CHARACTERIZATIONS OF p-ADIC CENTRAL CAMPANATO SPACES VIA COMMUTATOR OF p-ADIC HARDY TYPE OPERATORS

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Abstract. In this paper, we give some characterizations of p-adic central Campanato spaces via the boundedness of commutators of p-adic Hardy type operators. Besides, some further boundedness of p-adic Hardy operators and their commutators is also presented.

1. Introduction

Let f be a locally integrable function on $\mathbb{R}^n$. The well-known n-dimensional Hardy operator H (see [5]) is defined as

$$Hf(x) := \frac{1}{|x|^n} \int_{|y|<|x|} f(y)dy, \quad x \in \mathbb{R}^n.$$

The norm of H on $L^q(\mathbb{R}^n)$ was evaluated in [4] and was found to be equal to that of one-dimensional Hardy operator. One can see [10] for more famous results for the one-dimensional Hardy operators and Hardy inequalities. The dual operator of H is defined by

$$H^*f(x) := \int_{|y|\geq|x|} \frac{f(y)}{|y|^n}dy, \quad x \in \mathbb{R}^n\backslash\{0\}.$$

It is clear that H and H^* satisfy

$$\int_{\mathbb{R}^n} g(x)Hf(x)dx = \int_{\mathbb{R}^n} f(x)H^*g(x)dx$$

for a suitable function g. The commutators of H and H^* are defined by

$$H_b f := b(Hf) - H(bf)$$

and

$$H_b^* := b(H^*f) - H^*(bf),$$

respectively. For some known works about the boundedness of H_b and H_b^*, see [16] and [23]. The fact that both H and H^* are centrosymmetric motivates one to characterize central function spaces via the boundedness of H_b and H_b^*. In [7], Fu and Liu et al. gave some characterizations of central BMO space via the boundedness of H_b and H_b^* on Lesbesgue space. By different ideas comparing to that of [7], Zhao and Lu [30] characterized the λ-central BMO space via the boundedness of H_b and H_b^* on Lebesgue space under some suitable conditions on λ. In 2015, Shi and Lu [22] gave some characterizations of central Campanato space via the boundedness of H_b and H_b^* on the central Morrey space.

In recent years, p-adic analysis has received a lot of attention due to its application in Mathematical Physics (cf. [1,2,11,12,25] and [26]), and Harmonic analysis on p-adic field has been drawing more and more concern (cf. [13,14, 17,19–21] and references therein).

Now we are in a position to introduce p-adic field. For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$. If any non-zero rational number x is represented as $x = p^\gamma \frac{m}{n}$, where m and n are integers which are not divisible by p, and γ is an integer, then $|x_p| = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p |y|_p, \qquad |x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

It follows from the second property that when $|x|_p \neq |y|_p$, then $|x+y|_p = \max\{|x|_p, |y|_p\}$. From the standard p-adic analysis [25], we see that any non-zero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$(1) \qquad x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \qquad \gamma = \gamma(x) \in \mathbb{Z},$$

where a_j are integers, $0 \leq a_j \leq p-1$, $a_0 \neq 0$. The series (1) converges in the p-adic norm because $|a_j p^j|_p = p^{-\gamma}$.

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \ldots, n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \qquad x \in \mathbb{Q}_p^n.$$

Denote by $B_\gamma(a) = \{x \in \mathbb{Q}^n : |x-a|_p \leq p^\gamma\}$, the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, and by $S_\gamma(a) := \{x \in \mathbb{Q}_p^n : |x-a|_p = p^\gamma\}$ the sphere with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, $\gamma \in \mathbb{Z}$. It is clear that $S_\gamma(a) = B_\gamma(a) \backslash B_{\gamma-1}(a)$ and

$$(2) \qquad B_\gamma(a) = \bigcup_{k \leq \gamma} S_k(a).$$

We set $B_\gamma(0) = B_\gamma$ and $S_\gamma(0) = S_\gamma$. Set $\mathbb{Q}_p^{n,*} = \mathbb{Q}_p^n \backslash \{0\}$.

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a unique Harr measure dx on $\mathbb{Q}_p^n$ (up to positive constant multiple) which is translation invariant. We normalize the measure dx so that

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1,$$

where $|E|_H$ denotes the Harr measure of a measurable subset E of $\mathbb{Q}_p^n$. From this integral theory, it is easy to obtain that $|B_\gamma(a)|_H = p^{\gamma n}$ and $|S_\gamma(a)|_H = p^{\gamma n}(1 - p^{-n})$ for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to p-adic field, see [24] and [26].

Fu, Wu and Lu [8] introduced the p-adic Hardy operator defined by

$$\mathcal{H}^p f(x) := \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} f(y) dy, \qquad x \in \mathbb{Q}_p^n \backslash \{0\},$$

and $\mathcal{H}^{p,*}$, the dual operator of $\mathcal{H}^p$, which is defined as

$$\mathcal{H}^{p,*} f(x) := \int_{\mathbb{Q}_p^n \backslash B(0,|x|_p)} \frac{1}{|y|_p} f(y) dy, \qquad x \in \mathbb{Q}_p^n \backslash \{0\},$$

where $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0 \in \mathbb{Q}_p^n$ and radius $|x|_p$, and established the sharp of $\mathcal{H}^p$ and $\mathcal{H}^{p,*}$ on p-adic weighted Lebesgue spaces. Wu, Mi and Fu [29] obtained the sharp bounds of $\mathcal{H}^p$ on p-adic central Morrey spaces and p-adic λ-central BMO spaces. They also got the boundedness for commutators of $\mathbb{Q}_p^n$ on these spaces. The commutators of $\mathcal{H}^p$ and $\mathcal{H}^{p,*}$ are defined by

$$\mathcal{H}_b^p f := b(\mathcal{H}^p f) - \mathcal{H}^p(bf)$$

and

$$\mathcal{H}_b^{p,*} f := b(\mathcal{H}^{p,*} f) - \mathcal{H}^{p,*}(bf),$$

respectively.

