Commun. Korean Math. Soc. **34** (2019), No. 2, pp. 401–414
https://doi.org/10.4134/CKMS.c180181
pISSN: 1225-1763 / eISSN: 2234-3024

LEONARD PAIRS OF RACAH AND KRAWTCHOUK TYPE IN LB-TD FORM

Hasan Alnajjar

Abstract. Let $\mathcal{F}$ denote an algebraically closed field with characteristic not two. Fix an integer $d \geq 3$, let $\mathrm{Mat}_{d+1}(\mathcal{F})$ denote the $\mathcal{F}$-algebra of $(d+1) \times (d+1)$ matrices with entries in $\mathcal{F}$. An ordered pair of matrices A, A^* in $\mathrm{Mat}_{d+1}(\mathcal{F})$ is said to be LB-TD form whenever A is lower bidiagonal with subdiagonal entries all 1 and A^* is irreducible tridiagonal. Let A, A^* be a Leonard pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$ with fundamental parameter $\beta = 2$, with this assumption there are four families of Leonard pairs, Racah, Hahn, dual Hahn, Krawtchouk type. In this paper we show from these four families only Racah and Krawtchouk have LB-TD form.

1. Introduction

In this section we recall some facts concerning Leonard pairs. Leonard pairs are linear algebraic objects which provide insight into the relationship between finite-dimensional representations of certain nice algebras [2,6,17] and orthogonal polynomials [16]. In this paper we show that of the four families of Leonard pairs of classical type, precisely two, those of Racah and Krawtchouk types, admit particularly nice matrix representations. For both we describe these matrices explicitly. Fix an integer $d \geq 1$. Throughout this paper $\mathcal{F}$ shall denote an algebraically closed field with characteristic not two. Also, V shall denote an $\mathcal{F}$-vector space of dimension $d+1$, and $\mathrm{Mat}_{d+1}(\mathcal{F})$ shall denote the $\mathcal{F}$-algebra of $(d+1)\times(d+1)$ matrices with entries in $\mathcal{F}$ having rows and columns indexed by $1, 2, \ldots, d+1$. A square matrix is said to be *tridiagonal* whenever every nonzero entry appears on, immediately above, or immediately below the main diagonal. A tridiagonal matrix is said to be *irreducible* whenever all entries immediately above and below the main diagonal are nonzero. We now recall the notion of Leonard pair.

Definition 1.1. Let V denote a vector space over $\mathcal{F}$ with finite positive dimension. By a *Leonard pair on* V, we mean an ordered pair A, A^*, where

Received April 28, 2018; Revised June 1, 2018; Accepted August 22, 2018.
2010 *Mathematics Subject Classification.* 05E10, 05E30, 33C45, 33D45.
Key words and phrases. Leonard pair, Askey-Wilson relation, Racah polynomial, Hahn polynomial, dual Hahn polynomial, Krawtchouk polynomial.

$A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy both (i) and (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal.

Definition 1.2. An ordered pair of matrices A, A^* in $\mathrm{Mat}_{d+1}(\mathcal{F})$ is said to be LB-TD whenever A is lower bidiagonal with subdiagonal entries all 1 and A^* is irreducible tridiagonal.

Definition 1.3. A Leonard pair A, A^* on V is said to have LB-TD form whenever there exists a basis for V with respect to which the matrices representing A, A^* form a LB-TD pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$.

We remark that if A, A^* is a Leonard pair on V, then for any scalars α, α^*, the pair $A + \alpha I$, $A^* + \alpha^* I$ is also a Leonard pair on V.

For more details about Leonard pairs see [1, 2] and [7–14].

Definition 1.4. Let V denote a vector space over $\mathcal{F}$ with dimension $d+1$. Let $\{v_i\}_{i=0}^d$ denote a basis for V which satisfies condition (ii) of Definition 1.1. For $0 \leq i \leq d$, the vector v_i is an eigenvector of A, let θ_i denote the corresponding eigenvalues. Let $\{v_i^*\}_{i=0}^d$ denote a basis for V which satisfies condition (i) of Definition 1.1. For $0 \leq i \leq d$, the vector v_i^* is an eigenvector of A^*, let θ_i^* denote the corresponding eigenvalues. Let the sequence $\{a_i\}_{i=0}^d$ denote the diagonal of the matrix which represents A with respect to $\{v_i^*\}_{i=0}^d$. Let the sequence $\{a_i^*\}_{i=0}^d$ denote the diagonal of the matrix which represents A^* with respect to $\{v_i\}_{i=0}^d$.

The ordering of $\{\theta_i\}_{i=0}^d$ in Definition 1.4 is said to be standard. For a standard ordering $\{\theta_i\}_{i=0}^d$ of eigenvalues of A, the ordering $\{\theta_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result applies for $\{\theta_i^*\}_{i=0}^d$.

Theorem 1.5 ([14])**.** *Let A, A^* be a Leonard pair on V, let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) be standard ordering of the eigenvalues of A (resp. A^*). Then there exists a basis of V such that the matrices representing A, A^* with respect to this basis are respectively*

$$(1)\quad A = \begin{pmatrix} \theta_0 & & & & \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \varphi_1 & & & \\ & \theta_1^* & \varphi_2 & & \\ & & \theta_2^* & \ddots & \\ & & & \ddots & \varphi_d \\ & & & & \theta_d^* \end{pmatrix}$$

for some sequence of scalars $\varphi_1, \varphi_2, \ldots, \varphi_d$ *in* $\mathcal{F}$*, which we refer to as the first split sequence of* A, A^*.

