Commun. Korean Math. Soc. **34** (2019), No. 2, pp. 391–399
https://doi.org/10.4134/CKMS.c180172
pISSN: 1225-1763 / eISSN: 2234-3024

CHROMATIC NUMBER OF THE ZERO-DIVISOR GRAPHS OVER MODULES

Sang Cheol Lee and Rezvan Varmazyar

Abstract. Let R be a commutative ring with identity and let M be an R-module. The main purpose of this paper is to calculate the chromatic number of the zero-divisor graphs over modules.

1. Introduction

Let R be a commutative ring with identity and $Z(R)$ be its set of zero-divisors. The study of coloring of zero-divisor graphs of commutative rings dates back to [3]. For the information about the zero-divisor graphs of commutative rings, $\Gamma(R)$, which is an undirected (simple) graph with vertices $Z(R)^* = Z(R)\backslash\{0\}$ and with two distinct vertices x and y adjacent if and only if $xy = 0$, see [1–3], [7] and [9]. Recently, assigning a graph to a module has received a good deal of attention from many authors, see for instance [4], [6] and [8].

A graph is said to be *connected* if for each pair of distinct vertices x and y, there is a finite sequence of distinct vertices $x = x_1, \ldots, x_n = y$ such that each pair $\{x_i, x_{i+1}\}$ is an edge. Such a sequence is said to be a *path*, and the *distance*, $d(x, y)$, between vertices x and y is the length of the shortest path connecting them ($d(x, y) = \infty$ if there is no such path). The *diameter* of a connected graph G is

$$diam\,(G) = \sup\{\, d(x, y) \mid x \text{ and } y \text{ are distinct vertices of } G\,\}.$$

If $G = \emptyset$, then $diam\,(G) = -\infty$. The diameter is 0 if the graph consists of a single vertex. A connected graph with more than one vertex has diameter 1 if and only if it is complete, that is, there exists an edge between each pair of distinct vertices. We denote the complete graph with n vertices by K_n. A *cycle* is a path of edges and vertices (without repeat) such that a vertex is reachable from itself. In graph theory, the *girth* of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (that is,

Received April 23, 2018; Revised August 20, 2018; Accepted October 24, 2018.
2010 *Mathematics Subject Classification.* Primary 05C15, 13C12, 05C25, 05C38.
Key words and phrases. zero-divisor graphs of modules, chromatic number, clique number.

it is an acyclic graph), its girth is defined to be *infinity*. The girth of a graph G will be denoted by $g\,G$.

Throughout this paper, R will denote a commutative ring with identity and M will denote a nonzero unitary R-module. For a subset S of M, we denote the set of all nonzero elements of S by S^*.

Recall that an element x of M is called a:

- weak zero-divisor, if $x = 0$ or $(x :_R M)(y :_R M)M = 0$ for some nonzero $y \in M$ with $(y :_R M) \subset R$ ($(x :_R M) = \{r \in R \,|\, rM \subseteq Rx\}$).
- zero-divisor, if $x = 0$ or $(x :_R M) \neq 0$ and $(x :_R M)(y :_R M)M = 0$ for some nonzero $y \in M$ with $0 \neq (y :_R M) \subset R$.
- strong zero-divisor, if $x = 0$ or $(0 :_R M) \subset (x :_R M)$ and $(x :_R M)(y :_R M)M = 0$ for some nonzero $y \in M$ with $(0 :_R M) \subset (y :_R M) \subset R$.

We denote $\underline{Z}(_RM)$, $Z(_RM)$ and $\overline{Z}(_RM)$, respectively for the set of weak zero-divisors, zero-divisors and strong zero-divisors of M. It is clear that

$$\overline{Z}(_RM) \subseteq Z(_RM) \subseteq \underline{Z}(_RM).$$

We associate (simple) graphs, $\underline{\Gamma}(_RM)$, $\Gamma(_RM)$ and $\overline{\Gamma}(_RM)$ to M with vertices $\underline{Z}(_RM)^*$, $Z(_RM)^*$ and $\overline{Z}(_RM)^*$, respectively, and two distinct vertices x and y are adjacent if and only if $(x :_R M)(y :_R M)M = 0$. Hence

$$\overline{\Gamma}(_RM) \subseteq \Gamma(_RM) \subseteq \underline{\Gamma}(_RM).$$

Let M be an R-module. If $\underline{\Gamma}(_RM)$ contains a cycle, then we prove in Section 2 that $g\,\underline{\Gamma}(_RM) \leq 4$ (see Theorem 2.5). In Section 3, we deal with the chromatic number of the graph $\Gamma(_RM)$ of M. Our main result is as follows:

Let M_1 and M_2 be R-modules such that $(0 :_R M_1) + (0 :_R M_2) = R$. Then the following hold.

