ass}(R)| < \infty$. Therefore, we conclude that $Z_\ell(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and left annihilator of a nonzero element of $Z_r(R)$.

(2) Use a method similar to that we used in item (1).

(3) Since $Z(R) = Z_\ell(R) \cup Z_r(R)$, it follows from items (1) and (2). \square

Proposition 3.4. *Let $R = R_1 \times R_2$, where R_i is a Noetherian ring for $i = 1, 2$. If $\Gamma_{\text{Ann}}(R_i)$ is complete for $i = 1, 2$, then $\Gamma_{\text{Ann}}(R)$ is complete.*

Proof. Assume that $\Gamma_{\text{Ann}}(R_i)$ is complete for $i = 1, 2$. Let I and J be two distinct vertices of $\Gamma_{\text{Ann}}(R)$. Thus $I = I_1 \times I_2$ and $J = J_1 \times J_2$, where I_i and J_i are one-sided ideals of R_i for $i = 1, 2$. Without loss of generality, assume that I_1 and J_1 are distinct. Now we consider three following cases:

Case 1: Suppose that either I_1 or J_1 is zero. Then I is adjacent to J in $\Gamma_{\text{Ann}}(R)$.

Case 2: Suppose that either $I_1 = R_1$ or $J_1 = R_1$. Without loss of generality, assume that $I_1 = R_1$. We show that for all vertices K_1 of $\Gamma_{\text{Ann}}(R_1)$ we have $K_1 \subseteq Z(R_1)$. To see this, let L_1 be a vertex of $\Gamma_{\text{Ann}}(R_1)$ such that $L_1 \not\subseteq Z(R_1)$. Let $b \in L_1 \setminus Z(R_1)$. Without loss of generality, assume that $R_1 b \neq R_1$. Then $R_1 b$ is not adjacent to $R_1 b^2$ in $\Gamma_{\text{Ann}}(R_1)$, a contradiction. Hence for all vertices K_1 of $\Gamma_{\text{Ann}}(R_1)$ we have $K_1 \subseteq Z(R_1)$. Now by Lemma 3.3 and Proposition 3.2, we have $\ell_{R_1}(J_1) \cup r_{R_1}(J_1) \neq 0$. Thus I is adjacent to J in $\Gamma_{\text{Ann}}(R)$.

Case 3: Suppose that I_1 and J_1 are nontrivial one-sided ideals of R_1 . Then since $\Gamma_{\text{Ann}}(R_1)$ is complete, I is adjacent to J in $\Gamma_{\text{Ann}}(R)$. \square

The following corollary can be obtained directly from Proposition 3.1 and Proposition 3.4.

Corollary 3.5. *Assume that $R = R_1 \times R_2$, where R_i is a Noetherian ring for $i = 1, 2$. Then $\Gamma_{\text{Ann}}(R)$ is complete if and only if $\Gamma_{\text{Ann}}(R_i)$ is complete for $i = 1, 2$.*

In the following proposition, we investigate the relation between $\Gamma_{\text{Ann}}(R)$ and $\Gamma_{\text{Ann}}(R/J(R))$.

Proposition 3.6. *Let R be a Noetherian ring. If $\alpha(\Gamma_{\text{Ann}}(R)) < \infty$, then $\Gamma_{\text{Ann}}(R/J(R))$ is a complete graph.*

Proof. Assume that $x \in R \setminus Z(R)$. We show that x is unit. To see this, let $Rx \neq R$. Then the vertices of the set $\{Rx^i\}_{i=1}^\infty$ form an independent set which is a contradiction, since $\alpha(\Gamma_{\text{Ann}}(R)) < \infty$. Thus, x is unit for all $x \in R \setminus Z(R)$. Now if $Z(R) = 0$, then R is a division ring and hence $\Gamma_{\text{Ann}}(R/J(R))$ is a complete graph. Thus we may assume that $Z(R) \neq 0$. Now by Proposition 3.3, we have $Z(R) = \bigcup_{i \in \Theta} P_i$, where Θ is a finite set and each P_i is a completely prime ideal and left or right annihilator of a nonzero element of $Z(R)$. Hence, if M is a maximal left ideal of R , then by Proposition 3.2, we have $M = P_j$ for some $j \in \Theta$. Thus, $J(R) = \bigcap_{i \in \Theta} P_i$. Now by [7, Corollary 2.27 of Chapter 3],

we have $R/J(R) \cong K_1 \times \cdots \times K_n$, where $n = |\Theta|$ and K_i is a division ring for $i = 1, \dots, n$. Then one can see that $\Gamma_{\text{Ann}}(R/J(R))$ is a complete graph. \square