Definition 1.6 ([9])**.** Let d denote a nonnegative integer. By a *parameter array* over $\mathcal{F}$ of diameter d, we mean a sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ taken from $\mathcal{F}$ that satisfy the following conditions.

$$\theta_i \neq \theta_j \qquad (0 \leq i < j \leq d), \tag{2}$$

$$\theta_i^* \neq \theta_j^* \qquad (0 \leq i < j \leq d), \tag{3}$$

$$\varphi_i \neq 0 \qquad (1 \leq i \leq d), \tag{4}$$

$$\phi_i \neq 0 \qquad (1 \leq i \leq d), \tag{5}$$

$$\varphi_i = \phi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{i-1} - \theta_d) \qquad (1 \leq i \leq d), \tag{6}$$

$$\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d), \tag{7}$$

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{j-2}^* - \theta_{j+1}^*}{\theta_{j-1}^* - \theta_j^*} \qquad (2 \leq i, j \leq d-1). \tag{8}$$

Theorem 1.7 ([14])**.** *With reference to Definition 1.4,*

$$a_i = \theta_i + \frac{\varphi_i}{\theta_i^* - \theta_{i-1}^*} + \frac{\varphi_{i+1}}{\theta_i^* - \theta_{i+1}^*} \qquad (0 \leq i \leq d), \tag{9}$$

$$a_i^* = \theta_i^* + \frac{\varphi_i}{\theta_i - \theta_{i-1}} + \frac{\varphi_{i+1}}{\theta_i - \theta_{i+1}} \qquad (0 \leq i \leq d), \tag{10}$$

where $\varphi_0 = 0$ *and* $\varphi_{d+1} = 0$.

Theorem 1.8 ([17])**.** *Let* V *denote a vector space over* $\mathcal{F}$ *with finite positive dimension. Let* A, A^* *denote a Leonard pair on* V*. Then there exists a sequence of scalars* β, γ, γ^*, ϱ, ρ^*, ω, η, *and* η^* *taken from* $\mathcal{F}$ *such that*

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^* = \gamma^* A^2 + \omega A + \eta I, \tag{11}$$

$$A^{*^2}A - \beta A^*AA^* + AA^{*^2} - \gamma^*(A^*A + AA^*) - \varrho^* A = \gamma A^{*^2} + \omega A^* + \eta^* I. \tag{12}$$

The sequence is uniquely determined by the pair A, A^* *provided the dimension of* V *is at least* 4.

The relations (11) and (12) are called the *Askey-Wilson relations.*

2. Main result

In [15], the parameters arrays are classified into 13 families, each named for certain associated sequences of orthogonal polynomials. The four families which arise in this paper share certain property. Given a parameter array, let β be the common value of (8) minus one.

Definition 2.1. A parameter array is of classical type whenever $\beta = 2$.

Theorem 2.2 ([15])**.** *A parameter array is of classical type if and only if it is of Racah, Hahn, dual Hahn, or Krawtchouk type.*

In this paper, we give a solution in the case $\beta = 2$ of a problem given by Paul Terwilliger which is:

Problem 2.3 ([16])**.** *Find all Leonard pairs in* $\mathrm{Mat}_{d+1}(\mathcal{F})$ *that satisfy the following conditions*:

(i) A *is lower bidiagonal with subdiagonal entries all* 1.
(ii) A^* *is irreducible tridiagonal.*

The above problem is related to Leonard triple [3,4], adjacent Leonard pairs [5], and q-tetrahedron algebras [6]. In [7], Nomura gave a solution for this problem in the case $\beta = q^2 + q^{-2}$ where q is not a root of unity.

Consider the following LB-TD pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$

(13)
$$A = \begin{pmatrix} \theta_0 & & & & \\ 1 & \theta_1 & & & \\ & 1 & \theta_2 & & \\ & & \ddots & \ddots & \\ & & & 1 & \theta_d \end{pmatrix}, \quad A^* = \begin{pmatrix} x_0 & y_0 & & & & 0 \\ z_1 & x_1 & y_1 & & & \\ 0 & z_2 & x_2 & y_2 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & 0 & z_{d-1} & x_{d-1} & y_{d-1} \\ 0 & & & 0 & z_d & x_d \end{pmatrix}.$$

The main results in this paper are the following theorems.