(1) If $\underline{Z}(_RM_1)^* = \underline{Z}(_RM_2)^* = \emptyset$, then $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = 2$.

(2) If $\underline{Z}(_RM_1)^* = \emptyset$ and $\underline{Z}(_RM_2)^* \neq \emptyset$, then $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}(_RM_2)) + 1$.

(3) If $\underline{Z}(_RM_1)^* \neq \emptyset$ and $\underline{Z}(_RM_2)^* = \emptyset$, then $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}(_RM_1)) + 1$.

(4) If $\underline{Z}(_RM_1)^* \neq \emptyset$ and $\underline{Z}(_RM_2)^* \neq \emptyset$, then

$$\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}(_RM_1)) + \chi(\underline{\Gamma}(_RM_2)) + k_1k_2,$$

where k_i is the number of elements m_t of $\underline{Z}(_RM_i)^*$ with the property that $(m_t :_R M_i)(m_t :_R M_i)M_i = 0$ for $i = 1, 2$.

(5) Let $\underline{Z}(_RM_1) = M_1$ and $\underline{Z}(_RM_2) = M_2$. Let $\underline{\Gamma}(_RM_1)$ and $\underline{\Gamma}(_RM_2)$ be complete zero-divisor graphs. If $(x :_R M_i)(x :_R M_i)M_i = 0$ for every element x of M_i; $i = 1$, 2, then

$$\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}(_RM_1)) \cdot \chi(\underline{\Gamma}(_RM_2)) + \chi(\underline{\Gamma}(_RM_1)) + \chi(\underline{\Gamma}(_RM_2)) - 1.$$

Finally, in Section 4 we propose some results of the chromatic number of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module.

2. Zero-divisor graphs of modules

Let M be an R-module. If M is faithful, then $(0 : M) = 0$ and so $\Gamma({}_RM) = \overline{\Gamma}({}_RM)$. If M is not faithful, then $\Gamma({}_RM) = \underline{\Gamma}({}_RM)$.

Example 2.1. i) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Then

$$((0,1) :_R M) = ((1,0) :_R M) = ((1,1) :_R M) = 2\mathbb{Z} = ann_R(M).$$

Therefore $\Gamma({}_RM) = \underline{\Gamma}({}_RM)$ is a complete graph with three vertices and $\overline{\Gamma}({}_RM)$ is an empty graph.

ii) Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Then $(m :_R M) = ann_R(M)$ for every m of M. Hence $\Gamma({}_RM) = \overline{\Gamma}({}_RM)$ is an empty graph and $\underline{\Gamma}({}_RM)$ is complete with vertices M^*.

iii) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3$. Then $ann_R(M) = 6\mathbb{Z}$, $((1,0) :_R M) = 3\mathbb{Z}$, $((0,1) :_R M) = ((0,2) :_R M) = 2\mathbb{Z}$ and $((1,1) :_R M) = ((1,2) :_R M) = R$. Therefore $(0,1) - (1,0) - (0,2)$ is the graph of $\underline{\Gamma}({}_RM) = \Gamma({}_RM) = \overline{\Gamma}({}_RM)$.

Proposition 2.2. *Let M be an R-module. If x and y are adjacent in $\underline{\Gamma}({}_RM)$, then for every $u \in Rx$ and $v \in Ry$, u is adjacent to v.*

Proof. Since $(u :_R M) \subseteq (x :_R M)$ and $(v :_R M) \subseteq (y :_R M)$ we get $(u :_R M)(v :_R M)M = 0$, as needed. □

Next proposition shows when $\underline{\Gamma}({}_RM)$ has a cycle.

Proposition 2.3. *Let M be an R-module. If $\underline{\Gamma}({}_RM)$ contains a path of length 4, then $\underline{\Gamma}({}_RM)$ contains a cycle.*

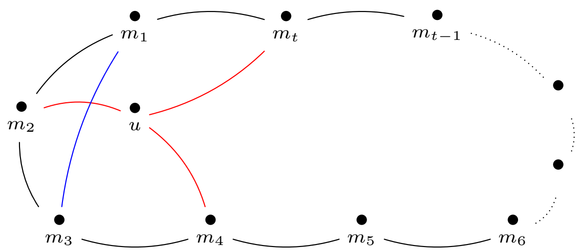
Proof. Let $x_1 - x_2 - x_3 - x_4 - x_5$ be a path of length 4 in $\underline{\Gamma}({}_RM)$. The proof is complete if x_2 is adjacent to x_4. Assume that $(x_2 :_R M)(x_4 :_R M)M \neq 0$. Hence $Rx_2 \cap Rx_4 \neq 0$. Let $m \in Rx_2 \cap Rx_4$. If $m = x_i$ for some $i; 1 \leq i \leq 5$, then by Proposition 2.2, we get a cycle. If $m \neq x_i$ for each $i; 1 \leq i \leq 5$, then $x_1 - m - x_3 - x_2 - x_1$ or $x_5 - m - x_3 - x_4 - x_5$ is a cycle in $\underline{\Gamma}({}_RM)$. □

