

## ARCHIMEDEAN SKEW GENERALIZED POWER SERIES RINGS

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**ABSTRACT.** Let  $R$  be a ring,  $(S, \leq)$  a strictly ordered monoid, and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism. In [18], Mazurek, and Ziembowski investigated when the skew generalized power series ring  $R[[S, \omega]]$  is a domain satisfying the ascending chain condition on principal left (resp. right) ideals. Following [18], we obtain necessary and sufficient conditions on  $R$ ,  $S$  and  $\omega$  such that the skew generalized power series ring  $R[[S, \omega]]$  is a right or left Archimedean domain. As particular cases of our general results we obtain new theorems on the ring of arithmetical functions and the ring of generalized power series. Our results extend and unify many existing results.

### 1. Introduction

Throughout this paper all monoids and rings are with identity element that is inherited by submonoids and subrings and preserved under homomorphisms, but neither monoids nor rings are assumed to be commutative.

A commutative ring  $R$  is said to satisfy the *ascending chain condition for principal ideals* (ACCP), if there does not exist an infinite strictly ascending chain of principal ideals of  $R$  (see, for example, Dumitrescu et al. [4] or Frohn, [7]). The ACCP is also called 1-ACC in Frohn [6]. In Anderson et al. [1] and Dumitrescu et al. ([4, Proposition 1.2]), the authors gave a necessary and sufficient condition under which the rings  $A + XB[[X]]$  and  $A + XB[X]$  satisfy ACCP where  $A \subseteq B$  are domains and  $X$  is an indeterminate.

A ring  $R$  is said to satisfy the *ascending chain condition on principal left ideals* (ACCPL) if there does not exist an infinite strictly ascending chain of principal left ideals of  $R$ . Rings satisfying the ascending chain condition on principal right ideals (ACCPR) are defined analogously. Obvious examples of rings satisfying ACCPL are left Noetherian rings. Also every left perfect ring satisfies ACCPL, since by a celebrated theorem of Bass (see [2]) the left perfect

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condition is equivalent to the descending chain condition on principal right ideals, which in turn implies ACCPL, by Jonah's theorem from [8].

A ring  $R$  is said to be *right Archimedean* if  $\bigcap_{n \in \mathbb{N}} Rr^n = 0$  for each nonunit element  $r \in R$ . Left Archimedean rings are defined similarly. Commutative Archimedean domains are studied in [30], where the author proves that if  $R$  is a commutative domain,  $a \in R$  and  $\bigcap_{t \in \mathbb{N}} a^t R = 0$ , then the quotient field  $Q(R[[x/a]])$  has infinite transcendence degree over the quotient field  $Q(R[[x]])$ . It is well-known that any commutative domain satisfying ACCP is Archimedean, but the converse is not true (see, for example, Dumitrescu et al. [4, p. 1127]).

In [17] a common generalization of the skew power series ring and the Mal'cev-Neumann ring constructions was introduced, the ring of skew generalized power series  $R[[S, \omega]]$ , where  $R$  is a ring,  $S$  is a strictly ordered monoid,  $\omega : S \rightarrow \text{End}(R)$  is a monoid homomorphism, and the pointwise addition and the skew convolution multiplication are performed on the set of all functions from  $S$  to  $R$  whose support is artinian and narrow (see Section 2 for particularities). Special cases of the construction are polynomial rings, monoid rings, skew polynomial rings, skew Laurent polynomial rings, skew monoid rings, skew power series rings, skew Laurent series rings, the Mal'cev-Neumann construction (see [3, p. 528]), the Mal'cev-Neumann construction of twisted Laurent series rings (see [9, p. 242]), generalized power series rings (see [29, Section 4]), and twisted generalized power series rings (see [12, Section 2]). Hence any result on skew generalized power series rings has its counterpart for each of these particular ring extensions, and these counterparts follow immediately from a single proof. This property makes skew generalized power series rings a useful tool for unifying results on the ring extensions listed above; such an approach was applied, e.g., in [14–19, 21–26].

In [18], Mazurek, and Ziembowski studied when the skew generalized power series ring  $R[[S, \omega]]$  satisfies the ascending chain condition on principal left (resp. right) ideals. Also, Nasr-Isfahani in [20], obtained characterizations of skew polynomial rings and skew power series rings that are ACCPL-domains or Archimedean domains. Motivated by results in [18] and [20], we obtain necessary and sufficient conditions on  $R$ ,  $S$  and  $\omega$  such that the skew generalized power series ring  $R[[S, \omega]]$  is a right (resp. left) Archimedean domain.

The paper is organized as follows. In Section 2 we recall the construction of a skew generalized power series ring  $R[[S, \omega]]$  and show how the aforementioned ring extensions can be obtained as special cases of the construction. In Section 3 we study what can be said about a ring  $R$ , a strictly ordered monoid  $(S, \leq)$  and a monoid homomorphism  $\omega : S \rightarrow \text{End}(R)$  such that the skew generalized power series ring  $R[[S, \omega]]$  is a right or left Archimedean domain. In Section 4 we prove that if the monoid  $S$  is strictly artinian totally ordered or  $S$  is commutative torsion-free cancellative artinian semisubtotally ordered, then the ring  $R[[S, \omega]]$  of skew generalized power series with coefficients in  $R$  and exponents in  $S$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain and each  $\omega(s)$  is injective and preserves nonunits of  $R$  for any  $s \in S$  (see Theorems

4.1 and 4.7). In particular, we obtain characterizations of power series rings, skew power series rings, the ring of arithmetical functions and generalized power series rings, that are right or left Archimedean domains (see Corollaries 4.4, 4.5 and 4.9). Finally, we pose natural open problems (Questions 4.2 and 4.8) on right (resp. left) Archimedean skew generalized power series domains.