Corollary 3.7. *Let R be a Noetherian ring. If $\Gamma_{\text{Ann}}(R)$ is a complete graph, then $\Gamma_{\text{Ann}}(R/J(R))$ is a complete graph.*

Proof. Since $\Gamma_{\text{Ann}}(R)$ is a complete graph, then either $\alpha(\Gamma_{\text{Ann}}(R)) = 0$ (whenever R is a division ring) or $\alpha(\Gamma_{\text{Ann}}(R)) = 1$. Thus we conclude that $\Gamma_{\text{Ann}}(R/J(R))$ is a complete graph. \square

In the next proposition, we study the case that $\Gamma_{\text{Ann}}(R)$ is complete. Before that, the following two lemmas are necessary.

Lemma 3.8. *Let R be a ring. Then the subgraph induced by nilpotent one-sided ideals of R is complete.*

Proof. Assume that I and J are two distinct vertices of $\Gamma_{\text{Ann}}(R)$ such that $I^n = J^m = 0$ for some $n, m \in \mathbb{N}$. Also, assume that I is not adjacent to J . We consider the following cases:

Case 1: Suppose that I and J are left ideals. Then $\ell_R(I) = \ell_R(J)$. Now since I is nilpotent, $I \cap \ell_R(I) \neq 0$ and hence I is adjacent to J , a contradiction.

Case 2: Suppose that I is a left ideal and J a right ideal. Then $\ell_R(I) = r_R(J)$. Now since I is nilpotent, $I \cap \ell_R(I) \neq 0$ and hence I is adjacent to J , a contradiction.

The other cases follow similarly. Therefore, we conclude that the subgraph induced by nilpotent one-sided ideals is complete. \square

Recall that a ring R is called *local* if R has a unique maximal left ideal, or equivalently, if R has a unique maximal right ideal.

Lemma 3.9. *Let R be a left (or right) Artinian local ring. Then $\Gamma_{\text{Ann}}(R)$ is complete.*

Proof. It follows from [10, Theorem 4.12] and Lemma 3.8. \square

Proposition 3.10. *Let R be a left (or right) Artinian ring. Then $\Gamma_{\text{Ann}}(R)$ is complete.*

Proof. Suppose that R is a left Artinian ring. We consider two following cases:

Case 1: Assume that R has no nontrivial idempotents. Then by [10, Corollary 19.19], R is a local ring. Then by Lemma 3.9, $\Gamma_{\text{Ann}}(R)$ is complete.

Case 2: Now assume that R has a nontrivial idempotent element, say e . Since R is a semicommutative ring, e is a central idempotent, by [10, Lemma 21.5]. Hence Re is an ideal of R . Thus we can suppose that $R \cong R_1 \times R_2$, where R_i is a left Artinian ring for $i = 1, 2$ (see [10, Exercise 1.7 and Corollary 21.13]). Now by a similar method, we conclude that $R \cong R'_1 \times \cdots \times R'_n$, where R'_i is a left Artinian local ring for $i = 1, \dots, n$. Then by a method similar to that we used in Proposition 3.4, one can see that $\Gamma_{\text{Ann}}(R)$ is complete. \square

Recall that a ring R is said to be reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Note that if R is a reversible ring and $X \subseteq R$, then $\ell_R(X) = r_R(X)$, and we denote it by $\text{Ann}(X)$. By [9, Lemma 1.4], reversible rings are semicommutative. We use the following lemma frequently.

Lemma 3.11. *Let R be a reversible ring. If I and J are non-adjacent in $\Gamma_{\text{Ann}}(R)$, then $\text{Ann}(I) = \text{Ann}(J)$.*

Proof. It follows from definition. \square

In the following proposition, we study the case that the degree of a vertex of $\Gamma_{\text{Ann}}(R)$ is finite.