Corollary 2.4. *Let M be an R-module. If $\overline{\Gamma}({}_RM)$ contains a path of length 4, then $\overline{\Gamma}({}_RM)$ contains a cycle.*

Theorem 2.5. *Let M be an R-module. Then $\underline{\Gamma}({}_RM)$ is a connected graph and $diam(\underline{\Gamma}({}_RM)) \leq 3$. Moreover, if $\underline{\Gamma}({}_RM)$ contains a cycle, then $g\,\underline{\Gamma}({}_RM) \leq 4$.*

Proof. The first part is similar to the proof of [6, Theorem 4.3].

Assume that $\underline{\Gamma}({}_RM)$ contains a cycle $m_1 - m_2 - \cdots - m_t - m_1$ of length t. If $t \leq 4$, then the proof is completed. Now assume that $t \geq 5$. Consider the following cycle of length t. We will show that it can be shorten to a cycle of length ≤ 4.



If $(m_1 : M)(m_3 : M)M = 0$, then we get a cycle of length 3: $m_1 - m_2 - m_3 - m_1$. Imagine $(m_1 : M)(m_3 : M)M \neq 0$. Now $(m_1 : M)(m_3 : M)M \subseteq Rm_1 \cap Rm_3$ implies that $Rm_1 \cap Rm_3 \neq 0$. Thus we can take a nonzero element u in $Rm_1 \cap Rm_3$. Then $Ru \subseteq Rm_1 \cap Rm_3$, so that $(u : M)(m_t : M)M = 0$, $(u : M)(m_2 : M)M = 0$, and $(u : M)(m_4 : M)M = 0$. Hence we get two cycles of length 4: $m_1 - m_2 - u - m_t - m_1$ and $m_2 - m_3 - m_4 - u - m_2$. This shows that $g\,\underline{\Gamma}(_RM) \leq 4$. $\square$

In view of the above theorem, we have the following result:

Corollary 2.6. *Let M be an R-module. If $\overline{Z}(_RM) \neq \emptyset$, then $\overline{\Gamma}(_RM)$ is a connected graph and diam$(\overline{\Gamma}(_RM)) \leq 3$. Moreover, If $\overline{\Gamma}(_RM)$ contains a cycle, then $g\,\overline{\Gamma}(_RM) \leq 4$.*

Recall that an R-module M is multiplication if $N = (N : M)M$ for every submodule N of M. Also, M is multiplication-like if $(0 :_R M) \subset (N :_R M)$. It is clear that every multiplication module is multiplication-like (See [4, 5] for more results of multiplication and multiplication-like modules). In [4] it was shown that $\overline{\Gamma}(_RM) = \Gamma(_RM) = \underline{\Gamma}(_RM)$ if and only if M is a multiplication-like R-module. Furthermore, if R is not a field and M is not a simple module, then $\overline{\Gamma}(_RM)$ is the empty graph and $\Gamma(_RM)$ is a complete graph with vertices M^*.

3. Coloring of modules

We recall a *coloring* of a graph G to be an assignment of colors to the vertices of G, one color to each vertex, so that adjacent vertices are assigned distinct colors. If n colors are used, then the coloring is referred to as an *n-coloring*. If there exists an n-coloring of a graph G, then G is called n-colorable. The minimum n for which a graph G is n-colorable is called the *chromatic number* of G, and is denoted by $\chi(G)$. A *clique* in a graph G, is a complete subgraph of G. A *maximum clique* of a graph G, is a clique, such that there is no clique with more vertices. The *clique number*, $\omega(G)$, of a graph G is the number of vertices in a maximum clique in G.

We may consider an R-module M as a graph $\Gamma_0(_RM)$ whose vertices are elements of M such that two distinct vertices x and y of M are adjacent if and only if $(x :_R M)(y :_R M)M = 0$. It is clear that $\Gamma_0(_RM)$ is connected with *diam*$(\Gamma_0(_RM)) \leq 2$ and $\underline{\Gamma}(_RM)$ is a subgraph of $\Gamma_0(_RM)$.

By the fact that $\underline{\Gamma}({}_RM)$ and $\Gamma_0({}_RM)$ are connected we can prove the following propositions by definitions of $\Gamma_0({}_RM)$, $\Gamma({}_RM)$ and [3, Proposition 2.2].