We will denote by  $\text{End}(R)$  the monoid of ring endomorphisms of  $R$ , and by  $\text{Aut}(R)$  the group of ring automorphisms of  $R$ . If  $S$  is a monoid or a ring, then the group of invertible elements of  $S$  is denoted by  $U(S)$ . When we consider an ordering relation  $\leq$  on a set  $S$ , then the word “order” means a partial ordering unless otherwise stated. The order  $\leq$  is *total* (respectively *trivial*) if any two different elements of  $S$  are comparable (respectively incomparable) with respect to  $\leq$ . We will use the symbol 1 to denote the identity elements of the monoid  $S$ , the ring  $R$ , and the ring  $R[[S, \omega]]$ , as well as the trivial monoid homomorphism  $1 : S \rightarrow \text{End}(R)$  that sends every element of  $S$  to the identity endomorphism. Also we use  $\mathbb{Z}$ ,  $\mathbb{N}$ ,  $\mathbb{Q}$  and  $\mathbb{R}$  for the integers, positive integers, rational numbers and the field of real numbers, respectively.

## 2. Preliminaries

A partially ordered set  $(S, \leq)$  is called *artinian* if every strictly decreasing sequence of elements of  $S$  is finite, and  $(S, \leq)$  is called *narrow* if every subset of pairwise order-incomparable elements of  $S$  is finite. Thus,  $(S, \leq)$  is artinian and narrow if and only if every nonempty subset of  $S$  has at least one but only a finite number of minimal elements. An *ordered monoid* is a pair  $(S, \leq)$  consisting of a monoid  $S$  and an order  $\leq$  on  $S$  such that for all  $a, b, c \in S$ ,  $a \leq b$  implies  $ca \leq cb$  and  $ac \leq bc$ . An ordered monoid  $(S, \leq)$  is said to be *strictly ordered* if for all  $a, b, c \in S$ ,  $a < b$  implies  $ca < cb$  and  $ac < bc$ .

For a strictly ordered monoid  $S$  and a ring  $R$ , Ribenboim [29] defined the ring of generalized power series  $R[[S]]$  consisting of all maps from  $S$  to  $R$  whose support is artinian and narrow with the pointwise addition and the convolution multiplication. This construction provided interesting examples of rings (e.g., Elliott and Ribenboim [5]; Ribenboim [27, 28]) and it was extensively studied by many authors.

In [17], Mazurek and Ziemkowski, introduced a “twisted” version of the Ribenboim construction and studied when it produces a von Neumann regular ring. Now we recall the construction of the skew generalized power series ring introduced in [17]. Let  $R$  be a ring,  $(S, \leq)$  a strictly ordered monoid, and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism. For  $s \in S$ , let  $\omega_s$  denote the image of  $s$  under  $\omega$ , that is  $\omega_s = \omega(s)$ . Let  $A$  be the set of all functions  $f : S \rightarrow R$  such that the support  $\text{supp}(f) = \{s \in S : f(s) \neq 0\}$  is artinian and narrow. Then for any  $s \in S$  and  $f, g \in A$  the set

$$X_s(f, g) = \{(x, y) \in \text{supp}(f) \times \text{supp}(g) : s = xy\}$$

is finite. Thus one can define the product  $fg : S \rightarrow R$  of  $f, g \in A$  as follows:

$$fg(s) = \sum_{(u,v) \in X_s(f,g)} f(u)\omega_u(g(v))$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above,  $A$  becomes a ring, called *the ring of skew generalized power series* with coefficients in  $R$  and exponents in  $S$  (one can think of a map  $f : S \rightarrow R$  as a formal series  $\sum_{s \in S} r_s s$ , where  $r_s = f(s) \in R$ ) and denoted either by  $R[[S^{\leq}, \omega]]$ , or by  $R[[S, \omega, \leq]]$ , or by  $R[[S, \omega]]$  (see [14] and [17]).

To each  $r \in R$  and  $s \in S$ , we associate elements  $c_r, \mathbf{e}_s \in R[[S, \omega]]$  defined by

$$c_r(x) = \begin{cases} r & x = 1, \\ 0 & x \in S \setminus \{1\}, \end{cases} \quad \mathbf{e}_s(x) = \begin{cases} 1 & x = s, \\ 0 & x \in S \setminus \{s\}. \end{cases}$$

It is clear that  $r \mapsto c_r$  is a ring embedding of  $R$  into  $R[[S, \omega]]$  and  $s \mapsto \mathbf{e}_s$  is a monoid embedding of  $S$  into the multiplicative monoid of the ring  $R[[S, \omega]]$ , and  $\mathbf{e}_s c_r = c_{\omega_s(r)} \mathbf{e}_s$ .

Below we quote from [15], how the classical constructions mentioned in Section 1 can be viewed as special cases of the skew generalized power series ring construction.