Assume that $1 \leq q < \infty$ and $-1/q < \lambda < 1/n$, and then the p-adic central Campanato space can be defined by

$$\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n) = \{f : \|f\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)} < \infty\},$$

where

$$\|f\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma|_H^\lambda} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |f(x) - f_{B_\gamma}|^q dx \right)^{1/q},$$

here $B_\gamma = B_\gamma(0)$. When $0 \leq \lambda < 1/n$, the space $\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)$ is just $\dot{\mathrm{CMO}}^{q,\lambda}(\mathbb{Q}_p^n)$ (λ-central bounded mean oscillation function space) which was introduced by Wu and Fu [28] with the equivalent norm

$$\|f\|_{\dot{\mathrm{CMO}}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \inf_{c \in \mathbb{C}} \frac{1}{|B_\gamma|_H^\lambda} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |f(x) - c|^q dx \right)^{1/q}.$$

For some information as regards the space $\mathrm{C\dot{M}O}^{q,\lambda}(\mathbb{Q}_p^n)$, see [3] for example. If $\lambda = 0$, then $\mathrm{C\dot{M}O}^{q,\lambda}(\mathbb{Q}_p^n) = \mathrm{C\dot{M}O}^q(\mathbb{Q}_p^n)$ which is defined in [8]. In [8,27], the authors obtained the boundedness of $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$ on Lebesgue spaces as:

$$b \in \mathrm{C\dot{M}O}^{\max(q,q')}(\mathbb{Q}_p^n) \Rightarrow \mathcal{H}_b^p(\mathcal{H}_b^{p,*}) : L^q(\mathbb{Q}_p^n) \to L^q(\mathbb{Q}_p^n).$$

For the case $-1/q < \lambda < 0$, $\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n) \supset \dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$. Here $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$ denote the central Morrey space with the following norm

$$\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} := \sup_{\gamma \in \mathbb{Z}} \frac{1}{|B_\gamma|_H^\lambda} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |f(x)|^q dx \right)^{1/q}.$$

It is remarkable that the Morrey space on $\mathbb{R}^n$ was first introduced in [18] by Morrey to study the local behavior of solutions of second order elliptic partial differential equations.

However, for the case $-1/q < \lambda < 0$, as a concept of highly independent interest, there has received nearly zero attention for characterization of $\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)$ by the boundedness of the commutator operators of Hardy type, to the best of our knowledge. In the paper, we will focus on this problem and give some characterizations of $\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)$ for $\lambda < 0$ via the boundedness of $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$ on $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$. As our previous studies, we settle this problem under the assumption that b satisfies the following mean value inequality. A function is said to satisfy the well known mean value inequality if there exists a constant $C > 0$ such that for any ball $B_\gamma(a) \in \mathbb{Q}_p^n$ with $\gamma \in \mathbb{Z}$,

$$(3) \qquad \sup_{B_\gamma(a) \ni x} |f(x) - f_{B_\gamma(a)}| \leq \frac{C}{|B_\gamma(a)|_H} \int_{B_\gamma(a)} |f(x) - f_{B_\gamma(a)}| dx.$$

The function class that satisfies (3) is also called the reverse Hölder class which contain many kinds of functions, such as polynomial functions [6] and harmonic functions [9]. For more information about the reverse Hölder classes, see also [15] for example.

We end this section with the outline of this paper. Section 2 is devote to give some characterizations of p-adic Campanato spaces via the boundedness of commutators of p-adic Hardy type operator. In Section 3, some interesting boundedness of p-adic Hardy operator is given. Throughout this paper, for $\gamma \in \mathbb{Z}$ and $B_\gamma = B_\gamma(0)$ denotes the ball centered at 0 with radius p^γ. $C > 0$ is a constant which may change from line to line.

2. Characterizations of p-adic central Campanato spaces via the boundedness of commutators of p-adic Hardy type operators

In this section, we give some characterizations of p-adic central Campanato spaces via the boundedness of commutator of p-adic Hardy operator.

Theorem 2.1. *Let* $1 < q < \infty$, $-1/q < \lambda < 0$, $-1/q_i < \lambda_i < 0$, $i = 1, 2$, $1/q = 1/q_1 + 1/q_2$, $\lambda = \lambda_1 + \lambda_2$ *and let* b *satisfy* (3). *Then the following statements are equivalent:*

(a) $b \in \dot{\mathcal{C}}^{q_1, \lambda_1}(\mathbb{Q}_p^n)$;

(b) *Both* $\mathcal{H}_b^p$ *and* $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q_2, \lambda_2}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q, \lambda}(\mathbb{Q}_p^n)$.

Under some stronger condition on λ, the following result can be deduced if we drop the assumption that b satisfies the condition (3).

Theorem 2.2. *Let* $1 < q < \infty$, $1/q + 1/q' = 1$ *and* $- \min\{1/(2q), 1/(2q')\} < \lambda < 0$. *Then the following statements are equivalent:*

(a) $b \in \dot{\mathcal{C}}^{\max(q, q'), \lambda}(\mathbb{Q}_p^n)$;

(b) *Both* $\mathcal{H}_b^p$ *and* $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q, \lambda}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q, 2\lambda}(\mathbb{Q}_p^n)$.

In addition, both $\mathcal{H}_b^p$ *and* $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q', \lambda}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q', 2\lambda}(\mathbb{Q}_p^n)$.

Let f be a locally integrable function in $\mathbb{Q}_p^n$ and $0 < \alpha < n$. The n-dimensional fractional operator can be defined by

$$\mathcal{H}_\alpha^p f(x) := \frac{1}{|x|_p^{n-\alpha}} \int_{B(0, |x|_p)} f(y) dy, \qquad x \in \mathbb{Q}_p^n \backslash \{0\}.$$

The dual operator of $\mathcal{H}_\alpha^p$ is $\mathcal{H}_\alpha^{p,*}$, which can be defined as

$$\mathcal{H}_\alpha^{p,*} f(x) := \int_{\mathbb{Q}_p^n \backslash B(0, |x|_p)} \frac{f(y)}{|x|_p^{n-\alpha}} dy, \qquad x \in \mathbb{Q}_p^n \backslash \{0\}.$$

The commutators of $\mathcal{H}_\alpha^p$ and $\mathcal{H}_\alpha^{p,*}$ are defined by

$$\mathcal{H}_{\alpha, b}^p f := b(\mathcal{H}_\alpha^p f) - \mathcal{H}_\alpha^p(bf)$$

and

$$\mathcal{H}_{\alpha, b}^{p,*} f := b(\mathcal{H}_\alpha^{p,*} f) - \mathcal{H}_\alpha^{p,*}(bf),$$

respectively. In [27], the author obtained the boundedness of $\mathcal{H}_{\alpha, b}^p$ and $\mathcal{H}_{\alpha, b}^{p,*}$ on both Lebesgue spaces and Herz spaces. In this paper, we give some characterizations of $\dot{\mathcal{C}}^{q, \lambda}(\mathbb{Q}_p^n)$ with $\lambda < 0$ via the boundedness of $\mathcal{H}_{\alpha, b}^p$ and $\mathcal{H}_{\alpha, b}^{p,*}$ on $\dot{\mathcal{M}}^{q, \lambda}(\mathbb{Q}_p^n)$.

