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SOME PROPERTIES OF SPECIAL POLYNOMIALS WITH EXPONENTIAL DISTRIBUTION

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Abstract. In this paper, we discuss special polynomials involving exponential distribution, which is related to life testing. We derive some identities of special polynomials such as the symmetric property, recurrence formula and so on. In addition, we investigate explicit properties of special polynomials by using their derivative and integral.

1. Introduction

The exponential distribution is one of the widely used continuous distributions. It is often used to model the time elapsed between events (see [1,2,4–6,8]).

Definition 1.1. For $\lambda > 0$, the probability density function of an exponential distribution is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

where X is a continuous random variable which is said to have an exponential distribution with parameter $\lambda > 0$.

An interesting property of the exponential distribution is that it can be viewed as a continuous analogue of the geometric distribution. The most important property of the exponential distribution is that it is memoryless, so we can state this formally as follows:

$$P(X > a + b \,|\, X > a) = P(X > b), \quad a, b \geq 0.$$

Properties 1.2. For $\lambda > 0$, an exponential distribution has

(i) (Mean) $E(X) = \frac{1}{\lambda}$,
(ii) (Variance) $V(X) = \frac{1}{\lambda^2}$,
(iii) (Moments) $E(X^n) = \frac{n!}{\lambda^n}$ for $n = 1, 2, \ldots$,
(iv) (Median) $m(X) = \frac{\ln(2)}{\lambda} < E(X)$,

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where X is a continuous random variable which is said to have an exponential distribution with parameter $\lambda > 0$.

Since the 1950s, many mathematicians have tried to find the properties of the exponential distribution by using various perspectives and methods (see [1,2,4–6,8]) and they have found some theorems relevant to life testing by using an exponential distribution. Nowadays, those who study life testing concentrate on the prediction of future records (see [3,7,9]).

Since an exponential distribution is very important in the probability theory, we feel that we need to study special polynomials including this distribution in detail. We hypothesized that special polynomials would have some characteristic properties when we combine the probability denseity function which is related to the exponential distribution.

Based on this idea, the main concern of this paper is to define special polynomials and study some of their formulae. Our paper is organized as follows: in Section 2, we define special polynomials including the probability density function which is related to the exponential distribution. From this definition, we investigate some interesting identities of polynomials and derive some relations. In Section 3, we consider some relations of special polynomials by using derivative and integral.

2. Some basic properties of special polynomials including exponential distribution

In this section, we define special polynomials and study some properties of these polynomials that are related to the exponential distribution. Also, we investigate a distinction of special polynomials by using the recurrence formula.

Definition 2.1. For $\lambda > 0$, we define special polynomials

$$\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda : x)\frac{t^n}{n!} = \frac{\lambda}{e^{\lambda t}}e^{tx}.$$

From Definition 2.1, we note that

$$\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda : 0)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda)\frac{t^n}{n!} = \frac{\lambda}{e^{\lambda t}}.$$

From the above equation, we know that $\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda)\frac{t^n}{n!}$ is the probability density function and we note that $E(X) = 1/\lambda$, $V(X) = 1/\lambda^2$.

Theorem 2.2. *Let $\lambda > 0$. Then we have*

(i) $\mathfrak{E}_n(\lambda : x) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k(\lambda)x^{n-k}$,

(ii) $\mathfrak{E}_n(\lambda : x + y) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k(\lambda : x)y^{n-k}$.

Proof. (i) From the generating function of specical polynomials, we find

$$\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda : x)\frac{t^n}{n!} = \frac{\lambda}{e^{\lambda t}} e^{tx} = \sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda)\frac{t^n}{n!} \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}.$$

Using Cauchy's product and a comparison of coefficients, we can complete the proof of Theorem 2.2(i).

(ii) It is a similar method to the proof of (i). □

Theorem 2.3. *Let $\lambda > 0$ and x be any real number. Then we find*

(i) $\mathfrak{E}_n(\lambda : x) = (-1)^{n+1}\mathfrak{E}_n(-\lambda : -x)$,
(ii) $\mathfrak{E}_n(\lambda : x) = 2\mathfrak{E}_n(\frac{\lambda}{2} : -\frac{\lambda}{2} + x)$.