Let  $R$  be a ring and  $\sigma$  an endomorphism of  $R$ . Then for the additive monoid  $S = \mathbb{N} \cup \{0\}$  of nonnegative integers, the map  $\omega : S \rightarrow \text{End}(R)$  given by

$$(2.1) \quad \omega(n) = \sigma^n \quad \text{for any } n \in S,$$

is a monoid homomorphism. If furthermore  $\sigma$  is an automorphism of  $R$ , then (2.1) defines also a monoid homomorphism  $\omega : S \rightarrow \text{Aut}(R)$  for  $S = \mathbb{Z}$ , the additive monoid of integers. We can consider two different orders on each of the monoids  $\mathbb{N} \cup \{0\}$  and  $\mathbb{Z}$ : the trivial order and the natural linear order. In both cases these monoids are strictly ordered, and thus in each of the cases we can construct the skew generalized power series ring  $R[[S, \omega]]$ . As a result, we obtain the following extensions of the ring  $R$ :

- (1) If  $S = \mathbb{N} \cup \{0\}$  and  $\leq$  is the trivial order, then the ring  $R[[S, \omega]]$  is isomorphic to the skew polynomial ring  $R[x, \sigma]$ .
- (2) If  $S = \mathbb{N} \cup \{0\}$  and  $\leq$  is the natural linear order, then  $R[[S, \omega]]$  is isomorphic to the skew power series ring  $R[[x; \sigma]]$ .
- (3) If  $S = \mathbb{Z}$  and  $\leq$  is the trivial order, and  $\sigma$  is an automorphism of  $R$ , then  $R[[S, \omega]]$  is isomorphic to the skew Laurent polynomial ring  $R[x, x^{-1}; \sigma]$ .
- (4) If  $S = \mathbb{Z}$  and  $\leq$  is the natural linear order, and  $\sigma$  is an automorphism of  $R$ , then  $R[[S, \omega]]$  is isomorphic to the skew Laurent series ring  $R[[x, x^{-1}; \sigma]]$ .

By applying the above points (1)-(4) to the case where  $\sigma$  is the identity map of  $R$ , we can see that also the following ring extensions are special cases of the skew generalized power series ring construction: the ring of polynomials  $R[x]$ ,

the ring of power series  $R[[x]]$ , the ring of Laurent polynomials  $R[x, x^{-1}]$ , and the ring of Laurent series  $R[[x, x^{-1}]]$ .

Furthermore, any monoid  $S$  is a strictly ordered monoid with respect to the trivial order on  $S$ . Hence if  $R$  is a ring,  $S$  is a monoid and  $\omega : S \rightarrow \text{End}(R)$  is a monoid homomorphism, then we can impose the trivial order on  $S$  and construct the skew generalized power series ring  $R[[S, \omega]]$ , which in this case will be denoted by  $R[S, \omega]$ . It is clear that the ring  $R[S, \omega]$  is isomorphic to the classical skew monoid ring built from  $R$  and  $S$  using the action  $\omega$  of  $S$  on  $R$ . If  $\omega$  is trivial, we write  $R[S]$  instead of  $R[S, \omega]$ . Obviously the ring  $R[S]$  is isomorphic to the ordinary monoid ring of  $S$  over  $R$ .

Also, the construction of skew generalized power series rings generalizes another classical ring constructions such as the Mal'cev-Neumann Laurent series rings  $((S, \leq)$  a totally ordered group and trivial  $\omega$ ; see [3, p. 528]), the Mal'cev-Neumann construction of twisted Laurent series rings  $((S, \leq)$  a totally ordered group; see [9, p. 242]), generalized power series rings  $R[[S]]$  (trivial  $\omega$ ; see [29, Section 4]), and twisted generalized power series rings (see [12, Section 2] and [17]).

We now recall some facts about units of skew generalized power series rings, which will be used later on in this paper.

Recall from [29] that an order  $\leq$  on a monoid  $S$  is said to be *subtotal* if for any  $s, t \in S$  there exists  $n \in \mathbb{N}$  such that  $s^n \leq t^n$  or  $t^n \leq s^n$ . A total order on a monoid is clearly subtotal, but the converse need not be true (see e.g. [18, Example 3.8] or [29, p. 371]).

From [18], an order  $\leq$  on a monoid  $S$  is said to be *semisubtotal* if for any  $s \in S$  there exists  $n \in \mathbb{N}$  such that  $s^n \geq 1$  or  $s^n \leq 1$ . If  $(S, \cdot, \leq)$  is an ordered monoid and  $\leq$  is semisubtotal, then we will say that  $(S, \cdot, \leq)$  is *semisubtotally ordered*. An ordered monoid  $(S, \leq)$  is called *positively ordered* if  $1 \leq s$  for any  $s \in S$ . It is also clear that any positively ordered monoid is semisubtotally ordered. If  $(S, \leq)$  is an ordered abelian group, then the order  $\leq$  is semisubtotal if and only if it is subtotal.

Recall that a monoid  $S$  is said to be *torsion-free* if for any  $n \in \mathbb{N}$  and  $s, t \in S$ ,  $s^n = t^n$  implies  $s = t$ . It is easy to see that if  $(S, \leq)$  is an ordered torsion-free commutative monoid such that  $\leq$  is subtotal, then the binary relation  $\preceq$  on  $S$  defined by

$$s \preceq t \text{ if and only if } s^n \leq t^n \text{ for some } n \in \mathbb{N}$$

is a total order on  $S$  and  $(S, \preceq)$  is an ordered monoid. The order  $\preceq$  will be called the *total order associated with  $\leq$* . Clearly,  $s \leq t$  implies  $s \preceq t$  for any  $s, t \in S$ , and thus by [18, Proposition 1.1], if a subset  $T$  of  $S$  is artinian and narrow with respect to  $\leq$ , then  $T$  is well-ordered with respect to  $\preceq$ . Hence for any  $f \in R[[S, \omega]] \setminus \{0\}$  there exists a smallest element  $s_0$  of  $\text{supp}(f)$  with respect to  $\preceq$ , which will be denoted by  $\pi(f)$ .