Proposition 3.12. *Let R be a reversible ring with $Z(R) \neq 0$. If $\Gamma_{\text{Ann}}(R)$ contains a vertex of finite degree, then $\Gamma_{\text{Ann}}(R)$ is a finite graph.*

Proof. Assume that $\deg(I) < \infty$, where I is a vertex of $\Gamma_{\text{Ann}}(R)$. Without loss of generality, we may assume that I is a left ideal of R . We claim that $\text{Ann}(I) \neq 0$. To prove the claim, let $\text{Ann}(I) = 0$. If J is a vertex of $\Gamma_{\text{Ann}}(R)$ such that $\text{Ann}(J) \neq 0$, then I is adjacent to J . Thus since $\deg(I) < \infty$, the number of one-sided ideals K of R such that $\text{Ann}(K) \neq 0$ is finite, and so R contains a minimal left ideal, say Rx . We show that R is a left Artinian ring. To see this, since $Rx \cong \frac{R}{\text{Ann}(x)}$ as a left R -module isomorphism and $0 \rightarrow \text{Ann}(x) \rightarrow R \rightarrow \frac{R}{\text{Ann}(x)} \rightarrow 0$ is an exact sequence of R -modules, R is a left Artinian ring. Now by a method similar to that we used in the proof of Proposition 3.10, one can show that $R \cong R_1 \times \cdots \times R_n$, where R_i is a local ring for $i = 1, \dots, n$. Then $\text{Ann}(I) \neq 0$, since $J(R_i) = Z(R_i)$ is nilpotent, which is a contradiction. Thus $\text{Ann}(I) \neq 0$.

Now suppose that $I = \text{Ann}(I)$. Then we show that I is adjacent to every other vertex. To see this, suppose J is a vertex of $\Gamma_{\text{Ann}}(R)$ such that J is not adjacent to I . Then by Lemma 3.11, $I = \text{Ann}(I) = \text{Ann}(J)$ which is impossible. Hence I is adjacent to every other vertex and so $\Gamma_{\text{Ann}}(R)$ is a finite graph. Now assume that $I \neq \text{Ann}(I)$. Then since $\deg(I) < \infty$ and $I \cap \text{Ann}(\text{Ann}(I)) \neq 0$, the number of R -submodules of $\text{Ann}(I)$ is finite and hence $\text{Ann}(I)$ is an Artinian left R -module. Moreover, $\frac{R}{\text{Ann}(I)}$ is an Artinian left R -module (because $\deg(I) < \infty$). Thus R is a left Artinian ring. Now by Proposition 3.10, we conclude that $\Gamma_{\text{Ann}}(R)$ is a finite graph. \square

Recall that a ring R is called *reduced* if it has no nonzero nilpotent elements. Clearly, a ring R is reduced if and only if R is a semiprime and reversible ring. In the next proposition, we study the case that $\gamma(\Gamma_{\text{Ann}}(R)) = 1$.

Proposition 3.13. *Let R be a ring. If R is not reduced, then $\gamma(\Gamma_{\text{Ann}}(R)) = 1$.*

Proof. Since R is not a reduced ring, there exists a vertex I of $\Gamma_{\text{Ann}}(R)$ such that $I^2 = 0$. We show that I is adjacent to every other vertex. To see this, assume that J is a vertex of $\Gamma_{\text{Ann}}(R)$ and J is not adjacent to I . We have the following two cases:

Case 1: Suppose that J is a left ideal. Since J is not adjacent to I , $\ell_R(I) \subseteq \ell_R(J)$ which is impossible.

Case 2: Suppose that J is a right ideal. Since J is not adjacent to I , $\ell_R(I) \subseteq r_R(J)$ which is impossible.

Therefore, I is adjacent to every other vertex and so $\gamma(\Gamma_{\text{Ann}}(R)) = 1$. \square

Example 3.14. Let R be a reduced ring. Then we cannot conclude that $\gamma(\Gamma_{\text{Ann}}(R)) = 2$. To see this, let R be a finite direct product of division rings. Then by Proposition 3.10, we have $\gamma(\Gamma_{\text{Ann}}(R)) = 1$.

In the following proposition, we find a dominating set in $\Gamma_{\text{Ann}}(R)$.

Proposition 3.15. *Let R be a ring and let I be a vertex of $\Gamma_{\text{Ann}}(R)$ such that $\ell_R(I) \cup r_R(I) \neq 0$. Then the set $\{I\}$, $\{I, \ell_R(I)\}$ or $\{I, r_R(I)\}$ is a dominating set.*

Proof. Without loss of generality, assume that $\ell_R(I) \neq 0$. If $I = \ell_R(I)$, then $I^2 = 0$. Thus one can see that I is adjacent to every other vertex and hence the set $\{I\}$ is a dominating set. Now assume that $I \neq \ell_R(I)$. We show that the set $\{I, \ell_R(I)\}$ is a dominating set. To see this, suppose that J is a vertex of $V(\Gamma_{\text{Ann}}(R)) \setminus \{I, \ell_R(I)\}$ such that J is not adjacent to I . Then by Proposition 2.1, we have $N(J) = N(I)$. Since $I \neq \ell_R(I)$, it is easy to see that I is adjacent to $\ell_R(I)$. Thus J is adjacent to $\ell_R(I)$. Therefore, the set $\{I, \ell_R(I)\}$ is a dominating set. \square