Proof. (i) Putting $t \to -t, \lambda \to -\lambda$, and $x \to -x$, we have

$$\sum_{n=0}^{\infty} \mathfrak{E}_n(-\lambda : -x)\frac{(-t)^n}{n!} = -\frac{\lambda}{e^{\lambda t}} e^{tx} = -\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda : x)\frac{t^n}{n!}.$$

The required relation now follows on comparing the coefficients of $t^n/n!$ on both sides.

(ii) By substituting λ and x with $\lambda/2$ and $-\lambda/2 + x$, respectively, we derive

$$\sum_{n=0}^{\infty} \mathfrak{E}_n(\frac{1}{2}\lambda : -\frac{1}{2}\lambda + x)\frac{t^n}{n!} = \frac{1}{2}\frac{\lambda}{e^{\frac{1}{2}\lambda t}} e^{t(-\frac{1}{2}\lambda + x)} = \frac{1}{2}\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda : x)\frac{t^n}{n!}.$$

The required relation now follows immediately. □

Theorem 2.4. *For $\lambda > 0$, we obtain*

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^{n-k-1}\mathfrak{E}_k(\lambda : x).$$

Proof. Using the generating function for $e^{\lambda t} \neq 0$, we have

$$\sum_{n=0}^{\infty} \mathfrak{E}_n(\lambda : x)\frac{t^n}{n!} \sum_{n=0}^{\infty} \lambda^n \frac{t^n}{n!} = \lambda \sum_{n=0}^{\infty} x^n \frac{t^n}{n!}.$$

To obtain the result, we note that

$$\lambda x^n = \sum_{k=o}^{n} \binom{n}{k} \lambda^{n-k}\mathfrak{E}_k(\lambda : x),$$

and the proof is completed. □

Theorem 2.5. *Let $\lambda > 0$ and x be any real number. Then the following holds*:

$$\sum_{k=0}^{n} \binom{n}{k} (\lambda - x)^k \mathfrak{E}_{n-k}(\lambda : x) = \begin{cases} \lambda & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Proof. From Definition 2.1, we can represent

$$\sum_{n=0}^{\infty}\mathfrak{E}_n(\lambda:x)\frac{t^n}{n!}\sum_{n=0}^{\infty}(\lambda-x)^n\frac{t^n}{n!}=\lambda.$$

Now using Cauchy's product, we obtain the relation

$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}(\lambda-x)^n\mathfrak{E}_{n-k}(\lambda:x)\right)\frac{t^n}{n!}=\lambda.$$

The required relation now follows immediately. □

From Theorem 2.5, we see:

Corollary 2.6.

$$\sum_{k=0}^{n}\binom{n}{k}\lambda^k\mathfrak{E}_{n-k}(\lambda)=\begin{cases}\lambda & \textit{if } n=0,\\ 0 & \textit{if } n\neq 0.\end{cases}$$

Example 2.7. From Theorem 2.5, we can calculate a few polynomials as follows:

$$\begin{aligned}
&\mathfrak{E}_0(\lambda:x)=\lambda,\\
&\mathfrak{E}_1(\lambda:x)=\lambda x-\lambda^2=-\lambda(\lambda-x),\\
&\mathfrak{E}_2(\lambda:x)=\lambda x^2-2\lambda^2x+\lambda^3=\lambda(\lambda-x)^2,\\
&\mathfrak{E}_3(\lambda:x)=\lambda x^3-3\lambda^2x^2+3\lambda^3x-\lambda^4=-\lambda(\lambda-x)^3,\\
&\qquad\qquad\vdots
\end{aligned}$$

Theorem 2.8. *Let a,b be integers. Then we derive*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{b}\right)^{n-2k}\mathfrak{E}_{n-k}\left(\frac{b\lambda}{a}:\frac{bx}{a}\right)\mathfrak{E}_k\left(\frac{a\lambda}{b}:\frac{ay}{b}\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b}{a}\right)^{n-2k}\mathfrak{E}_{n-k}\left(\frac{a\lambda}{b}:\frac{ax}{b}\right)\mathfrak{E}_k\left(\frac{b\lambda}{a}:\frac{by}{a}\right).
\end{aligned}$$

