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FINITE p-GROUPS ALL OF WHOSE SUBGROUPS OF CLASS 2 ARE GENERATED BY TWO ELEMENTS

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Abstract. We proved that finite p-groups in the title coincide with finite p-groups all of whose non-abelian subgroups are generated by two elements. Based on the result, finite p-groups all of whose subgroups of class 2 are minimal non-abelian (of the same order) are classified, respectively. Thus two questions posed by Berkovich are solved.

1. Introduction

In this note, the groups considered are finite p-groups (in brief, p-groups). p-groups is the groups of prime-power order. The subgroup of class 2 of a group means the subgroup of nilpotent class 2. Assume G is a p-group. We use $c(G)$ and $d(G)$ to denote the nilpotent class and the minimal number of generators of G respectively. Let

$$r(G) = \max\{d(H) \mid H \leqslant G\} \text{ and } r_i(G) = \max\{d(H) \mid H \leqslant G \text{ and } c(H) = i\}.$$

Obviously,

$$r(G) = \max\{r_i(G) \mid 1 \leqslant i \leqslant c(G) = c\}.$$

Moreover, if p is an odd prime, then Laffey in [5] have proved that

$$r(G) = \max\{r_1(G), r_2(G)\}.$$

Blackburn in [4, Theorem 4.1] classified p-groups G with $r_1(G) = 2$ and $p > 2$. Obviously, $r_2(G) \geqslant 2$. A natural question is: what can be said about p-groups G with $r_2(G) = 2$? The motivation of this note is to classify such p-groups. We prove that such p-groups coincide with the p-groups all of whose non-abelian subgroups are generated by two elements, which was classified by Xu et al. in [8]. The fact implies that

$$r_2(G) = 2 \Longleftrightarrow r_i(G) = 2 \text{ for all } i \text{ with } 2 \leqslant i \leqslant c.$$

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If $r_2(G) \geqslant 3$, then is it true that $r_i(G) \leqslant r_2(G)$ for all i with $3 \leqslant i \leqslant c$? We will give an example to show that there exists a group G of order 2^8 with $r_2(G) = 3$ and $r_3(G) = 4$. This above fact motivates us to consider such a question: how much difference are there between the p-groups determined by some property of their non-abelian subgroups and the p-groups determined by some property of their subgroups of class 2? Notice that if G is a minimal non-abelian p-group, then G is two-generator. Hence as a nontrivial application of the classification of the p-groups by Xu et al. in [8], p-groups all of whose subgroups of class 2 are minimal non-abelian (of the same order) are respectively classified in this note. Hence the following two questions posed by Berkovich are solved.

Problem 6 ([3, p_{337}])**.** Classify the p-groups all of whose subgroups of class 2 are two-generator.

Problem 372 ([1])**.** Study the p-groups all of whose subgroups of class 2 are minimal non-abelian.

2. Preliminaries

Following Berkovich and Janko [2], for a positive integer t, a finite p-group G is called an $\mathcal{A}_t$-group if its every subgroup of index p^t is abelian, but it has at least one non-abelian subgroup of index p^{t-1}. So $\mathcal{A}_1$-groups are nothing but the minimal non-abelian p-groups. For $t \leqslant 3$, all $\mathcal{A}_t$-groups are known (see [6, 11, 12]). We use $G \in \mathcal{A}_t$ to denote G is an $\mathcal{A}_t$-group.

Following Xu et al. [8], $\mathcal{B}_p$ denotes the class of p-groups whose non-abelian proper subgroups are two-generator, $\mathcal{B}'_p$ denotes the class of groups consisting of groups in $\mathcal{B}_p$ which are neither abelian nor minimal non-abelian, $\mathcal{D}_p = \{G \in \mathcal{B}'_p \mid G$ has an abelian maximal subgroup$\}$ and $\mathcal{M}_p = \{G \in \mathcal{B}'_p \mid G$ has no abelian maximal subgroup$\}$. $\mathcal{D}_p(2) = \{G \in \mathcal{D}_p \mid d(G) = 2\}$ and $\mathcal{D}_p(3) = \{G \in \mathcal{D}_p \mid d(G) = 3\}$, $\mathcal{D}'_p(2) = \{G \in \mathcal{D}_p(2) \mid G$ is not of maximal class$\}$ and $\mathcal{M}'_p = \{G \in \mathcal{M}_p \mid G$ is neither metacyclic nor 3-group of maximal class$\}$.

In terms of notation mentioned above, the [8, Main Theorem] can be restated as follows.

Theorem 2.1. *Suppose that G is a finite non-abelian p-group. If all non-abelian proper subgroups of G are two-generator, then G is one of the following groups*:

(1) $\mathcal{A}_1$*-groups*;

(2) $\mathcal{A}_2$*-groups*;

(3) *p-groups of maximal class with an abelian maximal subgroup*;

(4) 3*-groups of maximal class*;

(5) $\mathcal{D}'_p(2)$*-groups with $p \geqslant 3$*;

(6) $\mathcal{M}'_3$*-groups with a unique minimal non-abelian maximal subgroup*;

(7) $\mathcal{M}'_p$*-groups having no minimal non-abelian maximal subgroup, where $p \geqslant 3$*;

(8) *metacyclic groups.*

Remark 2.2. From the argument in [8] or a simple check, it is not difficult to get the converse of Theorem 2.1 is also true.