Proposition 3.1. *Let M be an R-module. Then the following hold.*

(1) $\chi(\Gamma_0({}_RM)) = 1$ *if and only if* $M = \{0\}$.

(2) *If M is multiplication, then* $\chi(\Gamma({}_RM)) = 1$ *if and only if* $M \cong \mathbb{Z}_4$ *or* $M = \mathbb{Z}_2[x]/(x^2)$.

In the graph $\Gamma_0(M)$ the zero element is adjacent to all $x \in M$ so that $\Gamma_0(M)$ is a star graph if and only if $Z({}_RM) = \{0\}$. Thus we have the following proposition:

Proposition 3.2. *Let M be an R-module. If $Z^*({}_RM) = \emptyset$, then $\chi(\Gamma_0(M)) = 2$.*

Proposition 3.3. *Let M be an R-module. If $\underline{Z}({}_RM)$ has at least 3 elements, then $\chi(\underline{\Gamma}({}_RM)) \geq 2$.*

Let M_1 and M_2 be R-modules. Then

$$\underline{Z}({}_RM_1 \times M_2) \cup (M_1 \times \underline{Z}({}_RM_2)) \subseteq \underline{Z}({}_R(M_1 \oplus M_2)),$$

but the equality does not hold in general even if $(0 :_R M_1) + (0 :_R M_2) = R$. The example of this is given below.

Example 3.4. It is clear that $(0 :_{\mathbb{Z}} \mathbb{Z}_4) + (0 :_{\mathbb{Z}} \mathbb{Z}_5) = \mathbb{Z}$. Notice that

$$\begin{aligned}
&\underline{Z}({}_{\mathbb{Z}}\mathbb{Z}_4)^* \times \mathbb{Z}_5 = \{(2,\,0),\,(2,\,1),\,(2,\,2),\,(2,\,3),\,(2,\,4)\},\\
&\mathbb{Z}_4 \times \underline{Z}({}_{\mathbb{Z}}\mathbb{Z}_5)^* = \emptyset,\\
&\underline{Z}({}_{\mathbb{Z}}(\mathbb{Z}_4 \oplus \mathbb{Z}_5))^* = \{(0,\,1),\,(0,\,2),\,(0,\,3),\,(0,\,4),\,(1,\,0),\\
&\qquad\qquad (2,\,0),\,(2,\,1),\,(2,\,2),\,(2,\,3),\,(2,\,4),\,(3,\,0)\}.
\end{aligned}$$

Then $(\underline{Z}({}_{\mathbb{Z}}\mathbb{Z}_4)^* \times \mathbb{Z}_5) \cup (\mathbb{Z}_4 \times \underline{Z}({}_{\mathbb{Z}}\mathbb{Z}_5)^*) \subsetneq \underline{Z}({}_{\mathbb{Z}}(\mathbb{Z}_4 \oplus \mathbb{Z}_5))^*$.

Theorem 3.5. *Let M_1 and M_2 be R-modules such that $(0 :_R M_1) + (0 :_R M_2) = R$. Then the following hold.*

(1) *If $\underline{Z}({}_RM_1)^* = \underline{Z}({}_RM_2)^* = \emptyset$, then $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = 2$.*

(2) *If $\underline{Z}({}_RM_1)^* = \emptyset$ and $\underline{Z}({}_RM_2)^* \neq \emptyset$, then $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}({}_RM_2)) + 1$.*

(3) *If $\underline{Z}({}_RM_1)^* \neq \emptyset$ and $\underline{Z}({}_RM_2)^* = \emptyset$, then $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}({}_RM_1)) + 1$.*

Proof. (1) Assume that $\underline{Z}({}_RM_1)^* = \underline{Z}({}_RM_2)^* = \emptyset$. Then for every $m_i \in M_1^*$ and $m_j \in M_2^*$ the element $(0, m_j)$ of $\underline{Z}({}_R(M_1 \oplus M_2))^*$ is adjacent to $(m_i,\,0)$. Also there is no edge between $(m_i, 0)$ and $(m_i',\,0)$ where $m_i' \in M_1^*$ and there is no edge between $(0,\,m_j)$ and $(0,\,m_j')$ where $m_j' \in M_2^*$. So we need two colors, one for all vertices of the form $(0,\,m_j)$ and another for all vertices of the form $(m_i,\,0)$. Therefore $\chi(\underline{\Gamma}(M_1 \oplus M_2)) = 2$.