Note that if R is a reduced ring with finitely many minimal prime ideals, then we have $Z(R) = \bigcup_{i=1}^n P_i$, where $P_i = \text{Ann}(x_i)$ for some $x_i \in R$ and P_i is completely prime for each i (see [10, Lemma 12.6 and Proposition 10.16]). In the following proposition, we study the case that $\gamma(\Gamma_{\text{Ann}}(R)) = 2$. Note that if $Z(R) \neq 0$, then by Proposition 3.15 we have $\gamma(\Gamma_{\text{Ann}}(R)) \leq 2$.

Proposition 3.16. *Let R be a reduced ring with finitely many minimal prime ideals and $Z(R) \neq 0$. Then for every vertex $I \in \Gamma_{\text{Ann}}(R)$ there exists a vertex $J \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I\}$ such that $\text{Ann}(I) = \text{Ann}(J)$ if and only if $\gamma(\Gamma_{\text{Ann}}(R)) = 2$.*

Proof. Suppose that for every vertex $I \in \Gamma_{\text{Ann}}(R)$ there exists a vertex $J \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I\}$ such that $\text{Ann}(I) = \text{Ann}(J)$. Assume to the contrary that I_1 is a vertex of $\Gamma_{\text{Ann}}(R)$ such that I_1 is adjacent to every other vertex. Now by our assumption, there exists a vertex $J_1 \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I_1\}$ such that $\text{Ann}(I_1) = \text{Ann}(J_1)$. Then I_1 is not adjacent to J_1 which is a contradiction. Therefore, $\gamma(\Gamma_{\text{Ann}}(R)) = 2$.

Conversely, let $\gamma(\Gamma_{\text{Ann}}(R)) = 2$. Then for every vertex $I \in \Gamma_{\text{Ann}}(R)$ there exists a vertex $J \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I\}$ such that I is not adjacent to J . Thus by Lemma 3.11, we have $\text{Ann}(I) = \text{Ann}(J)$. \square

The following corollary can be obtained directly from Proposition 3.16.

Corollary 3.17. *Let R be a reduced ring with finitely many minimal prime ideals. Then there exists a vertex $I \in \Gamma_{\text{Ann}}(R)$ such that $\text{Ann}(I) \neq \text{Ann}(J)$ for every vertex $J \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I\}$ if and only if $\gamma(\Gamma_{\text{Ann}}(R)) = 1$.*

Example 3.18. Let $R = \mathbb{Z} \times \mathbb{Z}$. Then one can easily see that for every vertex $I \in \Gamma_{\text{Ann}}(R)$ there exists a vertex $J \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I\}$ such that $\text{Ann}(I) = \text{Ann}(J)$. Also, $\gamma(\Gamma_{\text{Ann}}(R)) = 2$.

Example 3.19. Let $R = \mathbb{F} \times \mathbb{Z}$, where \mathbb{F} is a field. Put $I := \mathbb{F} \times 0$. Then it is easy to see that $\text{Ann}(I) \neq \text{Ann}(J)$ for every vertex $J \in V(\Gamma_{\text{Ann}}(R)) \setminus \{I\}$. Also, $\gamma(\Gamma_{\text{Ann}}(R)) = 1$.

In the next proposition, we characterize some reduced rings whose graphs are complete.

Proposition 3.20. *Let R be a reduced ring with finitely many minimal prime ideals. Then $\Gamma_{\text{Ann}}(R)$ is complete if and only if $R \cong K_1 \times \cdots \times K_n$, where K_i is a division ring for $i = 1, \dots, n$.*

Proof. Let P_1, \dots, P_n be the minimal prime ideals of R . First, assume that $\Gamma_{\text{Ann}}(R)$ is complete. We claim that if I is a vertex of $\Gamma_{\text{Ann}}(R)$, then $\text{Ann}(I) \neq 0$. To prove the claim, let $I \not\subseteq Z(R)$. Then we may suppose that there exists a regular element $x \in R$ such that Rx is a vertex of $\Gamma_{\text{Ann}}(R)$. Hence Rx is not adjacent to Rx^2 , which is a contradiction. Thus $I \subseteq Z(R)$. Since $Z(R) = \bigcup_{i=1}^n P_i$, by Proposition 3.2 we have $\text{Ann}(I) \neq 0$. On the other hand, P_1, \dots, P_n are the maximal ideals (maximal left ideals) of R . Now by [7, Corollary 2.27 of Chapter 3], $R \cong K_1 \times \cdots \times K_n$, where K_i is a division ring for $i = 1, \dots, n$.