Lemma 2.3 ([12, Lemma 2.6(1-2)]). *Assume $G \in \mathcal{A}_2$. Then $d(G) \leqslant 3$. If $d(G) = 3$, then $c(G) = 2$.*

Lemma 2.4 ([8, Lemma 2.2]). *Suppose that G is a finite non-abelian p-group. Then the following conditions are equivalent.*
(1) *G is minimal non-abelian*;
(2) *$d(G) = 2$ and $|G'| = p$*;
(3) *$d(G) = 2$ and $\Phi(G) = \mathrm{Z}(G)$.*

Proposition 2.5 ([7]). *Let G be a metabelian group and $a, b \in G$. For any positive integers i and j, let*

$$[ia, jb] = [a, b, \underbrace{a, \dots, a}_{i-1}, \underbrace{b, \dots, b}_{j-1}].$$

Then, for any positive integers m and n,

$$[a^m, b^n] = \prod_{i=1}^{m} \prod_{j=1}^{n} [ia, jb]^{\binom{m}{i}\binom{n}{j}}, \tag{1}$$

$$(ab^{-1})^m = a^m \left(\prod_{i+j \le m} [ia, jb]^{\binom{m}{i+j}} \right) b^{-m}, \ m \ge 2. \tag{2}$$

Lemma 2.6 ([1, Theorem 9.6(e)]). *Let G be a group of maximal class and order p^m, $p > 2$, $m > p+1$. Then one of maximal subgroups of G is the fundamental subgroup and the others are the subgroups of maximal class.*

Lemma 2.7 ([1, §9, Exercise 10]). *Let G be a 3-group of maximal class. Then the fundamental subgroup of G is either abelian or minimal non-abelian.*

Lemma 2.8 ([8, Theorem 5.4]). *Let $G \in \mathcal{M}_p'$, $|G| = p^n \geqslant p^6$, p be an odd prime and K be a maximal subgroup of G. Then*
(1) *K is not a group of maximal class*;
(2) *$K \in \mathcal{A}_1$ or $K \in \mathcal{D}_p'(2)$*;
(3) *$c(G) = n - 2$*;
(4) *If every maximal subgroup of G is not minimal non-abelian, then $|G| = p^6$.*

Lemma 2.9 ([8, Theorem 3.2(1)]). *Assume G is a $\mathcal{D}_p'(2)$-group and $c(G) = c$. If M is a non-abelian subgroup of G with $|G : M| = p^t$, then $c \geqslant 3$, $t \leqslant c - 2$, $c(M) = c - t$.*

3. The classification of finite p-groups G with $r_2(G) = 2$ and its application

Assume G is a finite non-abelian p-group. For convenience, we introduce the following notation.

$\mathcal{Q}_i = \{G \,|\, G$ is the p-group whose non-abelian subgroups have property $\mathcal{P}_i\}$;

$\mathcal{Q}_i^* = \{G \,|\, G$ is the p-group whose non-abelian proper subgroups have property $\mathcal{P}_i\}$;

$\mathcal{R}_i = \{G \,|\, G$ is the p-group whose subgroups of class 2 have property $\mathcal{P}_i\}$;

$\mathcal{R}_i^* = \{G \,|\, G$ is the p-group whose proper subgroups of class 2 have property $\mathcal{P}_i\}$.

In this note, $\mathcal{P}_1$ is "two-generator", $\mathcal{P}_2$ is "minimal non-abelian" and $\mathcal{P}_3$ is "the same order".

Obviously,

$$\mathcal{Q}_i \subseteq \mathcal{Q}_i^*, \;\; \mathcal{R}_i \subseteq \mathcal{R}_i^*, \;\; \mathcal{Q}_i \subseteq \mathcal{R}_i, \;\; \mathcal{Q}_i^* \subseteq \mathcal{R}_i^* \text{ and } \mathcal{Q}_i^* \cup \mathcal{R}_i = \mathcal{R}_i^*.$$

Moreover, in this note we will prove

$$\mathcal{Q}_1 = \mathcal{R}_1, \;\; \mathcal{Q}_1^* = \mathcal{R}_1^*, \;\; \mathcal{R}_3 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_1 \text{ and } \mathcal{R}_3^* \subseteq \mathcal{R}_2^* \subseteq \mathcal{R}_1^*.$$

A nature question is: is it true that $\mathcal{Q}_i = \mathcal{R}_i$ and $\mathcal{Q}_i^* = \mathcal{R}_i^*$ for $i = 2, 3$?

By determining the groups in $\mathcal{R}_2$ and $\mathcal{R}_3$, we can get the answer is false. That is,

$$\mathcal{Q}_i \subsetneqq \mathcal{R}_i \text{ and } \mathcal{Q}_i^* \subsetneqq \mathcal{R}_i^* \text{ for } i = 2, 3$$

Theorem 3.1. (1) $\mathcal{Q}_1 = \mathcal{R}_1$; (2) $\mathcal{Q}_1^* = \mathcal{R}_1^*$; (3) $\mathcal{R}_3 \subseteq \mathcal{R}_2 \subseteq \mathcal{R}_1$; (4) $\mathcal{R}_3^* \subseteq \mathcal{R}_2^* \subseteq \mathcal{R}_1^*$.