To characterize skew generalized power series rings that are Archimedean domains, we will need the following results, which play a key role in the sequel.

**Proposition 2.1** ([17, Proposition 2.2]). *Let  $R$  be a ring,  $(S, \leq)$  a strictly ordered monoid,  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism and  $A = R[[S, \omega]]$ . Let  $f \in A$  and assume that there exists a smallest element  $s_0$  in  $\text{supp}(f)$ . If  $s_0 \in U(S)$  and  $f(s_0) \in U(R)$ , then  $f \in U(A)$ .*

**Proposition 2.2** ([18, Proposition 3.12]). *Let  $R$  be a domain,  $(S, \cdot, \leq)$  a commutative torsion-free cancellative semisubtotally ordered monoid,  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism and  $A = R[[S, \omega, \leq]]$ . Assume that  $\omega_s$  is injective for any  $s \in S$ . Let  $\preceq$  be a total order associated with  $\leq$ , and let  $B = R[[S, \omega, \preceq]]$ . Then  $A$  is a subring of  $B$  and  $U(A) = A \cap U(B)$ .*

### 3. Necessary conditions for the ring $R[[S, \omega]]$ to be a right or left Archimedean domain

In this section we study what can be said about a ring  $R$ , a strictly ordered monoid  $(S, \leq)$  and a monoid homomorphism  $\omega : S \rightarrow \text{End}(R)$  such that the skew generalized power series ring  $R[[S, \omega]]$  is a right or left Archimedean domain. A monoid  $(S, \cdot)$  is said to satisfy the *ascending chain condition on principal left ideals* (ACCPL) if there does not exist an infinite strictly ascending chain of principal left ideals of  $S$ . Analogously monoids satisfying the ascending chain condition on principal right ideals (ACCPR) are defined. Monoids that satisfy ACCPL (resp. ACCPR) will be called *ACCPL-monoids* (resp. *ACCPR-monoids*). In [18, Example 2.6], the authors showed that ACCPL and ACCPR are independent conditions.

**Proposition 3.1** ([18, Proposition 2.1]). *For any cancellative monoid  $S$ , the following are equivalent:*

- (1)  $S$  satisfies ACCPL.
- (2) For any sequences  $(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$  of elements of  $S$  such that  $a_n = b_n a_{n+1}$  for all  $n \in \mathbb{N}$ , there exists  $m \in \mathbb{N}$  with  $b_n \in U(S)$  for all  $n \geq m$ .
- (3) For any sequences  $(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$  of elements of  $S$  such that  $a_n = b_n a_{n+1}$  for all  $n \in \mathbb{N}$ , there exists  $m \in \mathbb{N}$  with  $b_m \in U(S)$ .
- (4)  $\bigcap_{n \in \mathbb{N}} s_1 s_2 \cdots s_n S = \emptyset$  for any sequence  $(s_n)_{n \in \mathbb{N}}$  of nonunits of  $S$ .

A monoid  $(S, \cdot)$  is said to be a *left Archimedean monoid* if  $\bigcap_{n \in \mathbb{N}} s^n S = \emptyset$  for each nonunit element  $s$  of  $S$ . Right Archimedean monoids may be defined analogously. Obviously, any group, the multiplicative monoid  $\mathbb{N}$  and the additive monoid  $\mathbb{N} \cup \{0\}$  are Archimedean.

The following result is an immediate consequence of Proposition 3.1.

**Proposition 3.2.** *Let  $S$  be a monoid satisfying ACCPL. If  $S$  is cancellative, then  $S$  is left Archimedean.*

The following example shows that in Proposition 3.2 the cancellative condition on monoid  $S$  is not superfluous. This example shows also that a finite monoid need not be left Archimedean.

**Example 3.3.** Let  $S$  be a monoid generated by  $x$  with the defining relation  $x^2 = x$ . Then  $S = \{1, x\}$ ,  $x \notin U(S)$  and  $x \in \bigcap_{n \in \mathbb{N}} x^n S$ . Obviously, the monoid  $S$  satisfies ACCPL, but it is not left Archimedean.

*Remark 3.4.* [11, Example 2.5] shows that the converse of Proposition 3.2 is not true in general.