The converse is clear. \square

In the next proposition, we study the case that $\Gamma_{\text{Ann}}(R)$ is bipartite. Before that, the following lemma is necessary.

Lemma 3.21. *Let R be a reversible ring. If $\Gamma_{\text{Ann}}(R)$ contains a path of length three, then we have $\text{gr}(\Gamma_{\text{Ann}}(R)) = 3$.*

Proof. Suppose that $I_1 - I_2 - I_3 - I_4$ is a path of length three in $\Gamma_{\text{Ann}}(R)$. If I_1 is adjacent to I_3 , then $\text{gr}(\Gamma_{\text{Ann}}(R)) = 3$. Thus we may assume that I_1 is not adjacent to I_3 . Then by Proposition 2.1, $I_1 - I_2 - I_3 - I_4 - I_1$ is a cycle of length four in $\Gamma_{\text{Ann}}(R)$. Then without loss of generality, we can assume that $\text{Ann}(I_2) \neq 0$. If R is not reduced, then by Proposition 3.13, one can see that $\text{gr}(\Gamma_{\text{Ann}}(R)) = 3$. Thus we may assume that R is a reduced ring. Now if $\text{Ann}(I_1) = 0$, then $I_1 - I_2 - \text{Ann}(I_2) - I_1$ is a cycle of length three in $\Gamma_{\text{Ann}}(R)$ and so $\text{gr}(\Gamma_{\text{Ann}}(R)) = 3$. Otherwise, we suppose that $\text{Ann}(I_1) \neq 0$. Now we consider the following two cases:

Case 1: Assume that $I_1 + \text{Ann}(I_1) = R$. Then R is a decomposable ring (see [10, Exercise 1.7]). Thus $R \cong R_1 \times R_2$, where R_i is a ring for $i = 1, 2$. Since $\Gamma_{\text{Ann}}(R)$ contains a path of length three, we can suppose that R_1 is

not a division ring. Then $R_1 \times 0 - 0 \times R_2 - I \times R_2 - R_1 \times 0$, where I is a nontrivial one-sided ideal of R_1 , is a cycle of length three in $\Gamma_{\text{Ann}}(R)$ and so $\text{gr}(\Gamma_{\text{Ann}}(R)) = 3$.

Case 2: Assume that $I_1 + \text{Ann}(I_1) \neq R$. Then $I_1 - \text{Ann}(I_1) - I_1 + \text{Ann}(I_1) - I_1$ is a cycle of length three in $\Gamma_{\text{Ann}}(R)$ and so $\text{gr}(\Gamma_{\text{Ann}}(R)) = 3$. \square

Recall that a bipartite graph is one whose vertex set can be partitioned into two subsets so that no edge has both ends in any one subset. A complete bipartite graph is a bipartite graph such that every two vertices from different partitions classes are adjacent. Note that if a graph contains a cycle of odd length, then it is not bipartite (see [14, Theorem 1.2.18]).

Proposition 3.22. *Let R be a reversible ring. Then $\Gamma_{\text{Ann}}(R)$ is a bipartite graph with nonempty edge set if and only if the number of vertices of $\Gamma_{\text{Ann}}(R)$ is exactly two.*

Proof. Suppose that $\Gamma_{\text{Ann}}(R)$ is a bipartite graph. Since $\Gamma_{\text{Ann}}(R)$ contains an edge, R is not a domain. Then by Corollary 2.2, $\Gamma_{\text{Ann}}(R)$ is a complete bipartite graph (because, we have $\text{diam}(\Gamma_{\text{Ann}}(R)) \leq 2$). On the other hand, by Lemma 3.21, we conclude that $\Gamma_{\text{Ann}}(R)$ contains no path of length three. Thus $\gamma(\Gamma_{\text{Ann}}(R)) = 1$. Now by Proposition 3.12, $\Gamma_{\text{Ann}}(R)$ is a finite graph. Therefore, the number of vertices of $\Gamma_{\text{Ann}}(R)$ is exactly two, by Proposition 3.10.

The converse is clear. \square

In the following proposition, we study the clique number of $\Gamma_{\text{Ann}}(R)$, when R is decomposable.