Proof. (1) Obviously, $\mathcal{Q}_1 \subseteq \mathcal{R}_1$. We prove $\mathcal{Q}_1 \supseteq \mathcal{R}_1$. If not, then there exists G such that $G \in \mathcal{R}_1$ and $G \notin \mathcal{Q}_1$. Let $\mathcal{K} = \{K \leq G \mid d(K) \geqslant 3 \text{ and } K' \neq 1\}$. Since $G \notin \mathcal{Q}_1$, $\mathcal{K} \neq \emptyset$. Hence there exists $K \in \mathcal{K}$ such that $|K|$ is of smallest order. It follows that $K \in \mathcal{Q}_1^*$. Thus K is isomorphic to one of the groups in Theorem 2.1. By a simple check we get $d(K) = 2$ but $\mathcal{A}_2$-groups. Hence K is an $\mathcal{A}_2$-group and $d(K) \geqslant 3$. It follows by Lemma 2.3 that $c(K) = 2$. Notice that if $G \in \mathcal{R}_1$, then $H \in \mathcal{R}_1$ for all $H \leqslant G$. Hence $K \in \mathcal{R}_1$. This contradicts $d(K) \geqslant 3$. Thus the conclusion follows.

(2) Obviously, $\mathcal{Q}_1^* \subseteq \mathcal{R}_1^*$. We prove $\mathcal{Q}_1^* \supseteq \mathcal{R}_1^*$. Let $G \in \mathcal{R}_1^*$ and H is a non-abelian proper subgroup of G. Then $H \in \mathcal{R}_1$. It follows from (1) that $H \in \mathcal{Q}_1$. Hence $d(H) = 2$. Thus the conclusion follows.

(3) It follows from Lemma 2.4 that $\mathcal{R}_2 \subseteq \mathcal{R}_1$. We prove $\mathcal{R}_3 \subseteq \mathcal{R}_2$. Assume $G \in \mathcal{R}_3$, $H \leqslant G$ and $c(H) = 2$. Let $K < H$. Since $c(H) = 2$, $c(K) \leqslant 2$. Since $G \in \mathcal{R}_3$ and $c(H) = 2$, $c(K) \neq 2$. Hence K is abelian. It follows that H is minimal non-ableian. This means $G \in \mathcal{R}_2$. Thus the conclusion follows.

(4) It is a direct consequence of (3). □

Now the p-groups in $\mathcal{Q}_1^*$ were classified by Xu et al. in [8]. Thus, by Theorem 3.1(1),(2), Lemma 2.3 and the argument of Theorem 3.1(1) we get:

Theorem 3.2. *Suppose that G is a finite non-abelian p-group. Then*

(1) *$G \in \mathcal{R}_1^*$ if and only if G is one of the groups in Theorem 2.1.*

(2) *$G \in \mathcal{R}_1$ if and only if G is one of the groups in Theorem 2.1 except for $\mathcal{A}_2$-groups with three-generator.*

In following we determine the groups in $\mathcal{R}_2$ and $\mathcal{R}_3$.

Theorem 3.3. *$G \in \mathcal{R}_2$ if and only if all $\mathcal{A}_2$-subgroups of G are of class* 3.

Proof. ($\Longrightarrow$) Let $L \leqslant G$ and $L \in \mathcal{A}_2$. Then $c(L) \leqslant 3$ by [12, Lemma 2.6(1)]. Since $G \in \mathcal{R}_2$, $c(L) = 3$.

($\Longleftarrow$) If not, then there exists L such that $L \leqslant G$, $c(L) = 2$ and L is not minimal non-abelian. Without loss of generality assume L is an $\mathcal{A}_t$-group with $t \geqslant 2$. Let H be an non-abelian subgroup of smallest order of L. By the definition of $\mathcal{A}_t$ we get $|L : H| = p^{t-1}$. Thus there exists K satisfying $L \supseteq K \supseteq H$ and $|K : H| = p$. Thus K is an $\mathcal{A}_2$-group. Since $c(L) = 2$, $c(K) = 2$. This contradicts "all $\mathcal{A}_2$-subgroups of G are of class 3". □

Lemma 3.4. *Assume $G \in \mathcal{R}_1$ and $|G'| \geqslant p^2$. Then $G \in \mathcal{R}_2$ if and only if all subgroups H of G with $|H'| = p^2$ are of class* 3.

Proof. ($\Longrightarrow$) By hypothesis we get $2 \leqslant c(H) \leqslant 3$. Since $|H'| = p^2$, H is not minimal non-abelian by Lemma 2.4. It follow by $G \in \mathcal{R}_2$ that $c(H) = 3$.