Theorem 2.3. *Let* q, λ, q_i, λ_i, $i = 1, 2$ *and* b *be as in Theorem 2.1,* $0 < \alpha < \min\{n(1 - 1/q), n(\lambda_2 + 1/q_2)\}$ *and let* $\beta = \lambda_2 - \alpha/n$. *The the following statements are equivalent:*

(a) $b \in \dot{\mathcal{C}}^{q_1, \lambda_1}(\mathbb{Q}_p^n)$;

(b) *Both* $\mathcal{H}_{\alpha, b}^p$ *and* $\mathcal{H}_{\alpha, b}^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q_2, \beta}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q, \lambda}(\mathbb{Q}_p^n)$.

Theorem 2.4. *Let* $1 < q < \infty$, $1/q + 1/q' = 1$, $\max(q, q') = \bar{q}$, $-1/2\bar{q} < \lambda < 0$, $0 < \alpha < \min\{n(1 - 1/\bar{q}), n(\lambda + 1/\bar{q})\}$ *and let* $\beta = \lambda - \alpha/n$. *The the following statements are equivalent:*

(a) $b \in \dot{\mathcal{C}}^{\max(q, q'), \lambda}(\mathbb{Q}_p^n)$;

(b) *Both $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ are bounded operators from $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$ to $\dot{\mathcal{M}}^{q,2\lambda}(\mathbb{Q}_p^n)$.*
In addition, both $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ are bounded operators from $\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)$ to $\dot{\mathcal{M}}^{q',2\lambda}(\mathbb{Q}_p^n)$.

Proof of Theorem 2.1. This process can be divided into two steps.

(a) $\Rightarrow$ (b) Given a fixed ball $B_\gamma \in \mathbb{Q}_p^n$, the task is now to show that there exists a constant $C > 0$ such that

$$(4) \qquad \frac{1}{|B_\gamma|_H^\lambda} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |\mathcal{H}_b^p f(x)|^q dx \right)^{1/q} \le C \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}$$

and

$$(5) \qquad \frac{1}{|B_\gamma|_H^\lambda} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |\mathcal{H}_b^{p,*} f(x)|^q dx \right)^{1/q} \le C \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}.$$

The definition of $\mathcal{H}_b^p$ gives

$$\int_{B_\gamma} |\mathcal{H}_b^p f(x)|^q dx = \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} (b(x) - b(y)f(y)dy \right|^q dx$$

$$\le \sum_{k=-\infty}^{\gamma} \int_{S_k} \left| \frac{1}{|x|_p^n} \int_{B_k} (b(x) - b(y))f(y)dy \right|^q dx$$

$$\le C \sum_{k=-\infty}^{\gamma} \int_{S_k} \left| \frac{1}{|x|_p^n} \sum_{i=-\infty}^{k} \int_{S_i} |b(x) - b_{B_k}||f(y)|dy \right|^q dx$$

$$+ C \sum_{k=-\infty}^{\gamma} \int_{S_k} \left| \frac{1}{|x|_p^n} \sum_{i=-\infty}^{k} \int_{S_i} |b(y) - b_{B_k}||f(y)|dy \right|^q dx$$

$$=: I + II.$$

Applying Hölder's inequality (see [15]) to q_1/q and $(q_1/q)'$, the term I can be estimated as

$$I \le C \sum_{k=-\infty}^{\gamma} p^{-knq} \int_{S_k} |b(x) - b_{B_k}|^q dx \left| \sum_{i=-\infty}^{k} \int_{S_i} |f(y)|dy \right|^q$$

$$\le C \sum_{k=-\infty}^{\gamma} p^{-knq} \left(\int_{S_k} |b(x) - b_{B_k}|^{q_1} dx \right)^{q/q_1} |B_k|_H^{1/(q_1/q)'}$$

$$\times \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |f(y)|^{q_2} dy \right)^{1/q_2} |B_i|_H^{1/q_2'} \right|^q$$

$$\le C \|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} p^{-knq} |B_k|_H^{1+q\lambda_1} \left| \sum_{i=-\infty}^{k} |B_i|_H^{1+\lambda_2} \right|^q$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^{q}\|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^{q}\sum_{k=-\infty}^{\gamma}p^{kn(1+q\lambda)}$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^{q}\|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^{q}|B_\gamma|_H^{1+q\lambda}.$$

The fact $1/q = 1/q_1 + 1/q_2$ allows us to estimate the term II as

$$II \leq C\sum_{k=-\infty}^{\gamma}p^{-knq}\int_{S_k}\left|\sum_{i=-\infty}^{k}\int_{S_i}|b(y)-b_{B_k}||f(y)|dy\right|^q dx$$

$$\leq C\sum_{k=-\infty}^{\gamma}p^{-knq}\int_{S_k}\left|\sum_{i=-\infty}^{k}\left(\int_{B_i}(|b(y)-b_{B_k}||f(y)|)^q dy\right)^{1/q}|B_i|_H^{1/q'}\right|^q dx$$

$$\leq C\sum_{k=-\infty}^{\gamma}p^{-knq}\int_{S_k}\left|\sum_{i=-\infty}^{k}\left(\int_{B_i}|b(y)-b_{B_k}|^{q_1}dy\right)^{1/q_1}\right.$$

$$\left.\left(\int_{B_i}|f(y)|^{q_2}dy\right)^{1/q_2}|B_i|_H^{1/q'}\right|^q dx$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^{q}\|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^{q}\sum_{k=-\infty}^{\gamma}p^{kn(1+q\lambda)}$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^{q}\|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^{q}|B_\gamma|_H^{1+q\lambda}.$$

On account of the above estimate for I and II, (4) is obtained.

We are now in a position to (5). We note that

$$\int_{B_\gamma}|\mathcal{H}_b^{p,*}|^p dx = \int_{B_\gamma}\left|\int_{\mathbb{Q}_p^n\setminus B(0,|x|_p)}\frac{b(x)-b(y)}{|y|_p^n}f(y)dy\right|^q dx$$

$$\leq \int_{B_\gamma}\left|\int_{|x|_p\leq|y|_p\leq p^{n\gamma}}\frac{b(x)-b(y)}{|y|_p^n}f(y)dy\right|^q dx$$

$$+\int_{B_\gamma}\left|\int_{|y|_p>p^{n\gamma}}\frac{b(x)-b(y)}{|y|_p^n}f(y)dy\right|^q dx$$

$$=: I' + II'.$$

The term I' can be handled in a similar way as that of (4), the only difference being in the analysis of the term II'. Analysis of $\mathcal{H}_b^p$ shows

$$I' \leq \int_{B_\gamma}\left|\frac{1}{|x|_p^n}\int_{|y|_p\leq p^{n\gamma}}|b(x)-b(y)||f(y)|dy\right|^q dx$$

$$\leq C\sum_{k=-\infty}^{\gamma}p^{-knq}\int_{B_k}\left|\sum_{i=-\infty}^{k}\int_{B_i}|b(x)-b(y)||f(y)|dy\right|^q dx$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+q\lambda}.$$