Proposition 3.23. *Let R be a reversible ring and $R = R_1 \times \cdots \times R_n$, where R_i is a ring for $i = 1, \dots, n$ and $n \neq 1$. Then $\omega(\Gamma_{\text{Ann}}(R)) \geq 2^n - 2$. In particular, if R contains a nontrivial one-sided ideal as I such that $\text{Ann}(I) = 0$, then $\omega(\Gamma_{\text{Ann}}(R)) \geq 2^n - 1$.*

Proof. It is sufficient we choose the zero ideal of R_i for $i = 1, \dots, n$. Then the subgraph induced by vertices of the set

$$\Omega := \{I_1 \times \cdots \times I_n \in V(\Gamma_{\text{Ann}}(R)) \mid I_i \text{ is a trivial ideal of } R_i \text{ for } i = 1, \dots, n\}$$

is complete. Thus $\omega(\Gamma_{\text{Ann}}(R)) \geq \sum_{i=1}^{n-1} \binom{n}{i} = 2^n - 2$. The “in particular” statement is clear. \square

Corollary 3.24. *Let R be a reversible ring and $R = R_1 \times \cdots \times R_n$, where R_i is a ring for $i = 1, \dots, n$ and $n \neq 1$. Then*

- (1) $\omega(\Gamma_{\text{Ann}}(R)) = 2^n - 1$ if and only if R_i is a domain for $i = 1, \dots, n$ and R_i is not a division ring for some $i = 1, \dots, n$.
- (2) $\omega(\Gamma_{\text{Ann}}(R)) = 2^n - 2$ if and only if R_i is a division ring for $i = 1, \dots, n$.

Proof. (1) Suppose that R_i is a domain for $i = 1, \dots, n$ and R_i is not a division ring for some $i = 1, \dots, n$. By Proposition 3.23, we have $\omega(\Gamma_{\text{Ann}}(R)) \geq 2^n - 1$.

Now assume that Δ is a clique of $\Gamma_{\text{Ann}}(R)$ with 2^n vertices. Then there exist two distinct vertices $I, J \in \Delta$, where $I = I_1 \times \cdots \times I_n$, $J = J_1 \times \cdots \times J_n$, $I_i, J_i \subseteq R_i$ for $i = 1, \dots, n$, $I_i = J_i = 0$ for some $i \in \{1, \dots, n\}$, and for the other indices, I_i and J_i are nonzero. Then it is easy to see that I is not adjacent to J , which is a contradiction. Therefore, $\omega(\Gamma_{\text{Ann}}(R)) = 2^n - 1$.

Conversely, assume that $\omega(\Gamma_{\text{Ann}}(R)) = 2^n - 1$. If R is left (or right) Artinian, then by Proposition 3.10, we conclude that $\Gamma_{\text{Ann}}(R)$ is a complete graph. Thus the number of vertices of $\Gamma_{\text{Ann}}(R)$ is $2^n - 1$ which is impossible. Hence, we conclude that R is not left (or right) Artinian. We show that R_i is a domain for $i = 1, \dots, n$. To see this, without loss of generality, we may suppose that R_1 contains a nontrivial one-sided ideal as I such that $\text{Ann}(I) \neq 0$. Put

$$\Omega := \{I_1 \times \cdots \times I_n \in V(\Gamma_{\text{Ann}}(R)) \mid I_i \text{ is a trivial ideal of } R_i \text{ for } i = 1, \dots, n\}$$

and

$$\Lambda := \{I \times I_2 \times \cdots \times I_n \in V(\Gamma_{\text{Ann}}(R)) \mid I_i \text{ is a trivial ideal of } R_i \text{ for } i = 2, \dots, n\}.$$

Then, one can see that the subgraph induced by vertices of $\Omega \cup \Lambda$ is a clique of $\Gamma_{\text{Ann}}(R)$ with more than $2^n - 1$ vertices, which is a contradiction. Thus, we conclude that R_i is a domain for $i = 1, \dots, n$ and R_i is not a division ring for some $i = 1, \dots, n$.

(2) Suppose that $R = R_1 \times \cdots \times R_n$, where R_i is a division ring for $i = 1, \dots, n$. Then, it is easy to see that $\Gamma_{\text{Ann}}(R)$ is a complete graph with $2^n - 2$ vertices.

Conversely, assume that $\omega(\Gamma_{\text{Ann}}(R)) = 2^n - 2$. Then, by using a method similar to that we used in the proof of item (1), one can see that R_i is a division ring for $i = 1, \dots, n$. \square

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