Theorem 2.4. *Let* A*,* A^* *be a Leonard pair with fundamental parameter* $\beta = 2$*. If* A*,* A^* *has LB-TD form, then* A*,* A^* *is either of Racah type or Krawtchouk type.*

Theorem 2.5. *Let* A*,* A^* *be as in* (13)*. Fix nonzero* h*,* h^**,* x_0 *in* $\mathcal{F}$*, let* α*,* α^**,* μ*,* μ^* *in* $\mathcal{F}$*. Let*

(14) $\theta_i = \alpha + (\mu + h)i + hi^2, \ 0 \le i \le d,$

(15) $x_i = \alpha^* + i\mu^* + 2i(d - i + 1)h^* + (d - 2i)x_0/d, \ 1 \le i \le d,$

(16) $y_i = \frac{h(i+1)(d-i)}{d^2h^*}(x_0 + idh^*)(x_0 - d\mu^* - d(d - i + 1)h^*), \ 0 \le i \le d - 1,$

(17) $z_i = -h^{-1}h^*, \ 1 \le i \le d.$

Then the matrices A*,* A^* *form an LB-TD Leonard pair of Racah type in* $\mathrm{Mat}_{d+1}(\mathcal{F})$ *if and only if the following hold.*

$$\mu + kh \neq 0, \ 2 \le k \le 2d, \tag{18}$$

$$\mu^* + kh^* \neq 0, \ 2 \le k \le 2d, \tag{19}$$

$$x_0 + kdh^* \neq 0, \ 0 \le k \le d - 1, \tag{20}$$

$$x_0 - d\mu^* - kdh^* \neq 0, \ 2 \le k \le d + 1, \tag{21}$$

$$(x_0 - d\mu^* - kdh^*)h - d\mu h^* \neq 0, \ d + 3 \le k \le 2(d + 1), \tag{22}$$

$$(x_0 - kdh^*)h - d\mu h^* \neq 0, \ 2 \le k \le d + 1. \tag{23}$$

Theorem 2.6. *Let A, A^* be as in* (13)*. Fix nonzero μ, μ^*, x_0, z_1 in $\mathcal{F}$, let α, α^* in $\mathcal{F}$. Let*

$$\theta_i = \alpha + \mu i, \tag{24}$$

$$x_i = \alpha^* + ((d-2i)x_0 + di\mu^*)/d, \ 1 \le i \le d, \tag{25}$$

$$y_i = (i+1)(d-i)x_0(d\mu^* - x_0)/d^2 z_1, \ 0 \le i \le d-1, \tag{26}$$

$$z_i = z_1, \ 2 \le i \le d. \tag{27}$$

Then the matrices A, A^* form an LB-TD Leonard pair of Krawtchouk type in $\mathrm{Mat}_{d+1}(\mathcal{F})$ *if and only if the following hold.*

$$d\mu z_1 + x_0 - d\mu^* \neq 0, \tag{28}$$

$$d\mu z_1 + x_0 \neq 0, \tag{29}$$

$$d\mu^* - x_0 \neq 0. \tag{30}$$

3. Recurrent sequences

In this section we recall some facts concerning the recurrent sequences.

Definition 3.1 ([13])**.** Let $d \geq 3$ be an integer, let $\{\theta_i\}_{i=0}^d$ be a sequence in $\mathcal{F}$, let β, γ, ϱ denote scalars in $\mathcal{F}$.

(i) The sequence $\{\theta_i\}_{i=0}^d$ is said to be recurrent whenever $\theta_{i-1} - \theta_i \neq 0$ for $2 \le i \le d-1$ and the expression
$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}$$
is independent of i, for $2 \le i \le d-1$.

(ii) The sequence $\{\theta_i\}_{i=0}^d$ is said to be β-recurrent whenever
$$\theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_{i+1} = 0$$
for $2 \le i \le d-1$.

(iii) The sequence $\{\theta_i\}_{i=0}^d$ is said to be (β,γ)-recurrent whenever
$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma$$
for $1 \le i \le d-1$.

(iv) The sequence $\{\theta_i\}_{i=0}^d$ is said to be (β,γ,ϱ)-recurrent whenever
$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) = \varrho$$
for $1 \le i \le d$.

Lemma 3.2 ([13])**.** *With reference to Definition 3.1 the following are equivalent.*

(i) *The sequence $\{\theta_i\}_{i=0}^d$ is recurrent.*

(ii) *There exists $\beta \in \mathcal{F}$ such that The sequence $\{\theta_i\}_{i=0}^d$ is β-recurrent.*

Lemma 3.3 ([13])**.** *With reference to Definition 3.1 the following are equivalent.*

(i) *The sequence* $\{\theta_i\}_{i=0}^d$ *is* β*-recurrent.*
(ii) *There exists* $\gamma \in \mathcal{F}$ *such that The sequence* $\{\theta_i\}_{i=0}^d$ *is* (β, γ)*-recurrent.*

Lemma 3.4 ([13])**.** *With reference to Definition 3.1, if the sequence* $\{\theta_i\}_{i=0}^d$ *is* (β, γ)*-recurrent, then there exists* $\varrho \in \mathcal{F}$ *such that The sequence* $\{\theta_i\}_{i=0}^d$ *is* (β, γ, ϱ)*-recurrent.*