For the term II', we proceed to show that

$$II' \leq \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty \int_{S_k} \frac{|b(x) - b_{B_\gamma}|}{|y|_p^n} |f(y)| dy \right|^q dx$$

$$+ \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty \int_{S_k} \frac{|b(y) - b_{B_\gamma}|}{|y|_p^n} |f(y)| dy \right|^q dx$$

$$=: II_1' + II_2'.$$

By the Hölder's inequality, we have

$$II_1' \leq \int_{B_\gamma} |b(x) - b_\gamma|^q dx \left| \sum_{k=\gamma}^\infty \int_{S_k} \frac{|f(y)|}{|y|_p^n} dy \right|^q$$

$$\leq \left(\int_{B_\gamma} |b(x) - b_{B_\gamma}|^{q_1} dx \right)^{q/q_1} |B_\gamma|_H^{1/(q_1/q)'} \left| \sum_{k=\gamma}^\infty \left(\int_{S_k} \left| \frac{|f(y)|}{|y|_p^n} \right|^{q_2} \right)^{1/q_2} |B_k|_H^{1/q_2'} \right|^q$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+q\lambda_1} \left| \sum_{k=\gamma}^\infty |B_k|_H^{\lambda_2} \right|^q$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+q\lambda}.$$

To get the boundedness for the term II_2', we need the following decomposition

$$II_2' \leq \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty \int_{S_k} \frac{|b(y) - b_{B_k}||f(y)|}{|y|_p^n} \right| dx + \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty \int_{S_k} \frac{|b_{B_\gamma} - b_{B_k}||f(y)|}{|y|_p^n} \right| dx$$

$$=: II_{21}' + II_{22}'.$$

We first compute II_{21}'. To do this, the Hölder's inequality and $1/q = 1/q_1 + 1/q_2$ show that

$$II_{21}' \leq \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty \left(\int_{S_k} \left| \frac{|b(y) - b_{B_k}|}{|y|_p^n} |f(y)| \right|^q dy \right)^{1/q} |B_k|_H^{1/q'} \right|^q dx$$

$$\leq \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty |B_k|_H^{1/q'} \left(\int_{B_k} |b(y) - b_{B_k}|^{q_1} dy \right)^{1/q_1} \left(\int_{B_k} \left| \frac{|f(y)|}{|y|_p^n} \right|^{q_2} dy \right)^{1/q_2} \right|^q dx$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q \int_{B_\gamma} \left| \sum_{k=\gamma}^\infty |B_k|_H^\lambda \right|^q dx$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+q\lambda}.$$

For the term II'_{22}, we claim first that for $k > \gamma$

$$|b_{B_\gamma} - b_{B_k}| \leq C p^{(\gamma+1)n\lambda_1} \|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.$$

In fact,

$$
\begin{aligned}
|b_{B_\gamma} - b_{B_k}| &\leq \sum_{j=\gamma}^{k-1} |b_{B_j} - b_{B_{j+1}}| \leq \sum_{j=\gamma}^{k-1} \frac{1}{|B_j|_H} \int_{B_{j+1}} |b(y) - b_{B_{j+1}}| dy \\
&\leq \sum_{j=\gamma}^{k-1} \frac{1}{|B_j|_H} \left(\int_{B_{j+1}} |b(y) - b_{B_{j+1}}|^{q_1} dy \right)^{1/q_1} |B_{j+1}|_H^{1/q'_1} \\
&\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \sum_{j=\gamma}^{k-1} \frac{|B_{j+1}|_H^{1+\lambda_1}}{|B_j|_H} \leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \sum_{j=\gamma}^{k-1} p^{(j+1)n\lambda_1} \\
&\leq C p^{(\gamma+1)n\lambda_1} \|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
II'_{22} &\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \int_{B_\gamma} \left| \sum_{k=\gamma}^{\infty} \int_{S_k} \frac{p^{\gamma n\lambda_1}}{|y|_p^n} |f(y)| dy \right|^q dx \\
&\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \int_{B_\gamma} p^{\gamma n\lambda_1 q} \left| \sum_{k=\gamma}^{\infty} \frac{1}{p^{nk}} \left(\int_{S_k} |f(y)|^{q_2} dy \right)^{1/q_2} |B_k|_H^{1/q'_2} \right|^q dx \\
&\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q \int_{B_\gamma} p^{\gamma n\lambda_1 q} \left| \sum_{k=\gamma}^{\infty} |B_k|_H^{\lambda_2} \right|^q dx \\
&\leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+q\lambda}.
\end{aligned}
$$

Summarizing, we have

$$II' \leq C\|b\|_{\dot{\mathcal{C}}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+q\lambda}.$$

Which implies (5). This is the desired result.

(b) $\Rightarrow$ (a) In this case, the proof consists of constructions of a proper commutator. We are reduced to proving that for a fixed ball B_γ,

$$\frac{1}{|B_\gamma|_H^{1+q_1\lambda_1}} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^{q_1} dy \leq C.$$

We conclude from (3) and Hölder's inequality that

$$
\begin{aligned}
\frac{1}{|B_\gamma|_H^{1+q_1\lambda_1}} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^{q_1} dy &\leq \frac{1}{|B_\gamma|_H^{q_1\lambda_1}} \sup_{y \in B_\gamma} |b(y) - b_{B_\gamma}|^{q_1} \\
&\leq \frac{C}{|B_\gamma|_H^{q_1\lambda_1}} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |b(y) - b_{B_\gamma}| dy \right)^{q_1}
\end{aligned}
$$

$$\leq \frac{C}{|B_\gamma|_H^{q_1\lambda_1}} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^q dy \right)^{q_1/q}.$$

To deal with the above term, we note that

$$\int_{B_\gamma} |b(y) - b_{B_\gamma}|^q dy$$

$$\leq \frac{1}{|B_\gamma|_H^q} \int_{B_\gamma} \left| \int_{B_\gamma} (b(y) - b(z)dz \right|^q dy$$

$$\leq \frac{1}{|B_\gamma|_H^q} \int_{B_\gamma} |y|_p^{nq} \left| \frac{1}{|y|_p^n} \int_{\{z \in B_\gamma : |z|_p < |y|_p\}} (b(y) - b(z))\chi_{B_\gamma}(z)dz \right|^q dy$$

$$+ \frac{1}{|B_\gamma|_H^q} \int_{B_\gamma} \left| \int_{\{z \in B_\gamma : |z|_p \geq |y|_p\}} (b(y) - b(z))\chi_{B_\gamma}(z)dz \right|^q dy$$

$$=: J + JJ.$$

The $\left(\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_b^p$ allows us to estimate J as

$$J \leq \frac{1}{|B_\gamma|_H^q} \int_{B_\gamma} |y|_p^{nq} |\mathcal{H}_b^p \chi_{B_\gamma}(\cdot)|^q dy$$