(2) Assume that $\underline{Z}({}_RM_1)^* = \emptyset$, $\underline{Z}({}_RM_2)^* \neq \emptyset$ and $\chi(\underline{\Gamma}({}_RM_2)) = t$. So

$$\underline{Z}({}_R(M_1 \oplus M_2))^* = ((M_1 \times \underline{Z}({}_RM_2)) \cup (0 \times M_2)) \backslash \{(0,\,0)\}.$$

For every $m_j \in M_2$ we can color the vertices $(0,\, m_j)$ of $\underline{Z}({}_R(M_1 \oplus M_2))^*$ by t colors. Since for every $m_i \in M_1^*$ the vertex $(m_i,\, 0)$ of $\underline{Z}({}_R(M_1 \oplus M_2))^*$ is adjacent to all vertices of the form $(0,\, m_j)$ where $m_j \in M_2^*$, we have to use a new color to all vertices of the form $(m_i,\, 0)$. Also there is no edge between all vertices of the form $(m_i,\, 0)$ and all vertices of the form $(m_i,\, m_j)$ where $m_i \in M_1^*$ and $m_j \in M_2$. Hence we can use that new color for all the vertices of the form $(m_i,\, m_j)$. Therefore we color the graph with $t+1$ colors.

(3) The proof is similar to that of the part (2). □

Even if $\underline{\Gamma}(M_1)$ and $\underline{\Gamma}(M_2)$ are complete graphs, $\underline{\Gamma}(M_1 \oplus M_2)$ is not complete in general. If for every $x_i \in \underline{Z}({}_RM_1)^*$ and $y_j \in \underline{Z}({}_RM_2)^*$ we have $(x_i :_R M_1)(x_i :_R M_1)M_1 = 0$ and $(y_j :_R M_2)(y_j :_R M_2)M_2 = 0$, then $\underline{\Gamma}(M_1 \oplus M_2)$ is complete.

Theorem 3.6. *Let M_1 and M_2 be R-modules such that $(0 :_R M_1) + (0 :_R M_2) = R$. Let $\underline{Z}({}_RM_1)^* \neq \emptyset$ and $\underline{Z}({}_RM_2)^* \neq \emptyset$. Then*

$$\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}({}_RM_1)) + \chi(\underline{\Gamma}({}_RM_2)) + k_1k_2,$$

where k_i is the number of elements m_t of $\underline{Z}({}_RM_i)^*$ with the property that $(m_t :_R M_i)(m_t :_R M_i)M_i = 0$ for $i = 1, 2$.

Proof. Let's write $\chi(\underline{\Gamma}({}_RM_1))$ and $\chi(\underline{\Gamma}({}_RM_2))$ by t_1 and t_2, respectively. There are 8 cases to consider for coloring $\underline{\Gamma}(M_1 \oplus M_2)$.

Case 1. We color all vertices of the form $(0,\, y_j)$ by t_2 colors where $y_j \in M_2^*$.

Case 2. We color all vertices of the form $(x_i,\, 0)$ by t_1 colors where $x_i \in M_1^*$.

Case 3. Consider the vertex $(x_i,\, y_j)$ where $x_i \in M_1 \backslash \underline{Z}({}_RM_1)$ and $y_j \in \underline{Z}({}_RM_2)$. This vertex is not adjacent to any vertex of the form $(x_t,\, y_s)$ for every $x_t \in M_1^*$ and $y_s \in M_2$. So we can color this vertex by t_1 colors or by the color of vertex $(x_i,\, 0)$. It is clear that if $y_j \in M_2 \backslash \underline{Z}({}_RM_2)$, then $(x_i,\, y_j) \notin \underline{Z}({}_R(M_1 \oplus M_2))^*$.

Case 4. This is similar to case 3, we color all vertices of the form $(x_i,\, y_j)$ by t_2 colors or by the color of vertex $(0, y_j)$ where $x_i \in Z({}_RM_1)$ and $y_j \in M_2 \backslash \underline{Z}({}_RM_2)$. Now, let $x_i \in \underline{Z}({}_RM_1)^*$ and $y_j \in \underline{Z}({}_RM_2)^*$. We have the following cases.

Case 5. If $(x_i :_R M_1)(x_i :_R M_1)M_1 \neq 0$ and $(y_j :_R M_2)(y_j :_R M_2)M_2 \neq 0$, then the vertex $(x_i,\, y_j)$ is not adjacent to $(0,\, y_j)$ and $(x_i,\, 0)$. Hence we color all vertices of the form (x_i, y_j) by the color of vertex $(0,\, y_j)$ or by the color of vertex $(x_i,\, 0)$.

Case 6. If $(x_i :_R M_1)(x_i :_R M_1)M_1 = 0$ and $(y_j :_R M_2)(y_j :_R M_2)M_2 \neq 0$, then the vertex $(x_i,\, y_j)$ is not adjacent to $(0,\, y_j)$. Hence we color all vertices of the form $(x_i,\, y_j)$ by the color of vertex $(0,\, y_j)$.