Assume $\{\theta_i\}_{i=0}^d$ is β-recurrent with $\beta = 2$, by [8] there exist scalars α, μ and h such that

$$\theta_i = \alpha + \mu(i - d/2) + hi(d - i), \ 0 \leq i \leq d.$$

Lemma 3.5. *With reference to Definition 3.1 and above,*

$$\gamma = -2h, \tag{31}$$

$$\varrho = 4\alpha h + \mu^2 + (d+1)(d-1)h. \tag{32}$$

Lemma 3.6. *Let the scalars* γ*,* ϱ *be as in Definition 3.1, and let* $\{\tilde{\theta}_i\}_{i=0}^d$ *be a reordering of* $\{\theta_i\}_{i=0}^d$ *that satisfies both*

$$\gamma = \tilde{\theta}_{i-1} - \beta\tilde{\theta}_i + \tilde{\theta}_{i+1}, \ 1 \leq i \leq d-1, \tag{33}$$

$$\varrho = \tilde{\theta}_0^2 - \beta\tilde{\theta}_0\tilde{\theta}_1 + \tilde{\theta}_1^2 - \gamma(\tilde{\theta}_0 + \tilde{\theta}_1). \tag{34}$$

Then the sequence $\{\tilde{\theta}_i\}_{i=0}^d$ *is either* $\{\theta_i\}_{i=0}^d$ *or* $\{\theta_{d-i}\}_{i=0}^d$*.*

Proof. The sequence $\{\tilde{\theta}_i\}_{i=0}^d$ is β-recurrent by (33) and Lemma 3.3. Hence by [8] there exist scalars $\tilde{\alpha}$, $\tilde{\mu}$ and $\tilde{h}$ such that

$$\tilde{\theta}_i = \tilde{\alpha} + \tilde{\mu}(i - d/2) + \tilde{h}i(d - i), \ 0 \leq i \leq d.$$

Substitute $i = 1$ in (33) and compare with (31) to find $\tilde{h} = h$. Use (31), (32) and (34) to find

$$4h(\alpha - \tilde{\alpha}) + (\mu^2 - \tilde{\mu}^2) = 0.$$

From the assumption that $\{\tilde{\theta}_i\}_{i=0}^d$ is a permutation of $\{\theta_i\}_{i=0}^d$, we find $\tilde{\alpha} = \alpha$ and $\tilde{\mu} = \pm\mu$. It is straightforward to show that for $\tilde{\mu} = \mu$, $\tilde{\theta}_i = \theta_i$ and for $\tilde{\mu} = -\mu$, $\tilde{\theta}_i = \theta_{d-i}$. □

4. The type of Leonard pair

In this section we use Askey-Wilson relations to obtain some relations between the entries of A and A^* which help us to determine the type of the Leonard pair. We start by recalling some facts.

Lemma 4.1 ([17])**.** *Let* A*,* A^* *be a Leonard pair with fundamental parameter* β*, then there exist scalars* γ*,* γ^**,* ϱ*,* ϱ^* *such that*

(i) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}, \ 1 \leq i \leq d-1,$
(ii) $\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}, \ 1 \leq i \leq d-1,$
(iii) $\varrho = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i), \ 1 \leq i \leq d,$
(iv) $\varrho^* = {\theta^*_{i-1}}^2 - \beta\theta^*_{i-1}\theta^*_i + {\theta^*_i}^2 - \gamma^*(\theta^*_{i-1} + \theta^*_i), \ 1 \leq i \leq d.$

We remark, if $\beta = 2$, then by [15] there exist scalars α, α^*, μ, μ^*, h, h^*, and ξ in $\mathcal{F}$ such that

$$\theta_i = \alpha + (\mu + h)i + hi^2, \ 0 \le i \le d, \tag{35}$$

$$\theta^*_i = \alpha^* + (\mu^* + h^*)i + h^*i^2, \ 0 \le i \le d, \tag{36}$$

$$\varphi_i = i(d - i + 1)(\xi - (\mu h^* + h\mu^*)i - hh^*i(i + d + 1)), \ 1 \le i \le d, \tag{37}$$

$$\begin{aligned}\phi_i = {} & i(d - i + 1)(\xi + \mu\mu^* + h\mu^*(d + 1) \\ & + (\mu h^* - h\mu^*)i - hh^*i(d - i + 1)), \ 1 \le i \le d.\end{aligned} \tag{38}$$

Moreover the type of the Leonard pair depends on h and h^*. If $h \neq 0$ and $h^* \neq 0$, then the Leonard pair is of Racah type. If $h = 0$ and $h^* \neq 0$, then the Leonard pair is of Hahn type. If $h \neq 0$ and $h^* = 0$, then the Leonard pair is of dual Hahn type. If $h = 0$ and $h^* = 0$, then the Leonard pair is of Krawtchouk type.

Theorem 4.2 ([17])**.** *Let d denote a nonnegative integer and let V denote a vector space over $\mathcal{F}$ with dimension $d+1$. Let A, A^* denote a Leonard pair on V. Let β, γ, γ^*, ϱ, ρ^*, ω, η, and η^* denote a sequence of scalars taken from $\mathcal{F}$ which satisfy* (11) *and* (12)*. Let the scalars θ_i, θ^*_i, a_i, a^*_i be as in Definition* 1.4*. Then the following hold.*

(i) $\omega = a^*_i(\theta_i - \theta_{i+1}) + a^*_{i-1}(\theta_{i-1} - \theta_{i-2}) - \gamma^*(\theta_i + \theta_{i-1}), \ 1 \le i \le d,$
(ii) $\omega = a_i(\theta^*_i - \theta^*_{i+1}) + a_{i-1}(\theta^*_{i-1} - \theta^*_{i-2}) - \gamma(\theta^*_i + \theta^*_{i-1}), \ 1 \le i \le d,$
(iii) $\eta = a^*_i(\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) - \gamma^*\theta_i^2 - \omega\theta_i, \ 0 \le i \le d,$
(iv) $\eta^* = a_i(\theta^*_i - \theta^*_{i-1})(\theta^*_i - \theta^*_{i+1}) - \gamma{\theta^*_i}^2 - \omega\theta^*_i, \ 0 \le i \le d.$