($\Longleftarrow$) Let $L \leqslant G$ and $c(L) = 2$. We need to show $L \in \mathcal{A}_1$. Since $G \in \mathcal{R}_1$, $d(L) = 2$. Assume $L = \langle a, b\rangle$ without loss of generality. Since $c(L) = 2$, $L' = \langle [a, b]^g \mid g \in G\rangle = \langle [a, b]\rangle \leqslant Z(L)$. Let $|L'| = p^t$. If $t \geqslant 2$, then let $K = \langle a^{p^{t-2}}, b\rangle$. We get $K \leq L$ and $|K'| = p^2$. Hence $c(K) = 3$. This contradicts $c(L) = 2$. Hence $t = 1$. It follows by Lemma 2.4 that $L \in \mathcal{A}_1$. □

Lemma 3.5. *Assume G is a* 3*-group of maximal class which has no abelian subgroup of index* 3*. Then one of maximal subgroups of G is minimal non-abelian and the others are of maximal class with an abelian maximal subgroup.*

Proof. Notice that there exists an abelian maximal subgroup in a group of maximal class with order 3^4. Hence $|G| \geq 3^5$. By Lemma 2.6, all maximal subgroups of G are of maximal class except for the fundamental subgroup. The fundamental subgroup of G is minimal non-abelian by Lemma 2.7. It follows that $\Phi(G)$ is abelian. Moreover, $\Phi(G)$ is maximal in all maximal subgroups of G. □

Lemma 3.6. *Suppose that G is a finite non-abelian p-group. Then*

(1) *if $G \in \mathcal{A}_1$, then $G \in \mathcal{R}_3$;*

(2) *if $G \in \mathcal{A}_2$ and $c(G) = 3$, then $G \in \mathcal{R}_3$;*

(3) *if $G \in \mathcal{A}_2$ and $c(G) \neq 3$, then $G \notin \mathcal{R}_2$;*

(4) *if G is a p-group of maximal class with an abelian maximal subgroup, then $G \in \mathcal{R}_3$;*

(5) *if G is a 3-group of maximal class having no abelian maximal subgroup, then $G \in \mathcal{R}_2 \backslash \mathcal{R}_3$;*

(6) *if $G \in \mathcal{D}'_p(2)$, then $G \in \mathcal{R}_3$;*

(7) *if $G \in \mathcal{M}'_p$ and G has no minimal non-abelian maximal subgroup, where $p \geqslant 3$, then $G \in \mathcal{R}_3$;*

(8) *if $G \in \mathcal{M}'_3$ and G has a unique minimal non-abelian maximal subgroup, then $G \in \mathcal{R}_2 \backslash \mathcal{R}_3$.*

Proof. (1) and (2) are trivial. It follows by the definition of $\mathcal{A}_t$-groups.

(3) It follows by Theorem 3.3.

(4) By [9, Corollary 8.3.2] we know all non-abelian subgroups of G are of maximal class. Hence all subgroups of class 2 are of order p^3. That is, $G \in \mathcal{R}_3$.

(5) Let M be a subgroup of class 2 of G. Obviously, $c(G) > 2$. Hence M is contained in a maximal subgroup of G. By Lemma 3.5, one of maximal subgroups of G is minimal non-abelian and the others are of maximal class with an abelian maximal subgroup. If M is contained in a minimal non-abelian subgroup, then M is minimal non-abelian. If M is contained in a subgroup of maximal class with an abelian maximal subgroup, then, by the argument of (4), $|M| = 3^3$. Hence M is also minimal non-abelian. In either case, $G \in \mathcal{R}_2$.

Now G has a subgroup of class 2 of order 3^3 by the argument above paragraph. On the other hand, it follows by Lemma 3.5 that G has a maximal subgroup which is minimal non-abelian. Moreover, $|G| \geqslant 3^5$ by the argument of Lemma 3.5. Hence G has a subgroup of class 2 of order great than 3^3. So $G \notin \mathcal{R}_3$.

(6) Let M be a subgroup of class 2 of G. Then $|G : M| = p^{c(G)-2}$ by Lemma 2.9. That is, all subgroups of class 2 of G are of the same order. Thus $G \in \mathcal{R}_3$.

(7) Firstly, we claim that each maximal subgroup of G is of class 3. In fact, let K be a maximal subgroup of G. Since $G \in \mathcal{M}'_p$, we get $c(G) = 4$, $K \in \mathcal{D}'_p(2)$ and $c(K) \neq 4$ by Theorem 2.8. It follows by $c(G) = 4$ and $c(K) \neq 4$ that $c(K) \leq 3$. Since $K \in \mathcal{D}'_p(2)$, $c(K) = 3$ by Lemma 2.9.

Let M be a subgroup of class 2 of G. Since $c(G) = 4$, M is contained in a maximal subgroup H of G. Thus $|H : M| = p^{c(H)-2}$ by Lemma 2.9. Thus all subgroups of class 2 of G are of the same order. So $G \in \mathcal{R}_3$.

(8) Let M be a subgroup of class 2 of G. It follows by Lemma 2.8 that $c(G) > 2$, and one of maximal subgroups of G is minimal non-abelian and the others are $\mathcal{D}'_p(2)$ groups. Hence M is contained in a maximal subgroup of G. If M is contained in a minimal non-abelian subgroup, then M is minimal non-abelian. If M is contained in $\mathcal{D}'_p(2)$ group, then, by (6) and Theorem 3.1(3), M is also minimal non-abelian. In either case, $G \in \mathcal{R}_2$.