Proof. Consider that

$$A=\frac{\lambda^2e^{t(x+y)}}{e^{2\lambda t}}.$$

The form A can be turned to

$$\begin{aligned}
A&=\frac{\lambda}{e^{\lambda t}}e^{tx}\frac{\lambda}{e^{\lambda t}}e^{ty}\\
&=\sum_{n=0}^{\infty}\left(\frac{a}{b}\right)^n\mathfrak{E}_n\left(\frac{b\lambda}{a}:\frac{bx}{a}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^n\mathfrak{E}_n\left(\frac{a\lambda}{b}:\frac{ay}{b}\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{b}\right)^{n-2k}\mathfrak{E}_{n-k}\left(\frac{b\lambda}{a}:\frac{bx}{y}\right)\mathfrak{E}_k\left(\frac{a\lambda}{b}:\frac{ay}{b}\right)\right)\frac{t^n}{n!},
\end{aligned}$$

or equivalently,

$$A = \sum_{n=0}^{\infty} \left(\frac{b}{a}\right)^n \mathfrak{E}_n(\frac{a\lambda}{b} : \frac{ax}{b}) \frac{t^n}{n!} \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n \mathfrak{E}_n(\frac{b\lambda}{a} : \frac{by}{a}) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{a}\right)^{n-2k} \mathfrak{E}_{n-k}(\frac{a\lambda}{b} : \frac{ax}{b}) \mathfrak{E}_k(\frac{b\lambda}{a} : \frac{by}{a}) \right) \frac{t^n}{n!}.$$

The required relation now follows by comparing of the above equations. □

If $a = 1$ in Theorem 2.8, then we see:

Corollary 2.9.

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{b}\right)^{n-2k} \mathfrak{E}_{n-k}(b\lambda : bx) \mathfrak{E}_k(\frac{\lambda}{b} : \frac{y}{b})$$
$$= \sum_{k=0}^{n} \binom{n}{k} b^{n-2k} \mathfrak{E}_{n-k}(\frac{\lambda}{b} : \frac{x}{b}) \mathfrak{E}_k(b\lambda : by).$$

Theorem 2.10. *Let $\lambda > 0$. Then we have*

$$D\mathfrak{E}_n(\lambda : x) = n\mathfrak{E}_{n-1}(\lambda : x).$$

Proof. From Theorem 2.2, we find

$$D\mathfrak{E}_n(\lambda : x) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k(\lambda) \frac{d}{dx} x^{n-k}$$
$$= n \sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{E}_k(\lambda) x^{n-k-1}$$
$$= n\mathfrak{E}_{n-1}(\lambda : x).$$

Therefore we know that the above proof of Theorem 2.9 is clear. □

3. Some identities of special polynomials using derivative and integral

In this section, we use addition theorem to obtain some identities for special polynomials and use derivative and integral. Using two parameters, we investigate some interesting properties of special polynomials.

Theorem 3.1. *Let x, y be real numbers. Then the following holds*:

$$D_y \mathfrak{E}_n(\lambda : x + y) = n\mathfrak{E}_{n-1}(\lambda : x + y).$$

Proof. In Theorem 2.2(ii), we apply the derivative definition as

$$\begin{aligned} D_y\mathfrak{E}_n(\lambda : x+y) &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(\lambda : x)\frac{d}{dy}y^{n-k} \\ &= n\sum_{k=0}^{n-1}\frac{(n-1)!}{(n-1-k)!k!}\mathfrak{E}_k(\lambda : x)y^{n-k-1} \\ &= n\sum_{k=0}^{n-1}\binom{n-1}{k}\mathfrak{E}_k(\lambda : x)y^{n-k-1}. \end{aligned}$$

Comparing Theorem 2.2(ii) again, the required relation follows. □

Theorem 3.2. *Let x, y and λ be real numbers with $\lambda > 0$. Then we have*

$$\int_0^1 \mathfrak{E}_n(\lambda : x+y)dy = \frac{\mathfrak{E}_{n+1}(\lambda : 1+x) - \mathfrak{E}_{n+1}(\lambda : x)}{n+1}.$$