Definition 4.3. Let $\{\theta_i\}_{i=0}^d$, $\{x_i\}_{i=0}^d$, $\{y_i\}_{i=0}^{d-1}$, and $\{z_i\}_{i=1}^d$ be scalars such that $y_{i-1}z_i \neq 0$ for $1 \le i \le d$. Let A, A^* be two matrices as in (13), and assume that A, A^* is a Leonard pair of classical type in $\mathrm{Mat}_{d+1}(\mathcal{F})$. Let $\{\theta^*_i\}_{i=0}^d$ be a standard ordering of the eigenvalues of A^*. Let α, α^*, μ, μ^*, h and h^* be scalars that satisfy (35) and (36).

Let γ, γ^*, ϱ, ρ^*, ω, η, and η^* be from Lemma 4.1 and Theorem 4.2, let $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ be first and second split sequences of A and A^* associated with the ordering $\{\theta_i\}_{i=0}^d$, $\{\theta^*_i\}_{i=0}^d$ respectively. Let ξ be a scalar that satisfies (37) and (38).

Lemma 4.4. *With reference to Definition 4.3, $\{\theta_i\}_{i=0}^d$ is standard ordering of the eigenvalues of A.*

Proof. The sequence $\{\theta_i\}_{i=0}^d$ is an ordering of the eigenvalues of A. For ($2 \le i \le d$), compute the $(i-1, i+1)$-entries of (12) to find

$$y_iy_{i+1}(\theta_{i-1} - \beta\theta_i + \theta_{i+1}) = y_iy_{i+1}\gamma.$$

Hence $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$ for $2 \le i \le d$. Compute the $(1, 2)$-entry of (12) to find that

$$y_0\varrho = y_0(\theta_0^2 - \beta\theta_0\theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1)).$$

Hence $\varrho = \theta_0^2 - \beta\theta_0\theta_1 + \theta_1^2 - \gamma(\theta_0 + \theta_1)$. So the result hold by Lemma 3.6. $\square$

Lemma 4.5. *With reference to Definition 4.3, assume $\alpha = 0$ and $\alpha^* = 0$. Then*

$$\gamma = 2h, \tag{39}$$

$$\gamma^* = 2h^*, \tag{40}$$

$$\varrho = \mu^2 + 2h\mu, \tag{41}$$

$$\varrho^* = {\mu^*}^2 + 2h^*\mu^*, \tag{42}$$

$$\omega = -(2d(d+2)hh^* + 2d\mu h^* + 2d\mu^* h + \mu\mu^* + 2\xi), \tag{43}$$

$$\eta = d\mu(\xi - \mu h^* - \mu^* h - (d+2)hh^*), \tag{44}$$

$$\eta^* = d\mu^*(\xi - \mu h^* - \mu^* h - (d+2)hh^*). \tag{45}$$

Proof. Straightforward. □

Now we begin to obtain some relations between the entries of A and A^*.

Lemma 4.6. *With reference to Definition 4.3,*

$$z_i = z_1,\ 1 \leq i \leq d.$$

Proof. Compute the $(i+1, i-2)$-entries of (11) to find that

$$z_{i-1} - \beta z_i + z_{i+1} = 0,\ 3 \leq i \leq d-1. \tag{46}$$

Hence for $i = 3$, $z_3 = \beta z_2 - z_1$. Compute the $(4, 1)$-entry of (12) to find

$$z_1 z_2 - \beta z_1 z_3 + z_2 z_3 = 0. \tag{47}$$

Use $\beta = 2$ and eliminate z_3 in (47) to find $z_2 = z_1$. Substitute in (46) to get the result. □

Lemma 4.7. *With reference to Definition 4.3,*

$$x_{i+1} - 2x_i + x_{i-1} = -4h^*,\ 1 \leq i \leq d-1.$$

Proof. Compute the $(i+1, i-1)$-entries of (12) to find that

$$\begin{aligned} &z_i z_{i+1}(\theta_{i-1} - \beta\theta_i + \theta i + 1 - \gamma) + z_{i+1}(x_i + x_{i+1} - \beta x_{i-1} - \gamma^*) \\ &+ z_i(x_i + x_{i-1} - \beta x_{i+1} - \gamma^*) = 0. \end{aligned}$$

Simplify using Lemmas 4.5 and 4.6 to get the result. □

Lemma 4.8. *With reference to Definition 4.3, both h and h^* are zeros or both are non zeros.*

Proof. Compute the $(i+2, i)$-entries of (11) to find that $h^* + hz_1 = 0$. □

The proof of Theorem 2.4. Clear from Lemma 4.8 and the comment above Theorem 4.2. □

5. Leonard pair of Racah type

In this section we continue finding relations between the entries of A, A^* with assumption $h \neq 0$ and $h^* \neq 0$.