Case 7. If $(x_i :_R M_1)(x_i :_R M_1)M_1 \neq 0$ and $(y_j :_R M_2)(y_j :_R M_2)M_2 = 0$, then the vertex $(x_i,\, y_j)$ is not adjacent to $(x_i,\, 0)$. Hence we color all vertices of the form $(x_i,\, y_j)$ by the color of vertex $(x_i,\, 0)$.

Case 8. If $(x_i :_R M_1)(x_i :_R M_1)M_1 = 0$ and $(y_j :_R M_2)(y_j :_R M_2)M_2 = 0$, then we need a new color for all vertices of the form (x_i, y_j). Since the number of these vertices is k_1k_2 where k_i is the number of elements m_t of $\underline{Z}({}_RM_i)^*$ with the property that $(m_t :_R M_i)(m_t :_R M_i)M_i = 0$ for $i = 1, 2$, we have

$$\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}({}_RM_1)) + \chi(\underline{\Gamma}({}_RM_2)) + k_1k_2. \qquad \square$$

Corollary 3.7. *Let M_1 and M_2 be R-modules such that $(0 :_R M_1) + (0 :_R M_2) = R$. Let $\underline{Z}({}_RM_1) = M_1$ and $\underline{Z}({}_RM_2) = M_2$. Let $\underline{\Gamma}({}_RM_1)$ and $\underline{\Gamma}(M_2)$ be complete zero-divisor graphs. If $(x : M_i)(x : M_i)M_i = 0$ for every element x of M_i; $i = 1, 2$, then*

$$\chi(\underline{\Gamma}(M_1 \oplus M_2)) = \chi(\underline{\Gamma}({}_RM_1)) \cdot \chi(\underline{\Gamma}({}_RM_2)) + \chi(\underline{\Gamma}({}_RM_1)) + \chi(\underline{\Gamma}({}_RM_2)) - 1.$$

Proof. In this case $\underline{\Gamma}(M_1 \oplus M_2)$ is a complete graph. Since $(0,0) \notin \underline{Z}({}_R(M_1 \oplus M_2))^*$ this graph has $(\chi(\underline{\Gamma}({}_RM_1))+1)\cdot(\chi(\underline{\Gamma}({}_RM_2))+1)-1$ vertices, as needed. $\square$

4. Coloring of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module

Let $M = \mathbb{Z}_n$ and $R = \mathbb{Z}$. It is clear that $\mathbb{Z}_n$ is a multiplication $\mathbb{Z}$-module, so

$$\overline{\Gamma}({}_{\mathbb{Z}}\mathbb{Z}_n) = \Gamma({}_{\mathbb{Z}}\mathbb{Z}_n) = \underline{\Gamma}({}_{\mathbb{Z}}\mathbb{Z}_n).$$

For every prime number p, $\chi(\Gamma({}_{\mathbb{Z}}\mathbb{Z}_p)) = 0$.

In the following results, M will be treated as a $\mathbb{Z}$-module.

Theorem 4.1. *Let p be a prime number and $M = \mathbb{Z}_{p^n}$. Then $\chi(\Gamma({}_RM)) = p^k - 1$ if $n = 2k$ and $\chi(\Gamma({}_RM)) = p^k$ if $n = 2k+1$.*

Proof. It is clear that $Z(M) = Rp$ and

$$0 = Rp^n \subset Rp^{n-1} \subset Rp^{n-2} \subset \cdots \subset Rp^2 \subset Rp.$$

Let $n = 2k$. So $(x :_R Rp^k)(y :_R Rp^k)Rp^k = 0$ for every x, y of Rp^k. Hence Rp^k is an induced complete subgraph of $\Gamma({}_RM)$, that is, $\Gamma({}_RRp^k) = K_{p^k-1}$. It is clear that all elements of Rp^v are adjacent to all elements of Rp^u where $u+v \geq n$. There exist elements $x_i (1 \leq i \leq p^k - p^{k-1})$ of Rp^k such that $x_i \notin Rp^t$ where $k+1 \leq t < 2k$. Since $(z :_R M)(y :_R M)M = 0$ and $(z :_R M)(x_i :_R M)M \neq 0$ for every $z \in Rp^j$ with $1 \leq j < k$ and $y \in Rp^t$, we can use the color of all vertices x_i to all elements of Rp^j. Therefore $\chi(\Gamma({}_RM)) = \omega(\Gamma({}_RM)) = p^k - 1$.