$$\leq |B_\gamma|_H^{1+q\lambda} \|\mathcal{H}_b^p \chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q$$

$$\leq C |B_\gamma|_H^{1+q\lambda} \|\chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q$$

$$\leq C |B_\gamma|_H^{1+q\lambda_1}.$$

By the $\left(\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_b^{p,*}$, it is easy to check that

$$JJ \leq \frac{1}{|B_\gamma|_H^q} \int_{B_\gamma} \left| \int_{\{z \in B_\gamma : |z|_p \geq |y|_p\}} |z|_p^n \frac{(b(y) - b(z))}{|z|_p^n} \chi_{B_\gamma}(z)dz \right|^q dy$$

$$\leq C \int_{B_\gamma} |\mathcal{H}_b^{p,*} \chi_{B_\gamma}(\cdot)|^q dy \leq C |B_\gamma|^{1+q\lambda} \|\mathcal{H}_b^{p,*} \chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q$$

$$\leq C |B_\gamma|_H^{1+q\lambda} \|\chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q_2,\lambda_2}(\mathbb{Q}_p^n)}^q$$

$$\leq C |B_\gamma|_H^{1+q\lambda_1}.$$

We thus have established the following inequality if we combine the above estimates for J and JJ,

$$\frac{1}{|B_\gamma|_H^{1+q_1\lambda_1}} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^{q_1} dy \leq \frac{C}{|B_\gamma|_H^{q_1\lambda_1}} \left(\frac{|B_\gamma|_H^{1+q\lambda_1}}{|B_\gamma|_H} \right)^{q_1/q} \leq C.$$

The proof of Theorem 2.1 is completed. $\qquad\square$

Proof of Theorem 2.2. We begin the proof of Theorem 2.2 by proving the following lemma.

Lemma 2.5. *Let* $1 < q < \infty$, $-1/q < \lambda < 0$, $i, k \in \mathbb{Z}$ *and* $b \in \dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)$. *Then*

$$|b(y) - b_{B_k}| \leq |b_y - b_{B_i}| + C \max\{|B_k|_H^\lambda, |B_i|_H^\lambda\} \|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

Proof. Using the Hölder's inequality to q and q', one has

$$|b_{B_j} - b_{B_{j+1}}| \leq \frac{1}{|B_j|_H} \int_{B_{j+1}} |b(y) - b_{B_{j+1}}| dy$$

$$\leq \left(\frac{1}{|B_j|_H} \int_{B_{j+1}} |b(y) - b_{B_{j+1}}|^q dy \right)^{1/q}$$

$$\leq C|B_{j+1}|_H^\lambda \|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

The proof falls naturally into two cases. In the case $k < i$, we note that

$$|b(y) - b_{B_k}| \leq |b(y) - b_{B_i}| + \sum_{j=k}^{i-1} |b_{B_j} - b_{B_{j+1}}|$$

$$\leq |b(y) - b_{B_i}| + C \sum_{j=k}^{i-1} |B_{j+1}|_H^\lambda \|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}$$

$$\leq |b(y) - b_{B_i}| + C|B_k|_H^\lambda \|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

For the case $k > i$, it is easy to obtain

$$|b(y) - b_{B_k}| \leq |b(y) - b_{B_i}| + \sum_{j=i}^{k-1} |b_{B_j} - b_{B_{j+1}}|$$

$$\leq |b(y) - b_{B_i}| + C \sum_{j=i}^{k-1} |B_{j+1}|_H^\lambda \|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}$$

$$\leq |b(y) - b_{B_i}| + C|B_i|_H^\lambda \|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}. \qquad \square$$

Having disposed of this preliminary step, we can now return to the proof of Theorem 2.2.

(a) $\Rightarrow$ (b) The task is now to find a constant $C > 0$ such that for a fixed ball B_γ, the following inequalities are true

(6) $$\frac{1}{|B_\gamma|_H^{2\lambda}} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |\mathcal{H}_b^p f(x)|^q dx \right)^{1/q} \leq C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}$$

and

(7) $$\frac{1}{|B_\gamma|_H^{2\lambda}} \left(\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |\mathcal{H}_b^{p,*} f(x)|^q dx \right)^{1/q} \leq C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

To deal with (6), we note that

$$\int_{B_\gamma} |\mathcal{H}_b^p f(x)|^q dx \leq \int_{B_\gamma} \left| \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} |b(x) - b(y)||f(y)|dy \right|^q dx$$

$$\leq C \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|_H^q} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(x) - b_{B_k}||f(y)|dy \right|^q dx$$

$$+ C \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|_H^q} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y) - b_{B_k}||f(y)|dy \right|^q dx$$

$$=: K + KK.$$

Using Hölder's inequality to q_1/q and $(q_1/q)'$, the following boundedness for K can be shown:

$$K \leq C \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|_H^q} \int_{B_k} |b(x) - b_{B_k}|^q dx \left| \sum_{i=-\infty}^{k} \int_{B_i} |f(y)|dy \right|^q$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} |B_k|_H^{1+q\lambda-q} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |f(y)|^q dy \right)^{1/q} |B_i|_H^{1/q'} \right|^q$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} |B_k|_H^{1+q\lambda-q} \left| \sum_{i=-\infty}^{k} |B_i|_H^{1+\lambda} \right|^q$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} p^{kn(1+2q\lambda)}$$

$$\leq C\|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^{1+2q\lambda},$$

where the fact $\lambda > -1/(2q)$ has been used in the last inequality. Applying Lemma 2.5 to term KK can produce

$$KK \leq C \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|_H^q} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |b(y) - b_{B_i}||f(y)|dy \right|^q dx$$

$$+ C\|b\|_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|_H^q} \int_{B_k} \left| \sum_{i=-\infty}^{k} \int_{B_i} |B_i|_H^\lambda |f(y)|dy \right|^q dx$$

$$=: KK_1 + KK_2.$$

Repeated application of Hölder's inequality shows that

$$KK_1 \leq C \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|_H^q} \int_{B_k} \left| \sum_{i=-\infty}^{k} \left(\int_{B_i} |b(y) - b_{B_i}|^{q'} dy \right)^{1/q'} \left(\int_{B_i} |f(y)|^q dy \right)^{1/q} \right|^q dx$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q',\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|^q_H} \int_{B_k} \left|\sum_{i=-\infty}^{k} |B_i|^{1+2\lambda}_H\right|^q dx$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q',\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|^q_H} \int_{B_k} |B_k|^{(1+2\lambda)q}_H dx$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q',\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} |B_\gamma|^{1+2q\lambda}_H.$$