Lemma 5.1. *With reference to Definition 4.3, assume $h \neq 0$ and $h^* \neq 0$. Then $z_1 = -h^*/h$.*

Proof. Clear from Lemma 4.8. □

Lemma 5.2. *With reference to Definition 4.3,*

$$\begin{aligned} & y_i - 2y_{i-1} + y_{i-2} \\ &= \mu(x_1 - x_0 - \mu^*) - 2(d-1)\mu h^* - 2d\mu^* h - 6(i-1)hx_0 \\ &\quad + 2(3i-1)hx_1 - (12i(i-2) + 2d(d+2) + 8)hh^* - 2\xi, \quad 2 \leq i \leq d-1. \end{aligned}$$

Proof. Use the above lemmas to eliminate $\{z_i\}_{i=1}^d$ and $\{x_i\}_{i=2}^d$, then compute the $(i+1, i)$-entries of (11) to get the result. □

Lemma 5.3. *With reference to Definition 4.3,*

$$\begin{aligned} y_0 &= d\xi - d\mu h^* + x_0\mu - d\mu^* h + 2x_0 h - d(d+2)hh^*, \\ y_1 &= \mu(x_0 + x_1) + 4h(x_0 + x_1) - \mu\mu^* - 2(d-1)\xi - 4dh\mu^* \\ &\quad - 2(2d-1)\mu h^* - 4(d^2 + 2d - 1)hh^*. \end{aligned}$$

Proof. Use the above lemmas to eliminate $\{y_i\}_{i=2}^{d-1}$, $\{z_i\}_{i=1}^d$ and $\{x_i\}_{i=2}^d$, compute the $(1,1)$-entry of (11) to find y_0, then compute $(2,1)$-entry of (11) to find y_1. □

Lemma 5.4. *With reference to Definition 4.3,*

$$x_1 = (d\mu^* + 2d^2 h^* + (d-2)x_0)/d.$$

Proof. Use the above lemmas to eliminate $\{y_i\}_{i=0}^{d-1}$, $\{z_i\}_{i=1}^d$ and $\{x_i\}_{i=2}^d$, then compute the $(d+1, d)$-entry of (11) to find x_1. □

Lemma 5.5. *With reference to Definition 4.3,*

$$\xi = (x_0 - dh^*)(x_0 h - d\mu^* h - d\mu h^* - d(d+2)hh^*)/d^2 h^*.$$

Proof. Use the above lemmas to eliminate $\{y_i\}_{i=0}^{d-1}$, $\{z_i\}_{i=1}^d$ and $\{x_i\}_{i=1}^d$, compute the $(2,1)$-entry of (12) to find ξ. □

Lemma 5.6. *With reference to Definition 4.3, assume that A, A^* is a Leonard pair of Racah type. Then*

$$x_i = i\mu^* + 2i(d-i+1)h^* + (d-2i)x_0/d, \ 0 \leq i \leq d, \tag{48}$$

$$\begin{aligned} y_i = {} & h(i+1)(d-i)(x_0 + idh^*) \\ & (x_0 - d\mu^* - d(d-i+1)h^*)/d^2 h^*, \ 0 \leq i \leq d-1, \end{aligned} \tag{49}$$

$$z_i = -h^{-1}h^*, \ 1 \leq i \leq d. \tag{50}$$

Proof. Use Lemmas 4.7 and 5.4 to find (48), use Lemmas 5.2, 5.3 and 5.5 to find (49) and use Lemmas 4.6 and 5.1 to find (50). □

6. Leonard pair of Krawtchouk type

In this section we continue finding relations between the entries of A, A^* with assumption $h = 0$ and $h^* = 0$.

Lemma 6.1. *With reference to Definition 4.3, assume $\alpha = 0$ and $\alpha^* = 0$, $h = 0$ and $h^* = 0$. Then*

$$\gamma = 0, \tag{51}$$

$$\gamma^* = 0, \tag{52}$$

$$\varrho = \mu^2, \tag{53}$$

$$\varrho^* = {\mu^*}^2, \tag{54}$$

$$\omega = -(\mu\mu^* + 2\xi), \tag{55}$$

$$\eta = d\mu\xi, \tag{56}$$

$$\eta^* = d\mu^*\xi. \tag{57}$$

Proof. Clear from Lemma 4.1 and Theorem 4.2. □

Lemma 6.2. *With reference to Definition 4.3,*

$$y_{i+1} - y_i + y_{i-1} = \mu(x_1 - x_0) - \mu\mu^* - 2\xi, \quad 1 \le i \le d-2.$$

Proof. Use Lemmas 4.6 and 4.7 to eliminate $\{z_i\}_{i=2}^d$ and $\{x_i\}_{i=2}^d$, then compute the $(i+2, i+1)$-entries of (11) to find that

$$y_{i+1} - y_i + y_{i-1} = \mu(x_1 - x_0) + \varrho z_1 - \mu^2 z_1 + \omega, \quad 1 \le i \le d-2.$$