Since G has a maximal subgroup which is minimal non-abelian, G has a maximal subgroup M_1 of class 2. On the other hand, by the argument of above paragraph, we get that there exists $K \in \mathcal{D}'_p(2)$ and K is maximal in G. Then $c(K) \geqslant 3$ by Theorem 2.9. Thus there exits a subgroup M_2 of class 2 which is a proper subgroup of K. Obviously, $|M_1| \neq |M_2|$. So $G \notin \mathcal{R}_3$. $\square$

Theorem 3.7. *Suppose that G is a finite nonabelian p-group. Then $G \in \mathcal{R}_2$ if and only if G is one of the following groups*:

(1) *One of the groups* (1) *and* (3)-(7) *in Theorem* 2.1;

(2) $\mathcal{A}_2$*-groups with class* 3;

(3) *metacyclic groups*: $\langle a, b \mid a^{2^{r+s+v+t'+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s+v+t'}},\ a^b = a^{-1+2^{r+v}}\rangle$, *where r, s, v, t, t' and u are non-negative integers satisfying $r \geqslant 2$, $t' \leqslant r$, $u \leqslant 1$, $tt' = sv = tv = 0$, $0 \leqslant s + t' + u \leqslant 2$, and $u = 0$ if $t' \geqslant r - 1$.*

Proof. ($\Longrightarrow$) By Theorem 3.1(3) we get $\mathcal{R}_2 \subseteq \mathcal{R}_1$. By Theorem 3.2(2), G is one of the groups in Theorem 2.1 except for $\mathcal{A}_2$-groups with three-generator. If G is one of the groups (1)-(7) in Theorem 2.1, then, by Lemma 3.3, we get the groups (1)-(2) in the Theorem. The remains is the case of G being metacyclic.

Assume G is metacyclic. Then, by [10, Theorems 2.1, 2.2 and Remark 2.3], G is one of the following groups:

(i) groups with a cyclic subgroup of index p;

(ii) $\langle a, b \mid a^{p^{r+s+u}} = 1, b^{p^{r+s+t}} = a^{p^{r+s}}, a^b = a^{1+p^r}\rangle$, where r, s, t and u are non-negative integers satisfying $u \leq r$, and $r \geq 2$ if $p = 2$; $r \geq 1$ if $p > 2$;

(iii) $\langle a, b \mid a^{2^{r+s+v+t'+u}} = 1,\ b^{2^{r+s+t}} = a^{2^{r+s+v+t'}},\ a^b = a^{-1+2^{r+v}}\rangle$, where r, s, v, t, t' and u are non-negative integers satisfying $r \geqslant 2$, $t' \leqslant r$, $u \leqslant 1$, $tt' = sv = tv = 0$, and $u = 0$ if $t' \geqslant r - 1$.

If G is the group (i), then G is minimal non-abelian or a group of maximal class with an abelian maximal subgroup by [1, Theorem 1.2]. They are one of the groups (1) in the Theorem.

If G is the group (ii), then we will prove $r + s + u \leqslant 3$. If not, then let $K = \langle a, b^{p^{s+u-2}}\rangle$. By calculation, using Proposition 2.5(1), we get

$$[a, b^{p^{s+u-2}}] = [a, b]^{p^{s+u-2}} [a, b, b]^{\binom{p^{s+u-2}}{2}}.$$

Since $r+s+u > 3$, $[a, b, b]^{\binom{p^{s+u-2}}{2}} = 1$. Notice that $\langle [x, y]\rangle \trianglelefteq G$ for any $x, y \in G$. Thus

$$K' = \langle [a, b^{p^{s+u-2}}]\rangle = \langle [a, b]^{p^{s+u-2}}\rangle = \langle a^{p^{r+s+u-2}}\rangle.$$

Then $|K'| = p^2$. It follows by Lemma 3.4 that $c(K) = 3$. Hence $K_3 \neq 1$, where K_3 is the third term of the lower center series of G. Notice that

$$K_3 = \langle [a^{p^{r+s+u-2}}, b^{p^{s+u-2}}]\rangle = \langle a^{p^{2r+2s+2u-4}}\rangle.$$

Hence $2r+2s+2u-4 < r+s+u$. That is, $r+s+u \leqslant 3$. This is a contradiction.

Now it follows from $r + s + u \leqslant 3$ that $|G'| \leqslant p^2$. By Theorem 3.1(3), $G \in \mathcal{R}_2 \subseteq \mathcal{R}_1$. Hence non-abelian subgroups of G are generated by two elements. If $|G'| = p$, then $G \in \mathcal{A}_1$ by Lemma 2.4. Thus G is one of the groups (1) in the Theorem. If $|G'| = p^2$, then it is easy to get $|M'| = p$ for each non-abelian maximal subgroup M of G. It follows by Lemma 2.4 that $M \in \mathcal{A}_1$. Hence $G \in \mathcal{A}_2$. Since $G \in \mathcal{R}_2$, G is the group (2) in the Theorem by Theorem 3.3.