Proof. Using the integral from Theorem 2.2(ii), we get

$$\begin{aligned} \int_0^1 \mathfrak{E}_n(\lambda : x+y)dy &= \int_0^1\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(\lambda : x)y^{n-k}dy \\ &= \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(\lambda : x)\frac{1}{n-k+1}y^{n-k+1}\Big|_0^1 \\ &= \frac{1}{n+1}\sum_{k=o}^{n+1}\binom{n+1}{k}\mathfrak{E}_k(\lambda : x)y^{n+1-k}\Big|_0^1 \\ &= \frac{1}{n+1}\left(\mathfrak{E}_{n+1}(\lambda : x+1) - \mathfrak{E}_{n+1}(\lambda : x)\right). \end{aligned}$$

Therefore the required relation follows immediately. □

From Theorem 2.2(i) and Theorem 3.2, we see:

Corollary 3.3.

$$\int_0^1 \mathfrak{E}_n(\lambda : x)dx = \frac{\mathfrak{E}_{n+1}(\lambda : 1) - \mathfrak{E}_{n+1}(\lambda)}{n+1}.$$

Theorem 3.4. *For $\lambda, x, y > 0$, we obtain*

$$\mathfrak{E}_n(\lambda : x+y) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^{l-1}\mathfrak{E}_l(-\lambda)x^{k-l}y^{n-k}.$$

Proof. Replacing λ, t with $-\lambda, -t$, respectively, we get

$$\begin{aligned} &\sum_{n=0}^{\infty}\mathfrak{E}_n(-\lambda : x+y)\frac{(-t)^n}{n!} \\ &= \frac{-\lambda}{e^{\lambda t}}e^{-t(x+y)} \end{aligned}$$

$$= -\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathfrak{E}_l(\lambda)(-1)^{n-l}x^{n-l}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}(-y)^n\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^{n-l-1}\mathfrak{E}_l(\lambda)x^{k-l}y^{n-k}\right)\frac{t^n}{n!}.$$

From the above equation, the following holds

$$(-1)^n\mathfrak{E}_n(\lambda : x+y) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^{n-l-1}\mathfrak{E}_l(-\lambda)x^{k-l}y^{n-k}.$$

Therefore we complete the proof of Theorem 3.4. □

From Theorem 2.2(ii) and Theorem 3.4, we find:

Corollary 3.5.

$$\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k(\lambda : x)y^{n-k} = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^{l-1}\mathfrak{E}_l(-\lambda)x^{k-l}y^{n-k}.$$

From Theorem 3.4, we have:

Corollary 3.6.

$$\mathfrak{E}_n(\lambda : x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{k+1}\mathfrak{E}_k(-\lambda)x^{n-k}.$$

Theorem 3.7. *For $\lambda > 0$, we derive*

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\mathfrak{E}_k(\lambda : x) = \sum_{k=0}^{n-1}\sum_{l=0}^{k}\binom{n-1}{k}\binom{k}{l}(-1)^{l-1}\mathfrak{E}_l(-\lambda)x^{k-l}.$$

Proof. In Theorem 3.4, we can apply the derivative definition as

$$D_y\mathfrak{E}_n(\lambda : x+y) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}(-1)^{l-1}\mathfrak{E}_l(-\lambda)x^{k-l}D_y y^{n-k}$$
$$= \sum_{k=0}^{n-1}\sum_{l=0}^{k}\frac{n!(n-k)}{(n-k)!k!}\binom{k}{l}(-1)^{l-1}\mathfrak{E}_l(-\lambda)x^{k-l}y^{n-k-1}$$
$$= n\sum_{k=0}^{n-1}\sum_{l=0}^{k}\binom{n-1}{k}\binom{k}{l}(-1)^{l-1}\mathfrak{E}_l(-\lambda)x^{k-l}y^{n-k-1},$$

by comparing with Theorem 3.1 and Theorem 3.7 immediately gives the required relation. □

Furthermore, we can find the following corollary 3.8 by applying the processor of Theorem 3.4 to Corollary 3.6 and comparing the result with Theorem 2.9.

Corollary 3.8. *From Theorem 3.7, we have*

$$\sum_{k=0}^{n-1}\binom{n-1}{k}\mathfrak{E}_k(\lambda)=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{k+1}\mathfrak{E}_k(-\lambda).$$

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