Now, let $n = 2k+1$. It is clear that $\Gamma({}_RRp^{k+1})$ is an induced complete subgraph of $\Gamma({}_RM)$ and $|Rp^{k+1}| = p^k$. Hence $\chi(\Gamma({}_RRp^{k+1})) = p^k - 1$. All elements of Rp^k are adjacent to all elements of Rp^{k+1}, so we need a new color for elements $y \in Rp^k \setminus Rp^{k+1}$. Therefore $\omega(\Gamma({}_RM)) = p^k$. Also, there exist elements $x_i (1 \leq i \leq p^k)$ of Rp^{k+1} such that $x_i \notin Rp^t$ where $k+2 \leq t < 2k+1$. Since $(z : M)(y : M)M = 0$ and $(z : M)(x_i : M)M \neq 0$ for every $z \in Rp^j$ with $1 \leq j < k$, and $y \in Rp^t$ with $y \neq x_i$, we can use the color of all vertices x_i to all elements of R_p^j. Therefore $\chi(\Gamma({}_RM)) = p^k = \omega(\Gamma({}_RM))$. $\square$

Theorem 4.2. *Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers and $M = \mathbb{Z}_{p_1p_2\cdots p_n}$. Then*

$$\chi(\Gamma({}_RM)) = n.$$

Proof. Let $n = 2$. Then $Z({}_RM) = Rp_1 \cup Rp_2$ implies that $\Gamma({}_RM)$ is complete bipartite. Thus $\chi(\Gamma({}_RM)) = 2$.

Let $n \geq 3$. Then $\Gamma({}_RM)$ has $(p_1-1)(p_2-1)\cdots(p_{n-1}-1)(p_n-1)$ complete subgraphs of order n with vertices all elements of the submodules $Rp_1p_2p_3\cdots p_{n-3}p_{n-2}p_{n-1}$, $Rp_1p_2p_3\cdots p_{n-3}p_{n-2}p_n$, $Rp_1p_2p_3\cdots p_{n-3}p_{n-1}p_n, \ldots, Rp_1p_3\cdots p_{n-2}p_{n-1}p_n$ and $Rp_2p_3\cdots p_{n-2}p_{n-1}p_n$. So we need n colors for coloring these vertices. Since $(x_i :_R M)(t :_R M)M \neq 0$ where $x_i \in Rp_1p_2\cdots p_{i-1}p_{i+1}\cdots p_n$ and $t \in Rp_j (j \neq i), 1 \leq i, j \leq n$, we can use the color of x_i to all elements of Rp_j. Also, $(x_{ik} :_R M)(u :_R M)M \neq 0$ and $(x_i :_R M)(x_{ik} :_R M)M \neq 0$ where $x_{ik} \in Rp_1p_2\cdots p_{i-1}p_{i+1}\cdots p_{k-1}p_{k+1}p_n$ and $u \in Rp_ip_k$, we can use the color of x_i to all elements of Rp_jp_k and all elements of the form x_{ik}. By this way, we can color all elements of $Rp_1p_2\cdots p_{t_1-1}p_{t_1+1}\cdots p_{t_s-1}p_{t_s+1}\cdots p_{t_l-1}p_{t_l+1}\cdots p_{n-2}p_{n-1}p_n$ with the color of x_i or x_{t_m} where $t_m \in \{1, 2, \ldots, n\}$. Therefore we need exactly n colors for coloring $\Gamma({}_RM)$, that is, $\chi(\Gamma({}_RM)) = \omega(\Gamma({}_RM)) = n$. □

Corollary 4.3. *Let $p_1, p_2, \ldots, p_n$ be distinct prime numbers and $M = \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$. Then*

$$\chi(\Gamma({}_RM)) = n.$$

Proof. (Method I) By the Chinese Remainder Theorem,

$$\mathbb{Z}_{p_1p_2\cdots p_n} \cong \mathbb{Z}_{p_1} \oplus \mathbb{Z}_{p_2} \oplus \cdots \oplus \mathbb{Z}_{p_n}.$$

Hence the results follows from Theorem 4.2.

(Method II) The vertices $z_1 = (\alpha_1, 0, \ldots, 0)$, $z_2 = (0, \alpha_2, 0, \ldots, 0), \ldots, z_n = (0, \ldots, 0, \alpha_n)$ form a clique in $\Gamma({}_RM)$. So, we use color c_i for z_i. Now, the vertex $(u_1, u_2, \ldots, u_n)$ is adjacent to z_i if and only if $u_i = 0$, $1 \leq i \leq n$. Therefore $\chi(\Gamma({}_RM)) = \omega(\Gamma({}_RM)) = n$. □

Theorem 4.4. *Let p and q be distinct prime numbers and $M = \mathbb{Z}_{p^nq}$. Then $\chi(\Gamma({}_RM)) = p^k$ if $n = 2k$ and $\chi(\Gamma({}_RM)) = p^k + 1$ if $n = 2k + 1$.*

Proof. It is clear that $Z({}_RM) = Rp \cup Rq$ and $0 \subset Rp^{n-1}q \subset Rp^{n-2}q \subset \cdots \subset Rpq \subset Rq$.