If G is the group (iii), then we will prove $s+t'+u \leqslant 2$. If not, then let $K=\langle a, b^{2^{s+t'+u-2}}\rangle$. By calculation, using the formula in Proposition 2.5(1), we get

$$[a, b^{2^{s+t'+u-2}}]=a^{-1}a^{b^{2^{s+t'+u-2}}}=a^{-1}a^{(-1+2^{r+v})^{2^{s+t'+u-2}}}.$$

Since $s+t'+u>2$,

$$\langle a^{-1}a^{(-1+2^{r+v})^{2^{s+t'+u-2}}}\rangle=\langle a^{p^{r+s+v+t'+u-2}}\rangle.$$

Thus $\langle [a, b^{p^{s+t'+u-2}}]\rangle=\langle a^{p^{r+s+v+t'+u-2}}\rangle$. Hence

$$|K'|=|\langle [a, b^{p^{s+t'+u-2}}]\rangle|=|\langle a^{p^{r+s+v+t'+u-2}}\rangle|=p^2.$$

It follows by Lemma 3.4 that $c(K)=3$. Hence $K_3\neq 1$. Notice that

$$K_3=\langle [a^{p^{r+s+v+t'+u-2}}, b^{p^{s+t'+u-2}}]\rangle=\langle a^{p^{2(r+s+v+t'+u-2)}}\rangle.$$

Hence $2(r+s+v+t'+u-2)<r+s+v+t'+u$. That is, $r+s+v+t'+u\leqslant 3$. This is a contradiction. We get the groups (3) in the Theorem.

($\Longleftarrow$) If G is one of the groups (1)-(2), then $G\in\mathcal{R}_2$ by Theorem 3.6. We will prove all subgroups of class 2 in the groups (3) are minimal non-abelian. Assume G is the group (3), $H\leqslant G$ and $|H'|=4$. By Lemma 3.4 it is enough to show $c(H)=3$.

It is easy to see that $H'=\langle a^{2^{r+s+v+t'+u-2}}\rangle$. Assume $H=\langle a^{i_1}b^{j_1}, a^{i_2}b^{j_2}\rangle$ without loss of generality, where i_1, i_2, j_1, j_2 are integer numbers. Let $M=\langle a, b^2\rangle$. Then

$$[a, b^2]=a^{-1}a^{b^2}=a^{(-1+a^{r+v})^2-1}.$$

Obviously, $2^{r+v+1}\mid (-1+a^{r+v})^2-1$. Since $s+t'+u\leqslant 2$, $|M'|\leq 2$. If $2\mid j_1$ and $2\mid j_2$, then $H\leq M$. This contradicts $|H'|=4$. Hence $2\nmid j_1$ or $2\nmid j_2$. Assume $2\nmid j_1$ without loss of generality. It easy to see that

$$[a^{i_1}b^{j_1}, a^{2^{r+s+v+t'+u-2}}]=[b^{j_1}, a^{2^{r+s+v+t'+u-2}}].$$

Since $a^{2^{r+s+v+t'+u-2}}\notin Z(G)$, $[b^{j_1}, a^{2^{r+s+v+t'+u-2}}]\neq 1$. Hence $H_3\neq 1$. So $c(H)=3$. The proof is complete. □

Theorem 3.8. *Suppose that G is a finite nonabelian p-group. Then $G\in\mathcal{R}_3$ if and only if G is one of the following groups*:

(1) *One of the groups* (1), (3), (5) *and* (7) *in Theorem 2.1*;

(2) *the groups* (2) *in Theorem 3.7*;

(3) *the groups* (3) *in Theorem 3.7 with* $s+t'+u\leqslant 1$.

Proof. ($\Longrightarrow$) By Theorem 3.1(3) we get $\mathcal{R}_3\subseteq\mathcal{R}_2$. Thus G is one of the groups in Theorem 3.7. If G is one of the groups (1)-(2) in Theorem 3.7, then, by Lemma 3.6, we get the groups (1)-(2) in the Theorem. If G is the group (3) in Theorem 3.7, then we will prove $s+t'+u\leqslant 1$. If not, then let $H_1=\langle a^{2^{r+v}}, b\rangle$ and $H_2=\langle a, b^2\rangle$. It is easy to get $|H_1'|=|H_2'|=2$. Hence H_1 and H_2 are of class 2. Since $r \geqslant 2$, H_1 is not maximal in G. On the other hand, H_2 is maximal in G. Hence $|H_1| \neq |H_2|$. This contradicts $G \in \mathcal{R}_3$. So $s + t' + u \leqslant 1$. We get the group (3) in the Theorem.

($\Longleftarrow$) If G is one of the groups (1)-(2), then $G \in \mathcal{R}_3$ by Theorem 3.6. If G is the group (3), then each subgroup K of class 2 of G is minimal non-abelian. It follows by Lemma 2.4 that $|K'| = 2$. It is enough to show each subgroup H of G with $|H'| = 2$ is of the same order. Without loss of generality assume