Substitute ϱ and ω from Lemma 6.1 to get the result. □

Lemma 6.3. *With reference to Definition 4.3,*

$$y_0 = \mu x_0 + d\xi,$$
$$y_1 = \mu(x_0 + x_1) + 2(d-1)\xi - \mu\mu^*.$$

Proof. Use Lemmas 4.6, 4.7 and 6.2 to eliminate $\{y_i\}_{i=2}^{d-1}$, $\{z_i\}_{i=2}^d$ and $\{x_i\}_{i=2}^d$, compute the $(1,1)$-entry of (11) to find y_0, then compute the $(2,1)$-entry of (11) to find y_1. □

Lemma 6.4. *With reference to Definition 4.3,*

$$x_1 = ((d-2)x_0 + d\mu^*)/d.$$

Proof. Use Lemmas 4.6, 4.7, 6.2 and 6.3 to eliminate $\{y_i\}_{i=0}^{d-1}$, $\{z_i\}_{i=2}^d$ and $\{x_i\}_{i=2}^d$, then solve the $(d+1, d+1)$-entry of (11) for x_1 to get the result. □

Lemma 6.5. *With reference to Definition 4.3,*

$$\xi = -x_0(x_0 + d\mu z_1 - d\mu^*)/d^2 z_1.$$

Proof. Use Lemmas 4.6, 4.7, 6.2, 6.3 and 6.4 to eliminate $\{y_i\}_{i=0}^{d-1}$, $\{z_i\}_{i=2}^{d}$ and $\{x_i\}_{i=1}^{d}$, then solve the $(1,1)$-entry of (12) for ξ to get the result. □

Lemma 6.6. *With reference to Definition 4.3, assume that A, A^* is a Leonard pair of Krawtchouk type. Then*

$$x_i = ((d-2i)x_0 + di\mu^*)/d,\ 0 \le i \le d, \tag{58}$$

$$y_i = (i+1)(d-i)x_0(d\mu^* - x_0)/d^2 z_1,\ 0 \le i \le d-1, \tag{59}$$

$$z_i = z_1,\ 1 \le i \le d. \tag{60}$$

Proof. Use Lemmas 4.7 and 6.4 to find (58), use Lemmas 6.2, 6.3 and 6.5 to find (59), equation (60) holds from Lemma 4.6. □

7. Proof of Theorems 2.5 and 2.6

In this section we prove Theorems 2.5 and 2.6 by defining a parameter array related to the pair A, A^*. We start by recalling some facts concerning the parameter arrays.

Theorem 7.1 ([14]). *Let d denote a nonnegative integer, let B and B^* denote matrices in $\mathrm{Mat}_{d+1}(\mathcal{F})$. Assume B is lower bidiagonal and B^* is upper bidiagonal. Then the following are equivalent.*

(i) *The pair B, B^* is a Leonard pair in $\mathrm{Mat}_{d+1}(\mathcal{F})$.*
(ii) *There exists a parameter array $(\{\theta_i\}_{i=0}^{d}, \{\theta^*_i\}_{i=0}^{d}; \{\varphi_j\}_{j=1}^{d}, \{\phi_j\}_{j=1}^{d})$ over $\mathcal{F}$ such that*

$$B(i,i) = \theta_i, \qquad B^*(i,i) = \theta^*_i \qquad (0 \le i \le d),$$
$$B(j,j-1)B^*(j-1,j) = \varphi_j \qquad (1 \le j \le d).$$

Suppose (i), (ii) *hold. Then the parameter array in* (ii) *is uniquely determined by B, B^*.*

Lemma 7.2. *Let $\theta_i = \eta + (\mu+h)i + hi^2$ $(0 \le i \le d)$. Then for $i \ne j$, $\theta_i = \theta_j$ $(0 \le i,j \le d)$ if and only if $\mu + kh = 0$ $(2 \le k \le 2d)$.*

Proof. $\theta_i - \theta_j = (i-j)(\mu + h(1+i+j))$, hence the result hold. □

Lemma 7.3. *Let $\theta^*_i = \eta^* + (\mu^* + h^*)i + h^* i^2$ $(0 \le i \le d)$. Then for $i \ne j$, $\theta^*_i = \theta^*_j$ $(0 \le i,j \le d)$ if and only if $\mu^* + kh^* = 0$ $(2 \le k \le 2d)$.*

Proof. Similar to proof of Lemma 7.2. □

Lemma 7.4. *With reference to Lemmas 7.2, 7.3,*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta^*_{j-2} - \theta^*_{j+1}}{\theta^*_{j-1} - \theta^*_j} = 3 \qquad (2 \le i,j \le d-1).$$

Proof. Straightforward. □

Lemma 7.5. *Let*

$$\varphi_i = i(d-i+1)(x_0 + d(i-1)h^*)$$
$$(x_0 h - d\mu h^* - d\mu^* h - d(d+i+2)hh^*)/d^2 h^*,\ 1 \le i \le d.$$

Then $\varphi_i = 0$ *if and only if* $x_0 + kdh^* = 0$ $(0 \le k \le d-1)$ *or* $(x_0 - d\mu^* - kdh^*)h - d\mu h^* = 0$ $(d+3 \le k \le 2(d+1))$.

Proof. Straightforward. □

Lemma 7.6. *Let*

$$\phi_i = i(d-i+1)(x_0 - d\mu^* - d(i+1)h^*)$$
$$(x_0 h - d\mu h^* - d(d-i+2)hh^*)/d^2 h^*,\ 1 \le i \le d.$$

Then $\phi_i = 0$ *if and only if* $x_0 - d\mu^* - kdh^* = 0$ $(2 \le k \le d+1)$ *or* $(x_0 - kdh^*)h - d\mu h^* = 0$ $(2 \le k \le d+1)$.