**Proposition 3.5.** *Let  $T$  be a submonoid of a monoid  $S$  such that  $U(T) = T \cap U(S)$ . If  $S$  is a left Archimedean monoid, then  $T$  is a left Archimedean monoid.*

*Proof.* The proof is clear.  $\square$

**Proposition 3.6.** *Let  $\{S_i\}_{i \in I}$  be a family of monoids indexed by a nonempty set  $I$ . Then the cartesian product  $S = \prod_{i \in I} S_i$  of the monoids  $S_i$  is left Archimedean if and only if each monoid  $S_i$  is left Archimedean.*

*Proof.* Suppose that every  $S_i$  is a left Archimedean monoid and  $t = (t_i)_{i \in I}$  is in  $\bigcap_{n \in \mathbb{N}} s^n S$ , where  $s$  is a nonunit element of  $S$ . Then  $t = s^n v_n$  for some  $v_n \in S$  and for all  $n \in \mathbb{N}$ . Write  $s = (s_i)_{i \in I}$  and  $v_n = (v_{in})_{i \in I}$ . Then  $t_i = s_i^n v_{in}$  for all  $i \in I$  and  $n \in \mathbb{N}$ . Thus  $t_i \in \bigcap_{n \in \mathbb{N}} s_i^n S_i$  for all  $i \in I$ . Since  $s \notin U(S)$ , it follows that there exists  $j \in I$  such that  $s_j \notin U(S_j)$ . Therefore  $\bigcap_{n \in \mathbb{N}} s_j^n S_j = \emptyset$ , since  $S_j$  is a left Archimedean monoid. This is a contradiction. Thus  $\bigcap_{n \in \mathbb{N}} s^n S = \emptyset$ . Hence  $S$  is a left Archimedean monoid. Conversely, assume that  $S$  is a left Archimedean monoid. Then by Proposition 3.5, every  $S_i$  is a left Archimedean monoid.  $\square$

Applying Proposition 3.6 to a family  $S_1, S_2, \dots$  of commutative monoids, we obtain [11, Lemma 2.6].

**Corollary 3.7.** *Let  $S_1, S_2, \dots, S_n$  be monoids. Then  $S_1 \times S_2 \times \dots \times S_n$  is a left Archimedean monoid if and only if every  $S_i$  is a left Archimedean monoid.*

We will say that an endomorphism  $\alpha$  of a ring  $R$  preserves nonunit elements of  $R$  if  $\alpha(R \setminus U(R)) \subseteq R \setminus U(R)$ .

**Theorem 3.8.** *Let  $R$  be a ring,  $(S, \leq)$  a strictly ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism. If  $R[[S, \omega]]$  is a right Archimedean domain, then:*

- (i)  $R$  is a right Archimedean domain.
- (ii)  $S$  is a right Archimedean monoid.
- (iii)  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ .

*Proof.* (i) Since  $R$  is a subring of  $R[[S, \omega]]$ ,  $R$  is a domain. Now, let  $a$  be a nonunit element of  $R$  and  $b \in \bigcap_{n \in \mathbb{N}} Ra^n$ . Then  $c_b \in \bigcap_{n \in \mathbb{N}} R[[S, \omega]](c_a)^n$  and it is clear that  $c_a$  is a nonunit element of  $R[[S, \omega]]$ . This implies that  $c_b = 0$ . Thus  $b = 0$ . Therefore  $R$  is a right Archimedean domain.

(ii) Let  $s \in S \setminus U(S)$  and suppose  $t \in \bigcap_{n \in \mathbb{N}} Ss^n$ . Then

$$\mathbf{e}_t \in \bigcap_{n \in \mathbb{N}} R[[S, \omega]](\mathbf{e}_s)^n$$

and it is obvious that  $\mathbf{e}_s$  is a nonunit element of  $R[[S, \omega]]$ . This implies that  $\mathbf{e}_t = 0$ . Hence  $0 = \mathbf{e}_t(t) = 1$ , and this contradiction completes the proof of (ii).

(iii) Let  $s \in S$ . To prove that  $\omega_s$  is injective, we repeat the argument of the first part of the proof of [13, Proposition 3.2]. Suppose that  $\omega_s$  is not injective. Choose  $a \in R \setminus \{0\}$  such that  $\omega_s(a) = 0$ . Then for the nonzero elements  $\mathbf{e}_s, c_a$  of the domain  $R[[S, \omega]]$  we obtain  $\mathbf{e}_s c_a = c_{\omega_s(a)} \mathbf{e}_s = 0$ , a contradiction. Now, suppose that  $\omega_{s_0}(r) \in U(R)$  for some  $s_0 \in S$  and a nonunit element  $r$  of  $R$ . For all  $n \in \mathbb{N}$  define  $f_n \in R[[S, \omega]]$  by  $f_n = c_{(\omega_{s_0}(r))^{-n}} \mathbf{e}_{s_0}$ . Clearly  $\mathbf{e}_{s_0} = f_n(c_r)^n$  for any  $n \in \mathbb{N}$ . Consequently  $\mathbf{e}_{s_0} \in \bigcap_{n \in \mathbb{N}} R[[S, \omega]](c_r)^n$ . This contradiction completes the proof of part (iii).  $\square$

**Theorem 3.9.** *Let  $R$  be a ring,  $(S, \leq)$  a strictly ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism. If  $R[[S, \omega]]$  is a left Archimedean domain, then:*

- (1)  $R$  is a left Archimedean domain.
- (2)  $S$  is a left Archimedean monoid.
- (3)  $\omega_s$  is injective for any  $s \in S$ .

*Proof.* The proof is similar to the proof of Theorem 3.8.  $\square$

#### 4. Archimedean domains of skew generalized power series

In this section we study when the skew generalized power series ring  $R[[S, \omega]]$  is a right (resp. left) Archimedean domain. We first consider the case when  $S$  is artinian and the order  $\leq$  is total (Theorem 4.1), and next the case when  $S$  is commutative torsion-free cancellative artinian and the order  $\leq$  is semisubtotal (Theorem 4.7).