Let $n = 2k$. So $(x :_R Rp^kq)(y :_R Rp^kq)Rp^kq = 0$ for every x, y of Rp^kq. Hence Rp^kq is an induced complete subgraph of $\Gamma({}_RM)$, that is, $\Gamma({}_RRp^kq) = K_{p^k-1}$. There exist elements $x_i (1 \leq i \leq p^k - p^{k-1})$ of Rp^kq such that $x_i \notin Rp^tq$ where $k + 1 \leq t < 2k$. Since $(z :_R M)(y :_R M)M = 0$ and $(z :_R M)(x_i :_R M)M \neq 0$ for every $z \in Rp^jq$ with $1 \leq j < k$, and $y \in Rp^tq$ with $y \neq x_i$, we can use the color of all vertices x_i to all elements of Rp^jq. Since $(a :_R M)(b :_R M)M = 0$ for every $a \in Rp^kq$ and $b \in Rp^k$, we need a new color for all elements of Rp^k. Also, the elements of Rp^{2k} are not adjacent to the elements of Rp^k, implies that we can use the color of vertices of Rp^k to coloring vertices of Rp^{2k}. Therefore $\chi(\Gamma({}_RM)) = p^k + 1 - 1 = p^k = \omega(\Gamma({}_RM))$.

Now, let $n = 2k + 1$. It is clear that $\Gamma({}_RRp^{k+1}q)$ is an induced complete subgraph of $\Gamma({}_RM)$ and $\chi(\Gamma({}_RRp^{k+1}q)) = p^k - 1$. Since $(x :_R M)(y :_R M)M =$

$(z :_R M)(y :_R M)M = (z :_R M)(x :_R M)M = 0$ for every $x \in Rp^k q$, $y \in Rp^{k+1}q$ and $z \in Rp^s$ where $k \leq s < 2k+1$ we need two new colors for elements of $Rp^k q$ and Rp^s. Also, there exist elements $x_i (1 \leq i \leq p^k - p^{k-1})$ of $Rp^{k+1}q$ such that $x_i \notin Rp^t q$ where $k+2 \leq t < 2k+1$. Since $(z :_R M)(y :_R M)M = 0$ and $(z :_R M)(x_i :_R M)M \neq 0$ for every $z \in Rp^j q$ with $1 \leq j < k+1$, and $y \in Rp^t$ with $y \neq x_i$, we can use the color of all vertices x_i to all elements of $Rp^j q$. Therefore $\chi(\Gamma(_R M)) = p^k + 1 = \omega(\Gamma(_R M))$. □

In this section, we calculated the chromatic numbers of the zero-divisor graphs of $\mathbb{Z}_n$ as a $\mathbb{Z}$-module, in cases that n has several kinds of standard prime factorizations. We think that there is a formal method to calculate $\chi(\Gamma(_R \mathbb{Z}_n))$.

References

[1] S. Akbari and A. Mohammadian, *On zero-divisor graphs of finite rings*, J. Algebra **314** (2007), no. 1, 168–184.

[2] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[3] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.

[4] M. Behboodi, *Zero divisor graphs for modules over commutative rings*, J. Commut. Algebra **4** (2012), no. 2, 175–197.

[5] S. C. Lee and R. Varmazyar, *Semiprime submodules of graded multiplication modules*, J. Korean Math. Soc. **49** (2012), no. 2, 435–447.

[6] ———, *Zero-divisor graphs of multiplication modules*, Honam Math. J. **34** (2012), no. 4, 571–584.

[7] D. Lu and T. Wu, *On bipartite zero-divisor graphs*, Discrete Math. **309** (2009), no. 4, 755–762.

[8] S. Pirzada and R. Raja, *On graphs associated with modules over commutative rings*, J. Korean Math. Soc. **53** (2016), no. 5, 1167–1182.

[9] T. Wu, *On directed zero-divisor graphs of finite rings*, Discrete Math. **296** (2005), no. 1, 73–86.

Sang Cheol Lee
Department of Mathematics Education
and Institute of Pure and Applied Mathematics
Chonbuk National University
Jeonju 54896, Korea
Email address: `scl@jbnu.ac.kr`

Rezvan Varmazyar
Department of Mathematics
Khoy Branch
Islamic Azad University
Khoy 58168-44799, Iran
Email address: `varmazyar@iaukhoy.ac.ir`