The term KK_2 can be bounded by

$$KK_2 = C\|b\|^q_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|^q_H} \int_{B_k} \left|\sum_{i=-\infty}^{k} \int_{B_i} |B_i|^\lambda_H |f(y)| dy\right|^q dx$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|^q_H} \int_{B_k} \left|\sum_{i=-\infty}^{k} |B_i|^{1/q'+\lambda}_H \left(\int_{B_i} |f(y)|^q dy\right)^{1/q}\right|^q dx$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{\gamma} \frac{1}{|B_k|^q_H} \int_{B_k} \left|\sum_{i=-infty}^{k} |B_i|^{1+2\lambda}_H\right|^q dx$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} \sum_{k=-\infty}^{\gamma} |B_k|^{1+2q\lambda}_H$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} |B_\gamma|^{1+2q\lambda}_H.$$

Summarizing, we have

$$KK \leq C\|b\|^q_{\dot{\mathcal{C}}^{q',\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} |B_\gamma|^{1+2q\lambda}_H + C\|b\|^q_{\dot{\mathcal{C}}^{q,\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} |B_\gamma|^{1+2q\lambda}_H$$

$$\leq C\|b\|^q_{\dot{\mathcal{C}}^{\max(q,q'),\lambda}(\mathbb{Q}_p^n)}\|f\|^q_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)} |B_\gamma|^{1+2q\lambda}_H.$$

Based on the above estimates for K and KK, we obtain (6).

With a slight modification of the proofs for (5) and (6), (7) can be obtained easily, we omit its proof here for the similarity.

(b) $\Rightarrow$ (a) This step will be divided into two cases.

Case 1: $q > q'$. In this case, we want to show that there is a constant $C > 0$ such that for a fixed ball B_γ, there holds

$$(8) \qquad \frac{1}{|B_\gamma|^{1+q\lambda}_H} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^q dy \leq C.$$

To reach this inequality, we note that

$$\frac{1}{|B_\gamma|^{1+q\lambda}_H} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^q dy$$

$$\leq \frac{1}{|B_\gamma|^{1+q+q\lambda}_H} \int_{B_\gamma} \left|\int_{B(0,|y|_p)} (b(y) - b(z))\chi_{B_\gamma}(z) dz\right|^q dy$$

$$+ \frac{1}{|B_\gamma|_H^{1+q+q\lambda}} \int_{B_\gamma} \left| \int_{\mathbb{Q}_p^n \setminus B(0,|y|_p)} (b(y) - b(z)) \chi_{B_\gamma}(z) dz \right|^q dy$$

$$=: L + LL.$$

The rest of the proof runs as that of Theorem 2.1, so we apply that argument again. The $\left(\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q,2\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_b^p$ produces the following estimate for the term L,

$$L \leq \frac{C}{|B_\gamma|_H^{1++q+q\lambda}} \int_{B_\gamma} |y|_p^{nq} |\mathcal{H}_b^p \chi_{B_\gamma}(\cdot)|^q dy$$

$$\leq C |B_\gamma|_H^{q\lambda} \|\mathcal{H}_b^p \chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,2\lambda}(\mathbb{Q}_p^n)}^q$$

$$\leq C |B_\gamma|_H^{q\lambda} \|\chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \leq C.$$

Applying the similar argument to term LL, the following can be confirmed easily

$$LL \leq \frac{1}{|B_\gamma|_H^{1+q+q\lambda}} \int_{B_\gamma} \left| \int_{\mathbb{Q}_p^n \setminus B(0,|y|_p)} |z|_p^n \frac{(b(y) - b(z))}{|z|_p^n} \chi_{B_\gamma}(z) dz \right|^q dy$$

$$\leq \frac{C}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}_b^{p,*} \chi_{B_\gamma}(\cdot)|^q dy$$

$$\leq C |B_\gamma|_H^{q\lambda} \|\mathcal{H}_b^{p,*} \chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,2\lambda}(\mathbb{Q}_p^n)}^q$$

$$\leq C |B_\gamma|_H^{q\lambda} \|\chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \leq C.$$

So (8) is a by-product of the estimates for L and LL.

Case 2: $q < q'$. With the $\left(\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q',2\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_b^p$ and $\mathcal{H}_b^{p,*}$, the similar arguments of Case 1 can be applied to this and show that

$$\frac{1}{|B_\gamma|_H^{1+q'\lambda}} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^{q'} \leq C,$$

which completes the proof Theorem 2.2. □

Proofs of Theorem 2.3 and Theorem 2.4. The methods used in the proofs of Theorem 2.1 and Theorem 2.2 remain valid for that of Theorem 2.3 and Theorem 2.4 with only a slight modification. We omit their proofs here for their similarity. □

3. Further boundedness for Hardy type operators and their commutators on *p*-adic central Morrey spaces

In this section, some further boundedness for *p*-adic Hardy type operators and their commutators on central Morrey spaces will be given. Now we formulate our main results as follows:

Theorem 3.1. *Let* $1 < p < \infty$, $-\min(1/q, 1/q') < \lambda < 0$ *and let* $1/q + 1/q' = 1$. *Then*

(a) *Both* $\mathcal{H}^p$ *and* $\mathcal{H}^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$.

(b) *Both* $\mathcal{H}^p$ *and* $\mathcal{H}^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)$.

Theorem 3.2. *Let* $1 < r \leq q < \infty$, $1/r + 1/r' = 1$, $1/q + 1/q' = 1$, $-\min(1/q, 1/r') < \lambda < 0$, $1/r - 1/q = \alpha/n$, $0 < \alpha < \min\{n(\lambda + 1/r), n(\lambda + 1/q')\}$ *and let* $\beta = \lambda - \alpha/n$. *Then the following statements are equivalent:*

(a) $b \in \dot{\mathrm{CMO}}^{\max(q,r')}(\mathbb{Q}_p^n)$;

(b) *Both* $\mathcal{H}_{\alpha,b}^p$ *and* $\mathcal{H}_{\alpha,b}^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$.

In addition, both $\mathcal{H}_{\alpha,b}^p$ *and* $\mathcal{H}_{\alpha,b}^{p,*}$ *are also bounded operators from* $\dot{\mathcal{M}}^{q',\beta}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{r',\lambda}(\mathbb{Q}_p^n)$.

When $r = q$ in Theorem 3.2, we have the following result.

Corollary 3.3. *Let* $1 < q < \infty$, $1/q + 1/q' = 1$ *and* $-\min(1/q, 1/q') < \lambda < 0$. *Then the following statements are equivalent:*

(a) $b \in \dot{\mathrm{CMO}}^{\max\{q,q'\}}(\mathbb{Q}_p^n)$;

(b) *Both* $\mathcal{H}_b^p$ *and* $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)$.

In addition, both $\mathcal{H}_b^p$ *and* $\mathcal{H}_b^{p,*}$ *are bounded operators from* $\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)$ *to* $\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)$.