**Theorem 4.1.** *Let  $R$  be a ring,  $(S, \leq)$  an artinian strictly totally ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism.*

- (i) *The ring  $R[[S, \omega]]$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ .*
- (ii)  *$R[[S, \omega]]$  is a left Archimedean domain if and only if  $R$  is a left Archimedean domain and  $\omega_s$  is injective for any  $s \in S$ .*

*Proof.* (i) Set  $A = R[[S, \omega]]$ . Assume that  $R$  is a right Archimedean domain,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ . It is clear that  $A$  is a domain, by [18, Proposition 3.1(ii)]. Assume to the contrary that  $f$  is a nonunit element of  $A$  for which there is a nonzero element  $g$  in  $\bigcap_{n \in \mathbb{N}} A f^n$ . Then for each  $n \in \mathbb{N}$  there exists  $h_n \in A$  such that  $g = h_n f^n$ . Using [18, Proposition 3.1(i)], we get  $\pi(g) = \pi(h_n) \pi(f^n)$  and  $\pi(f^n) = (\pi(f))^n$  for any



$n \in \mathbb{N}$ , since  $R$  is a domain and  $\omega_s$  is injective for any  $s \in S$ . So  $g(\pi(g)) = h_n(\pi(h_n))\omega_{\pi(h_n)}(f^n(\pi(f^n)))$ . There are three cases.

Case 1. First, let  $\pi(f) = 1$ . Hence by using again [18, Proposition 3.1(i)],  $\pi(g) = \pi(h_n)$ ,  $g(\pi(g)) = h_n(\pi(g))\omega_{\pi(g)}(f^n(1))$  and  $f^n(1) = (f(1))^n$  for each  $n \in \mathbb{N}$ . Therefore, we obtain:

$$g(\pi(g)) = h_n(\pi(g))(\omega_{\pi(g)}(f(1)))^n \in R(\omega_{\pi(g)}(f(1)))^n \quad \text{for any } n \in \mathbb{N}.$$

This yields that  $g(\pi(g)) \in \bigcap_{n \in \mathbb{N}} R(\omega_{\pi(g)}(f(1)))^n$ . Also  $f(1)$  is not a unit, otherwise 1 and  $f(1)$  would be both units and  $f$  will be a unit of  $A$ , by Proposition 2.1. Since  $\omega_{\pi(g)}$  preserves nonunit elements of  $R$ ,  $\omega_{\pi(g)}(f(1))$  is not a unit of  $R$ . Since  $R$  is right Archimedean, it follows that  $g(\pi(g)) = 0$ , which contradicts the fact that  $\pi(g) \in \text{supp}(g)$ .

Case 2.  $\pi(f) > 1$ . We know that  $\pi(g) = \pi(h_n)(\pi(f))^n$  for any  $n \in \mathbb{N}$ . So  $\pi(h_n) < \pi(h_{n-1})$  for each  $n$ . Thus  $\{\pi(h_n)\}_{n \in \mathbb{N}}$  forms an infinite strictly descending chain of elements of  $S$ , which is a contradiction.

Case 3. Now, assume that  $\pi(f) < 1$ . Hence  $(\pi(f))^n < (\pi(f))^{n-1}$  for any  $n$ . Thus  $\{(\pi(f))^n\}_{n \in \mathbb{N}}$  forms an infinite strictly descending chain of elements of  $S$ . This is also a contradiction. Consequently  $g = 0$  and the result follows.

Conversely, if  $A$  is a right Archimedean domain, then  $R$  is a right Archimedean domain,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$  by Theorem 3.8.

(ii) The proof is similar to the proof of part (i). □

From the preceding results, it is natural to raise the following question.

**Question 4.2.** *Let  $R$  be a ring,  $(S, \leq)$  a strictly totally ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism.*

- (i) *Is it true that the ring  $R[[S, \omega]]$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain,  $S$  is a right Archimedean monoid,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ ?*
- (ii) *Is it true that the ring  $R[[S, \omega]]$  is a left Archimedean domain if and only if  $R$  is a left Archimedean domain,  $S$  is a left Archimedean monoid and  $\omega_s$  is injective for any  $s \in S$ ?*

Any submonoid of the additive monoid  $\mathbb{N} \cup \{0\}$  is called a *numerical monoid*. We have:

**Corollary 4.3.** *Let  $S$  be a numerical monoid,  $\leq$  the usual natural order of  $\mathbb{N} \cup \{0\}$ , and  $R$  be a ring. Then  $R$  is a right (resp. left) Archimedean domain if and only if the ring  $R[[S]]$  is a right (resp. left) Archimedean domain.*

**Corollary 4.4.** ([20, Theorem 2.9]) *Let  $R$  be a ring and  $\alpha$  an endomorphism of the ring  $R$ . Then:*

- (1) *The ring  $R[[x; \alpha]]$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain,  $\alpha$  is injective and preserves nonunits of  $R$ .*

- (2)  $R[[x; \alpha]]$  is a left Archimedean domain if and only if  $R$  is a left Archimedean domain and  $\alpha$  is injective.