$$H = \langle b^{j_1} a^{i_1}, b^{j_2} a^{i_2} \rangle,$$

where i_1, i_2, j_1, j_2 are integer numbers. Notice that

$$[a, b^2] = a^{-1} a^{b^2} = a^{(-1+a^{r+v})^2 - 1}.$$

Obviously, $2^{r+v+1} \mid (-1 + a^{r+v})^2 - 1$. Since $s + t' + u \leqslant 1$, $b^2 \in Z(G)$. If $2 \mid j_1$ and $2 \mid j_2$, then H is abelian. This contradicts $|H'| = 2$. Hence $2 \nmid j_1$ or $2 \nmid j_2$. Assume $2 \nmid j_1$ without loss of generality. By calculation we have that there exists k_1 such that $(b^{j_1} a^{i_1})^{j_1^{-1}} = ba^{k_1}$. Then $H = \langle ba^{k_1}, b^{j_2} a^{i_2} \rangle$. Moreover, there exists k_2 such that $(ba^{k_1})^{j_2^{-1}} b^{j_2} a^{i_2} = a^{k_2}$. Thus $H = \langle ba^{k_1}, a^{k_2} \rangle$. Now

$$H' = \langle [ba^{k_1}, a^{k_2}] \rangle = \langle [b, a^{k_2}] \rangle = \langle a^{2k_2} \rangle.$$

On the other hand, since $|H'| = 2$, $H' = \langle a^{2^{r+s+v+t'+u-1}} \rangle$.

Let $n = r + s + v + t' + u$. Then $2k_2 \equiv 2^{n-1} (\mathrm{mod}\ 2^n)$. That is, $k_2 \equiv 2^{n-2} (\mathrm{mod}\ 2^{n-1})$. Hence

$$H = \langle ba^{k_1}, a^{2^{n-2}} \rangle.$$

By calculation we get

$$(ba^{k_1})^2 = b^2 a^{k_1 2^{r+v}} \neq 1, (ba^{k_1})^4 = (b^2 a^{k_1 2^{r+v}})^2 = b^4 a^{k_1 2^{r+v+1}} = b^4.$$

Hence

$$|H| = |\langle ba^{k_1}, a^{2^{n-2}} \rangle| = \frac{|\langle a^{2^{n-2}} \rangle||\langle ba^{k_1} \rangle|}{|\langle a^{2^{n-2}} \rangle \cap \langle ba^{k_1} \rangle|} = \frac{|\langle a^{2^{n-2}} \rangle||\langle b \rangle|}{|\langle a^{2^{n-2}} \rangle \cap \langle b \rangle|}.$$

By the arbitrary of H, the conclusion follows. □

Corollary 3.9. *Suppose that G is a finite non-abelian p-group. Then*

(1) *if G is non-metacyclic, then $G \in \mathcal{R}_2$ if and only if $G \in \mathcal{R}_1$;*

(2) *If G has no minimal non-abelian maximal subgroup, then $G \in \mathcal{R}_3$ if and only if $G \in \mathcal{R}_2$.*

Proof. (1) By Theorem 3.2 and Theorem 3.7, it is enough to check non-metacyclic $\mathcal{A}_2$-groups G with $d(G) \neq 3$ are of class 3. $\mathcal{A}_2$-groups are listed in [11] or [12, Lemma 2.5]. This is a routine work.

(2) It follows by Theorem 3.7, Theorem 3.8 and Lemma 3.5. □

Corollary 3.10. *$\mathcal{Q}_i \subsetneqq \mathcal{R}_i$ and $\mathcal{Q}_i^* \subsetneqq \mathcal{R}_i^*$ for $i = 2, 3$.*

Proof. Let G be a maximal class group of order 3^5 and G have an abelian maximal subgroup. Then $G \in \mathcal{R}_i$ for $i = 2, 3$ by Theorem 3.7 and Theorem 3.8. Thus $G \in \mathcal{R}_i^*$ for $i = 2, 3$. It is obvious that $|Z(G)| = p$. Thus there is a non-abelian subgroup H of order 3^4 of G. By [9, Corollary 8.3.2] we know all non-abelian subgroups of G are of maximal class. Hence $c(H) = 3$. So H is not a minimal non-abelian group by Lemma 2.4. Then $G \notin \mathcal{Q}_2^*$. It follows by $\mathcal{Q}_3^* \subseteq \mathcal{Q}_2^*$ that $G \notin \mathcal{Q}_3^*$. Obviously, $G \notin \mathcal{Q}_i$ for $i = 2, 3$. □

4. An example of a p-group G with $r_2(G) = 3$ and $r_3(G) = 4$

Theorem 3.1(1) means such a fact that $r_2(G) = 2 \iff r_i(G) = 2$ for all i with $2 \leqslant i \leqslant c(G)$. In other words, if $r_2(G) = 2$, then $r_i(G) \leq r_2(G)$ for all i with $3 \leqslant i \leqslant c(G)$. However, if $r_2(G) \geqslant 3$, then the fact is not true. Here we give an example to show that there exists a group G of order 2^8 with $r_2(G) = 3$ and $r_3(G) = 4$. First we give a lemma as follows.

Lemma 4.1. *Let* $G = \langle a, b, c, d \mid a^4 = b^4 = c^4 = 1, d^2 = b^2c^2, [a, b] = [a, c] = 1, [a, d] = [b, d] = a^2, [b, c] = a^2b^2, [c, d] = a^2c^2\rangle$. *Then* $d(H) \leqslant 3$ *for* $H < G$.