The proofs of Theorem 3.1 and Theorem 3.2 based on following lemma.

Lemma 3.4 (See [8])**.** *Let* $1 < q < \infty$ *and* $1/q + 1/q' = 1$. *Then*

(a) *Both* $\mathcal{H}^p$ *and* $\mathcal{H}^{p,*}$ *are bounded operators from* $L^q(\mathbb{Q}_p^n)$ *to* $L^q(\mathbb{Q}_p^n)$;

(b) *Both* $\mathcal{H}^p$ *and* $\mathcal{H}^{p,*}$ *are bounded operators from* $L^{q'}(\mathbb{Q}_p^n)$ *to* $L^{q'}(\mathbb{Q}_p^n)$;

Lemma 3.5 (See [28])**.** *Let* $1 < q < \infty$, $0 \leq \lambda < 1/n$, $i, k \in \mathbb{Z}$, *and assume that* $b \in \dot{\mathrm{CMO}}^{q,\lambda}(\mathbb{Q}_p^n)$.

(a) *If* $\lambda > 0$, *then*

$$|b_y - b_{B_k}| \leq |b(y) - b_{B_i}| + \frac{p^n(1 + p^{-|k-i|n\lambda})}{1 - p^{-n\lambda}}\|b\|_{\dot{\mathrm{CMO}}^{q,\lambda}(\mathbb{Q}_p^n)} \max\{|B_i|_H^\lambda, |B_k|_H^\lambda\}.$$

(b) *If* $\lambda = 0$, *then*

$$|b_y - b_{B_k}| \leq |b(y) - b_{B_i}| + p^n|i - k|\|b\|_{\dot{\mathrm{CMO}}^{q,\lambda}(\mathbb{Q}_p^n)} \max\{|B_i|_H^\lambda, |B_k|_H^\lambda\}.$$

Lemma 3.6 (See [27])**.** *Let* $0 < \alpha < n$, $1 < r \leq q < \infty$, $1/r + 1/r' = 1$, $1/q + 1/q' = 1$, $1/r - 1/q = \alpha/n$, *and let* $b \in \dot{\mathrm{CMO}}^{\max\{q,r'\}}(\mathbb{Q}_p^n)$. *Then there exist constants* $C > 0$ *such that*

(a) $\|\mathcal{H}_{\alpha,b}^p f\|_{L^q(\mathbb{Q}_p^n)} \leq C\|f\|_{L^r(\mathbb{Q}_p^n)}$;

(b) $\|\mathcal{H}_{\alpha,b}^{p,*} f\|_{L^q(\mathbb{Q}_p^n)} \leq C\|f\|_{L^r(\mathbb{Q}_p^n)}$.

Furthermore, by duality, we can deduce that exist constants $C > 0$ *such that*

(c) $\|\mathcal{H}_{\alpha,b}^p f\|_{L^{r'}(\mathbb{Q}_p^n)} \leq C\|f\|_{L^{q'}(\mathbb{Q}_p^n)}$;

(d) $\|\mathcal{H}_{\alpha,b}^{p,*}f\|_{L^{r'}(\mathbb{Q}_p^n)} \le C\|f\|_{L^{q'}(\mathbb{Q}_p^n)}$.

Proof of Theorem 3.1. Following the notations of Section 2, we need to show that for fixed ball B_γ with $\gamma \in \mathbb{Z}$, there exist constants $C > 0$ such that

$$(9) \qquad \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^p f(x)|^q dx \le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q;$$

$$(10) \qquad \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^{p,*} f(x)|^q dx \le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q;$$

$$(11) \qquad \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^p f(x)|^{q'} dx \le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^{q'};$$

and

$$(12) \qquad \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^{p,*} f(x)|^{q'} dx \le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^{q'}.$$

Let $f = f_{\chi_{B_{2\gamma}}} + f_{\chi_{B_{2\gamma}^c}} =: f_1 + f_2$ to produce

$$\frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^p f(x)|^q dx$$

$$\le \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^p f_1(x)|^q dx + \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}^p f_2(x)|^q dx$$

$$=: L_1 + L_2.$$

By Lemma 3.4 the following estimate for L_1 can be proven

$$L_1 \le \frac{C}{|B_\gamma|_H^{1+q\lambda}} \int_{B_{2\gamma}} |f(x)|^q dx \le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q.$$

Next, the boundedness of L_2 can be shown. Since

$$|\mathcal{H}^p f_2(x)| = \left| \frac{1}{|x|_p^n} \int_{B(0,|x|_p)} f_2(y) dy \right|$$

$$\le \sum_{k=2\gamma}^{\infty} \frac{1}{|x|_p^n} \int_{S_k} |f(y)| dy$$

$$\le \sum_{k=2\gamma}^{\infty} \frac{1}{|B_k|_H} \left(\int_{B_k} |f(y)|^q dy \right)^{1/q} |B_k|_H^{1/q'}$$

$$\le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \sum_{k=2\gamma}^{\infty} |B_k|_H^\lambda$$

$$\le C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q |B_\gamma|_H^\lambda,$$

therefore,

$$L_2 \leq C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |B_\gamma|_H^{q\lambda} dx \leq C\|f\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q,$$

which yield (9). (10), (11) and (12) follow by same methods as that of (9). $\quad\square$

Proof of Theorem 3.2. (a) $\Rightarrow$ (b) In this case, the task is to show for a fixed ball B_γ with $\gamma \in \mathbb{Z}$, there exist constants $C > 0$ such that

$$(13) \qquad \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^p f(x)|^q dx \leq C\|f\|_{\dot{\mathcal{M}}^{r,\lambda}(\mathbb{Q}_p^n)}^q;$$

$$(14) \qquad \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^{p,*} f(x)|^q dx \leq C\|f\|_{\dot{\mathcal{M}}^{r,\lambda}(\mathbb{Q}_p^n)}^q;$$

$$(15) \qquad \frac{1}{|B_\gamma|_H^{1+r'\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^p f(x)|^{r'} dx \leq C\|f\|_{\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)}^{r'};$$

and

$$(16) \qquad \frac{1}{|B_\gamma|_H^{1+r'\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^{p,*} f(x)|^{r'} dx \leq C\|f\|_{\dot{\mathcal{M}}^{q',\lambda}(\mathbb{Q}_p^n)}^{r'}.$$