Proof. Straightforward. □

Lemma 7.7. *With reference to Lemmas 7.2, 7.3, 7.5, and 7.6, the following hold.*

$$\varphi_i = \phi_1 \sum_{k=0}^{i-1} \frac{\theta_k - \theta_{d-k}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{i-1} - \theta_d) \qquad (1 \le i \le d), \tag{61}$$

$$\phi_i = \varphi_1 \sum_{k=0}^{i-1} \frac{\theta_k - \theta_{d-k}}{\theta_0 - \theta_d} + (\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) \quad (1 \le i \le d). \tag{62}$$

Proof. To prove (61), note that

$$\sum_{k=0}^{i-1} \frac{\theta_k - \theta_{d-k}}{\theta_0 - \theta_d} = i(d-i+1)/d,$$
$$(\theta^*_i - \theta^*_0)(\theta_{d-i+1} - \theta_0) = -i(d-i+1)(\mu^* + h^*(i+1))(\mu + h(i+d)).$$

Substitute ϕ_1 and simplify to get the result. (62) similar to (61). □

Lemma 7.8. *With reference to Theorem 2.5, assume A, A^* is an LB-TD Leonard pair of Racah type in* $\mathrm{Mat}_{d+1}(\mathcal{F})$. *Define the scalars*

$$\theta^*_i = \alpha^* + (\mu^* + h^*)i + h^* i^2,\ 0 \le i \le d, \tag{63}$$

$$\varphi_i = \tfrac{i(d-i+1)}{d^2 h^*}(x_0 + d(i-1)h^*)$$
$$(x_0 h - d\mu h^* - d\mu^* h - d(d+i-2)hh^*),\ 1 \le i \le d, \tag{64}$$

$$\phi_i = \tfrac{i(d-i+1)}{d^2 h^*}(x_0 - d\mu^* - d(i+1)h^*)$$
$$(x_0 h - d\mu h^* - d(d-i+2)hh^*),\ 1 \le i \le d. \tag{65}$$

Then $\{\theta_i\}_{i=0}^d$, $\{\theta^*_i\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ *is a parameter array of A, A^*.*

Proof. Clear from Lemmas 7.2, 7.3, 7.4, 7.5, 7.6, and 7.7 and the equations (18)-(23). $\square$

The proof of Theorem 2.5. Let P be $(d+1)\times(d+1)$ matrix indexed $1, 2, \ldots, d+1$ such that the (i, j)-entry is

$$P_{ij} = \begin{cases} \dfrac{h^{*^{i-j}}\ d^{i-j}\ P_{11}\ \Pi_{k=1}^{j-1}(d\mu h^* + d\mu^* h + d(d+k+2)hh^* - x_0 h)}{h^{i-1}\ (i-j)!\ \Pi_{s=1}^{i-1}(d\mu^* + d(d-s+2)h^* - x_0)}, & j \le i, \\ 0, & j > i. \end{cases}$$

Assume $\alpha = 0$, $\alpha^* = 0$, let the matrices A, A^* be from (13) with the entries as in (14)-(17). Then the matrices representing $P^{-1}AP$, $P^{-1}A^*P$ are as in (1), where $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ are as in Lemma 7.8. Hence by Definition 1.6 and Theorem 7.1, $P^{-1}AP$, $P^{-1}A^*P$ is a Leonard pair if and only if (18)-(23) hold, which implies that the pair A, A^* is a Leonard pair if and only if (18)-(23) hold. $\square$

Lemma 7.9. *With reference to Theorem 2.6, assume A, A^* is an LB-TD Leonard pair of Krawtchouk type in* $\mathrm{Mat}_{d+1}(\mathcal{F})$. *Define the scalars*

(66) $$\theta_i^* = \alpha^* + \mu^* i,\ 0 \le i \le d,$$

(67) $$\varphi_i = \tfrac{-i(d-i+1)}{d^2 z_1}(d\mu z_1 + x_0 - d\mu^*),\ 1 \le i \le d,$$

(68) $$\phi_i = \tfrac{i(d-i+1)}{d^2 z_1}(x_0 + d\mu z_1)(d\mu^* - x_0),\ 1 \le i \le d.$$

Then $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\varphi_i\}_{i=1}^d$, $\{\phi_i\}_{i=1}^d$ is a parameter array of A, A^*.

Proof. Similar to proof of Lemma 7.8. $\square$

The proof of Theorem 2.6. Similar to proof of Theorem 2.5 with

$$P_{ij} = \begin{cases} \dfrac{(-1)^{i-1}\ P_{11}\ (dz_1)^{i-j}\ (d\mu z_1 + x_0 - d\mu^*)^{j-1}}{(i-j)!\ (d\mu^* - x_0)^{i-1}}, & j \le i, \\ 0, & j > i. \end{cases}$$ $\square$

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Hasan Alnajjar
Department of Mathematics
The University of Jordan
Amman 11942, Jordan
Email address: h.najjar@ju.edu.jo