Let  $R$  be a ring, and consider the multiplicative monoid  $\mathbb{N}^{\geq 1}$ , endowed with the usual order  $\leq$ . Then  $A = R[[\mathbb{N}^{\geq 1}]]$  is the ring of arithmetical functions with values in  $R$ , endowed with the Dirichlet convolution:

$$fg(n) = \sum_{d|n} f(d)g(n/d) \quad \text{for each } n \geq 1.$$

**Corollary 4.5.** *Let  $R$  be a ring. Then  $R$  is a right (resp. left) Archimedean domain if and only if the ring of arithmetical functions  $R[[\mathbb{N}^{\geq 1}]]$  is a right (resp. left) Archimedean domain.*

Let  $(S, \leq)$  be a strictly totally ordered monoid which is also artinian and narrow. Then the set  $X_s = \{(u, v) \mid uv = s, u, v \in S\}$  is finite for any  $s \in S$ . Let  $V$  be a free abelian additive group with the base consisting of elements of  $S$ . It was noted in [10, Remark 1.2] that  $V$  is a coalgebra over  $\mathbb{Z}$  with the comultiplication map and the counit map as follows:

$$\Delta(s) = \sum_{(u,v) \in X_s} u \otimes v, \quad \epsilon(s) = \begin{cases} 1 & s = 1 \\ 0 & s \neq 1, \end{cases}$$

and  $R[[S]] \cong \text{Hom}(V, R)$ , the dual algebra with multiplication

$$f * g = (f \otimes g) \Delta \quad \text{for each } f, g \in \text{Hom}(V, R).$$

**Corollary 4.6.** *Let  $R$  be a ring,  $(S, \leq)$  be a strictly totally ordered monoid which is also artinian and narrow. Let  $\text{Hom}(V, R)$  be the dual algebra defined as above. Then  $\text{Hom}(V, R)$  is a right (resp. left) Archimedean domain if and only if  $R$  is a right (resp. left) Archimedean domain.*

Below we provide another characterization of Archimedean domains of skew generalized power series  $R[[S, \omega]]$  in the case where  $(S, \leq)$  is a commutative torsion-free cancellative artinian semisubtotally ordered monoid.

**Theorem 4.7.** *Let  $R$  be a ring,  $(S, \leq)$  a commutative torsion-free cancellative artinian semisubtotally ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism.*

- (i) *The ring  $R[[S, \omega]]$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ .*
- (ii)  *$R[[S, \omega]]$  is a left Archimedean domain if and only if  $R$  is a left Archimedean domain and  $\omega_s$  is injective for any  $s \in S$ .*

*Proof.* We adapt the proof of [18, Theorem 3.13]. Let  $\preceq$  be the total order associated with  $\leq$ ,  $A = R[[S, \omega, \leq]]$  and  $B = R[[S, \omega, \preceq]]$ . The “only if” parts of (i) and (ii) follow by analogous arguments as the “only if” parts of (i) and (ii) in the proof of Theorem 4.1.

To prove the “if” part of (i), assume that  $R$  is a right Archimedean domain,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ . Then  $B$  is a right Archimedean domain by Theorem 4.1(i). Now applying Proposition 2.2 and Proposition 3.5 we deduce that  $A$  is a right Archimedean domain. Similar arguments apply to the proof of the “if” part of (ii), and therefore, the result follows.  $\square$

From the preceding results, we can suggest the following question.

**Question 4.8.** *Let  $R$  be a ring,  $(S, \leq)$  a commutative torsion-free semisubtotally ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism.*

- (i) *Is it true that the ring  $R[[S, \omega]]$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain,  $S$  is a cancellative right Archimedean monoid,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ ?*
- (ii) *Is it true that the ring  $R[[S, \omega]]$  is a left Archimedean domain if and only if  $R$  is a left Archimedean domain,  $S$  is a cancellative left Archimedean monoid, and  $\omega_s$  is injective for any  $s \in S$ ?*

We will see in the following five corollaries that Theorem 4.7 provides a rich source of examples of Archimedean domains.

**Corollary 4.9.** *Let  $R$  be a ring and  $(S, \leq)$  be a commutative torsion-free cancellative artinian semisubtotally ordered monoid. Then the generalized power series ring  $R[[S]]$  is a right (resp. left) Archimedean domain if and only if  $R$  is a right (resp. left) Archimedean domain.*

**Corollary 4.10.** *Let  $R$  be a ring,  $(S, \leq)$  a commutative torsion-free cancellative artinian positively ordered monoid and  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism.*

- (i) *The ring  $R[[S, \omega]]$  is a right Archimedean domain if and only if  $R$  is a right Archimedean domain,  $\omega_s$  is injective and preserves nonunits of  $R$  for any  $s \in S$ .*
- (ii)  *$R[[S, \omega]]$  is a left Archimedean domain if and only if  $R$  is a left Archimedean domain and  $\omega_s$  is injective for any  $s \in S$ .*

*Proof.* It is clear that  $(S, \leq)$  is a semisubtotally ordered monoid. Thus the result follows from Theorem 4.7.  $\square$

**Corollary 4.11.** *Let  $(S_1, \leq_1), \dots, (S_n, \leq_n)$  be commutative torsion-free cancellative artinian positively ordered monoids. Denote by  $(\text{lex } \leq)$  and  $(\text{relex } \leq)$  the lexicographic order, the reverse lexicographic order, respectively, on the ordered monoid  $S_1 \times \dots \times S_n$ . If  $R$  is a ring and  $\omega : S_1 \times \dots \times S_n \rightarrow \text{Aut}(R)$  is a monoid homomorphism, then the following statements are equivalent:*