Decomposing $f = f_{\chi_{B_{2\gamma}}} + f_{\chi_{B_{2\gamma}^c}} =: f_1 + f_2$ derives that

$$\frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^p f(x)|^q dx$$

$$\leq \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^p f_1(x)|^q dx + \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^p f_2(x)|^q dx$$

$$=: LL_1 + LL_2.$$

We conclude from Lemma 3.6 that

$$LL_1 \leq \frac{1}{|B_\gamma|_H^{1+q\lambda}} \|\mathcal{H}_{\alpha,b}^p f \chi_{B_{2\gamma}}\|_{L^q(\mathbb{Q}_p^n)}^q$$

$$\leq \frac{C}{|B_\gamma|_H^{1+q\lambda}} \|f \chi_{B_{2\gamma}}\|_{L^r(\mathbb{Q}_p^n)}^q$$

$$\leq C\|f\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)}^q.$$

For the term LL_2, by the definition of $\mathcal{H}_{\alpha,b}^p f$ and $\dot{\mathrm{CMO}}^q(\mathbb{Q}_p^n)$, the following can be shown

$$LL_2 \leq \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} \left| \frac{1}{|x|_p^{n-\alpha}} \int_{B(0,|x|_p)} (b(x) - b(y)) f_2(y) dy \right|^q dx$$

$$\leq \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} \left| \sum_{k=2\gamma}^{\infty} \frac{1}{|B_k|_H^{1-\alpha/n}} \int_{B_k} (b(x) - c) f(y) dy \right|^q dx$$

$$+ \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} \left| \sum_{k=2\gamma}^\infty \frac{1}{|B_k|_H^{1-\alpha/n}} \int_{B_k} (b(y) - c)f(y)dy \right|^q dx$$

$$=: LL_{21} + LL_{22}.$$

Applying Hölder inequality to q and q' can produce the estimate for LL_{21} as

$$LL_{21} \leq \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} \left| \sum_{k=2\gamma}^\infty (b(x) - c)|B_k|_H^{\alpha/n-1/r} \left(\int_{B_k} |f(y)|^r dy \right)^{1/r} \right|^q dx$$

$$\leq C\|f\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)}^q \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} |b(x) - c|^q \left| \sum_{k=2\gamma}^\infty |B_k|_H^\lambda \right|^q dx$$

$$\leq C\|b\|_{\dot{\mathrm{CMO}}^q(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)}^q.$$

The same conclusion can be drawn for the term LL_{22} as

$$LL_{22} \leq \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} \left| \sum_{k=2\gamma}^\infty \frac{1}{|B_k|_H^{1-\alpha/n}} \left(\int_{B_k} |b(y) - c|^{r'} dy \right)^{1/r'} \left(\int_{B_k} |f(y)|^r dy \right)^{1/r} \right|^q dx$$

$$\leq C\|b\|_{\dot{\mathrm{CMO}}^{r'}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)}^q \frac{1}{|B_\gamma|_H^{1+q\lambda}} \int_{B_\gamma} \left| \sum_{k=2\gamma}^\infty |B_K|_H^\lambda \right|^q dx$$

$$\leq C\|b\|_{\dot{\mathrm{CMO}}^{r'}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)}^q \frac{1}{|B_\gamma|_H^{q\lambda}} \left| \sum_{k=2\gamma}^\infty |B_k|_H^\lambda \right|^q$$

$$\leq C\|b\|_{\dot{\mathrm{CMO}}^{r'}(\mathbb{Q}_p^n)}^q \|f\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)}^q.$$

Combining the above estimates for LL_{21} and LL_{22}, (13) is proved. In the same manner, we can show (14). (15) and (16) can be handled in much the same way as that of (13) and (14), and we only need to substitute Lemma 3.6(a) and (b) into Lemma 3.6(c) and (d). This produces the desired results.

(b) $\Rightarrow$ (a) This step will be divided into two cases.

Case 1: $q > r'$. In this case, we want to show that there is a constant $C > 0$ such that for a fixed ball B_γ with $\gamma \in \mathbb{Z}$, there holds

$$(17) \qquad \frac{1}{|B_\gamma|_H} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^q dy \leq C.$$

To reach this inequality, we note that

$$\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |b(y) - b_{B_\gamma}|^q dy$$

$$\leq \frac{1}{|B_\gamma|_H^{1+q}} \int_{B_\gamma} \left| \int_{B(0,|x|_p)} (b(y) - b(z))\chi_{B_\gamma}(z)dz \right|^q dy$$

$$+ \frac{1}{|B_\gamma|_H^{1+q}} \int_{B_\gamma} \left| \int_{\mathbb{Q}_p^n \setminus B(0,|x|_p)} (b(y)-b(z))\chi_{B_\gamma}(z)dz \right|^q dy$$
$$=: K_1 + K_2.$$

By the $\left(\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_{\alpha,b}^p$, the following can be shown

$$K_1 \leq \frac{C}{|B_\gamma|_H^{1+q}} \int_{B_\gamma} |y|_p^{nq-q\alpha} |\mathcal{H}_{\alpha,b}^p \chi_{B_\gamma}(\cdot)|^q dy$$
$$\leq C|B_\gamma|_H^{q(\lambda-\alpha/n)} \|\mathcal{H}_{\alpha,b}^p \chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q$$
$$\leq C|B_\gamma|_H^{q(\lambda-\alpha/n)} \|\chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)} \leq C.$$

Applying the $\left(\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_{\alpha,b}^{p,*}$, the following can be confirmed

$$K_2 \leq \frac{C}{|B_\gamma|_H^{1+q}} \int_{B_\gamma} \left| \int_{\mathbb{Q}_p^n \setminus B(0,|x|_p)} |z|_p^{n-\alpha} \frac{(b(y)-b(z))}{|z|_p^{n-\alpha}} \chi_{B_\gamma}(z)dz \right|^q dx$$
$$\leq \frac{C}{|B_\gamma|_H^{1+q\alpha/n}} \int_{B_\gamma} |\mathcal{H}_{\alpha,b}^{p,*} \chi_{B_\gamma}(\cdot)|^q dx$$
$$\leq C|B_\gamma|_H^{q(\lambda-\alpha/n)} \|\mathcal{H}_{\alpha,b}^{p,*} \chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n)}^q$$
$$\leq C|B_\gamma|_H^{q(\lambda-\alpha/n)} \|\chi_{B_\gamma}(\cdot)\|_{\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n)} \leq C.$$

So (17) is a by-product of the estimates for K_1 and K_2.

Case 2: $q < r'$. With the $\left(\dot{\mathcal{M}}^{r,\beta}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{q,\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$ being replaced by the $\left(\dot{\mathcal{M}}^{q',\beta}(\mathbb{Q}_p^n), \dot{\mathcal{M}}^{r',\lambda}(\mathbb{Q}_p^n) \right)$ boundedness of $\mathcal{H}_{\alpha,b}^p$ and $\mathcal{H}_{\alpha,b}^{p,*}$, the similar arguments of Case 1 can be applied to this case and show that

$$\frac{1}{|B_\gamma|_H} \int_{B_\gamma} |b(y)-b_{B_\gamma}|^{r'} dy \leq C,$$

which completes the proof of Theorem 3.2. $\qquad\square$

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