Proof. By a simple checking we know that $G \in \mathcal{A}_4$ and $|G| = 2^7$, and

$$\Omega_1(G) = \mho_1(G) = Z(G) = G' \cong C_2^3.$$

It follows that $d(H) \leqslant 3$ if H is abelian. By Lemma 2.4 we get $d(H) = 2$ if $H \in \mathcal{A}_1$. It follows that $d(H) \leqslant 3$ if $H \in \mathcal{A}_2$. So it needs only to show $d(H) \leqslant 3$ for any $\mathcal{A}_3$-subgroup H of G. If not, then there exists $M \in \mathcal{A}_3$ and $d(M) \geqslant 4$. Let $\overline{G} = G/\langle a^2\rangle$. Then $\overline{G} = \langle \bar{a}\rangle \times \langle \bar{b}, \bar{c}, \bar{d}\rangle$, where $\langle \bar{b}, \bar{c}, \bar{d}\rangle$ is a minimal non-metacyclic group of order 2^5. Obviously, all maximal subgroups of $\overline{G}$ are three-generator. It follows that $d(\overline{M}) = 3$. It follows from $d(M) > d(\overline{M})$ that $a^2 \notin \Phi(M)$. Hence $a \notin M$. Thus $M = \langle ba^i, ca^j, da^k, a^2\rangle$, where $i, j, k \in \{0, 1\}$. Let $K = \langle ba^i, ca^j, da^k\rangle$. Since $d(M) \geqslant 4$, $a^2 \notin K$. On the other hand, $[ca^j, da^k](ca^j)^2 = (c^2a^2a^{2j})(c^2a^{2j}) = a^2 \in K$. This is a contradiction. □

Example 4.2. Let $G = \langle a, b, c, d \mid a^8 = b^4 = c^4 = 1, d^2 = a^4b^2c^2, [a, b] = [a, c] = [b, c^2] = 1, [a, d] = [b, d] = a^2, [b, c] = a^2b^2, [c, d] = a^{-2}c^2\rangle$ and H be a non-abelian proper subgroup of G. Then $|G| = 2^8, c(G) = 3, d(G) = 4$ and $d(H) \leqslant 3$.

Proof. Let $K = \langle a, b, c^2 \mid a^8 = b^4 = c^4 = 1, [a, b] = [a, c^2] = [b, c^2] = 1\rangle$. Then $K \cong C_8 \times C_4 \times C_2$. Let

$$M = \langle K, c\rangle = \langle a, b, c \mid a^8 = b^4 = c^4 = 1, [a, b] = [a, c] = [b, c^2] = 1, [b, c] = a^2b^2\rangle.$$

Then M is an extension of K by C_2. It is easy to verify that G is an extension of M by C_2. Thus $|G| = 2^8$.

By calculation we get

$$G' = \mho_1(G) = \Phi(G) = \langle a^2\rangle \times \langle b^2\rangle \times \langle c^2\rangle \cong C_4 \times C_2 \times C_2 \text{ and } G_3 = \langle a^4\rangle \cong C_2,$$

where G_3 is the third term of the lower central series of G. Thus $d(G) = 4$ and $c(G) = 3$.

In following we prove $d(H) \leq 3$. First we have the following facts:

(1) $\Omega_1(G) \cong C_2^3$;

(2) $\Omega_2(C_G(\Omega_1(G)) \cong C_4^2 \times C_2$;

(3) $\mho_2(G) = G_3 = \langle a^4 \rangle \cong C_2$;

(4) $\overline{G} = G/\mho_2(G) \cong L$, where L is the group described in Lemma 4.1.

Assume the conclusion is false. Then there exists $H < G$ such that $d(H) \geqslant 4$. If $\mho_2(G) \nleqslant H$ or $\mho_2(G) \leqslant \Phi(H)$, then it follows by Lemma 4.1 that $d(H) \leqslant 3$. This contradicts $d(H) \geqslant 4$. If $\mho_2(G) \in H\backslash\Phi(H)$, then we may assume $H = K \times \mho_2(G)$. Since $d(H) \geqslant 4$, $d(K) \geqslant 3$. Then K has a normal subgroup N of type $(2, 2)$. It follows from N/C-theorem that $|K : C_K(N)| \leqslant 2$. Notice that $\Omega_1(G) = N \times \mho_2(G)$. Then $\mho_2(G) \nleqslant C_K(N) \leqslant C_G(\Omega_1(G))$. In particular, $C_K(N) \leqslant \Omega_2(C_G(\Omega_1(G))$. From (2) we get $\mho_1(\Omega_2(C_G(\Omega_1(G)))) \cong C_2^2$. Obviously, $\mho_2(G) \leqslant \mho_1(\Omega_2(C_G(\Omega_1(G))))$. Hence $\mho_1(\Omega_2(C_K(N))) \leqslant C_2$. This means $C_K(N) \lesssim C_2 \times C_2$. It follows that $|K| \leqslant 2^4$. From (1) we know H is non-abelian. Hence K is non-abelian. Since $d(K) \geqslant 3$, K has an $\mathcal{A}_1$-subgroup of order 8. Moreover, $K \cong K\mho_2(G)/\mho_2(G) \leqslant \overline{G} \cong L$. This contradicts $L \in \mathcal{A}_4$. □

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