- (1)  *$R[[S_1 \times \dots \times S_n, \omega, \text{lex } \leq]]$  is a right (resp. left) Archimedean domain.*
- (2)  *$R[[S_1 \times \dots \times S_n, \omega, \text{relex } \leq]]$  is a right (resp. left) Archimedean domain.*
- (3)  *$R$  is a right (resp. left) Archimedean domain.*

*Proof.* It is clear that  $(S_1 \times \cdots \times S_n, \omega, \text{lex} \leq)$  (resp.  $(S_1 \times \cdots \times S_n, \omega, \text{relex} \leq)$ ) is a commutative torsion-free cancellative artinian positively ordered monoid. Thus the result follows from Corollary 4.10.  $\square$

**Corollary 4.12.** *Let  $S$  be a submonoid of  $(\mathbb{N} \cup \{0\})^n$  ( $n \geq 2$ ), endowed with the usual natural order  $\leq$  induced by the product order, or lexicographic order or reverse lexicographic order. Let  $R$  be a ring and  $\omega : S \rightarrow \text{Aut}(R)$  a monoid homomorphism. Then  $R[[S, \omega]]$  is a right (resp. left) Archimedean domain if and only if  $R$  is a right (resp. left) Archimedean domain.*

Let  $\alpha$  and  $\beta$  be endomorphisms of  $R$  such that  $\alpha \circ \beta = \beta \circ \alpha$ . Assume that  $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$  is endowed with the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of  $\mathbb{N} \cup \{0\}$ , and define  $\omega : S \rightarrow \text{End}(R)$  a monoid homomorphism via  $\omega(m, n) = \alpha^m \beta^n$  for any  $m, n \in \mathbb{N} \cup \{0\}$ . Then  $R[[S, \omega]] \cong R[[x, y; \alpha, \beta]]$ , in which  $(ax^m y^n)(bx^p y^q) = \alpha \alpha^m \beta^n(b)x^{m+p}y^{n+q}$  for any  $m, n, p, q \in \mathbb{N} \cup \{0\}$ .

**Corollary 4.13.** *Let  $\alpha$  and  $\beta$  be automorphisms of a ring  $R$  such that  $\alpha \circ \beta = \beta \circ \alpha$ . Then  $R[[x, y; \alpha, \beta]]$  is a right (resp. left) Archimedean domain if and only if  $R$  is a right (resp. left) Archimedean domain.*

We close this paper with the following example of an Archimedean ACCP-ring  $R$  with an endomorphism  $\alpha$  for which the skew polynomial ring  $R[x; \alpha]$  is not an ACCPL-ring. Note that, an element  $r_i = v_{i_1} v_{i_2} \cdots v_{i_k}$  is considered as a monomial of degree  $k$ . The degree of  $r = \sum_{i \in \mathbb{N}} r_i$  is the maximum of  $\deg(r_i)$ .

**Example 4.14.** Suppose that  $K$  is a field and consider the ring  $R$  as follows:  $R = \{K + v_1 K + v_2 K + \cdots + v_1 v_2 K + v_1 v_3 K + \cdots \mid v_i^2 = 0, v_i v_j = v_j v_i \text{ for every } i, j\}$ . In fact,  $R$  also can be defined as  $R := \frac{K[v_1, v_2, \dots]}{\langle v_1^2, v_2^2, \dots \rangle}$ . Define  $\alpha$  on  $R$  as follows:

$$\alpha(v_i) = \begin{cases} v_{i+1}, & i \text{ is a positive even number,} \\ 0, & i \text{ is odd.} \end{cases}$$

It is obvious that  $\alpha(1) = 1$ . We claim that  $R$  satisfies ACCP. Suppose that  $R$  does not satisfy ACCP. So there exists a non-stabilizing chain  $a_1 R \subseteq a_2 R \subseteq \cdots$ , with  $a_i \in R$ . One can easily see that  $\deg(a_i) > \deg(a_{i+1})$  and that  $\deg(a_1) = \infty$ , which is a contradiction. Thus,  $R$  is an ACCP-ring. Now, we claim that  $R$  is Archimedean. Let there exists  $a \in R$  such that  $\bigcap_{n \in \mathbb{N}} Ra^n \neq 0$ . Then there is  $t \in Ra^n$  for each  $n$  which means that the degree of  $t$  should be  $\infty$  and it is also a contradiction. Hence  $R$  is Archimedean.

We claim that  $R[x; \alpha]$  is not an ACCPL-ring. To do this, consider the following sequence:

$$f_0 = a_0 x + b_0, \quad f_n = a_n x + b_n \quad \text{for any } n \in \mathbb{N},$$

where  $a_0 = v_1 + v_3 + \cdots$ ,  $b_0 = 1 + v_2 + v_4 + \cdots$  and

$$a_n = (p_0)^{-1} (a_{n-1} - p_1 \alpha((p_0)^{-1}) \alpha(b_{n-1})), \quad b_n (p_0)^{-1} = b_{n-1}$$

such that  $p_0 = 1 + v_1$  and  $p_1 = v_1 + v_3 + \cdots$ . One can see that  $f_{n-1} = (p_1x + p_0)f_n$ . So we have  $Rf_0 \subseteq Rf_1 \subseteq \cdots$ . Note that  $p_1$  is not nilpotent. Since the degree of  $p^k$  is  $k$ , so  $p_1x + p_0$  is not a unit and we see that  $Rf_0 \subsetneq Rf_1 \subsetneq \cdots$ , which shows that the skew polynomial ring  $R[x; \alpha]$  does not satisfy ACC on principal left ideals.

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