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REVERSIBILITY AND SYMMETRY OVER CENTERS

Kwang-Jin Choi, Tai Keun Kwak, and Yang Lee

Abstract. A property of reduced rings is proved in relation with centers, and our argument in this article is spread out based on this. It is also proved that the Wedderburn radical coincides with the set of all nilpotents in symmetric-over-center rings, implying that the Jacobson radical, all nilradicals, and the set of all nilpotents are equal in polynomial rings over symmetric-over-center rings. It is shown that reduced rings are reversible-over-center, and that given reversible-over-center rings, various sorts of reversible-over-center rings can be constructed. The structure of radicals in reversible-over-center and symmetric-over-center rings is also investigated.

Throughout this note every ring is an associative ring with identity unless otherwise stated. A nilpotent element in a ring is said to be a *nilpotent* for short. Let R be a ring. We denote the center of R by $Z(R)$, and use $N(R)$, $J(R)$, $N_*(R)$, $N^*(R)$, and $W(R)$ to denote the set of all nilpotents, Jacobson radical, lower nilradical (i.e., prime radical), upper nilradical (i.e., the sum of all nil ideals), and the Wedderburn radical (i.e., the sum of all nilpotent ideals) of R, respectively. It is well-known that $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$.

The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n) and use $\mathbb{Q}$ (resp., $\mathbb{R}$) for the field of rational (resp., real) numbers. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $\mathrm{Mat}_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere.

A ring is usually called *reduced* if it has no nonzero nilpotents. Lambek [10] introduced the concept of a symmetric right ideal of a ring, unifying the sheaf representation of commutative rings and reduced rings. Lambek called a right ideal I of a ring R *symmetric* if $abc \in I$ implies $acb \in I$ for all $a, b, c \in R$. If the zero ideal is symmetric, then R is usually called *symmetric*; while Anderson-Camillo [2] used the term ZC_3 for this concept. It is proved by Lambek that a

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ring R is symmetric if and only if $r_1 r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 1$ and $r_i \in R$ for all i (see [10, Proposition 1]). Anderson-Camillo also obtained this result independently in [2, Theorem I.1]. Commutative rings clearly symmetric. Reduced rings are symmetric by [2, Theorem I.3].

Following Cohn [3], a ring R is called *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Anderson-Camillo [2] used the term ZC_2 for the reversible property. It is well-known that $W(R) = N_*(R) = N^*(R) = N(R)$ for a reversible ring R. Symmetric rings are clearly reversible, but not conversely as in [11]. A ring is called *Abelian* if every idempotent is central. Reversible rings are easily shown to be Abelian.

1. Reversible-over-center rings

In this section we deal with a generalization of a symmetric-over-center ring, which shall be said to be *reversible-over-center*. We show that reduced rings are reversible-over-center, and next that the class of reversible-over-center rings is quite large. We investigate the structure of centers of $D_n(R)$ for $n \geq 2$, and apply this to observe the relation between the commutativity of R and the reversible-over-center ring property of $D_3(R)$, where R is a given ring.

We start our study by observing a property of reduced rings.

Theorem 1.1. *Let R be a reduced ring. If $ab \in Z(R)$ for $a, b \in R$, then $ab = ba$.*

Proof. We first claim that the result holds for domains. Let A be a domain and suppose $ab \in Z(A)$ for $a, b \in A$. If a or b is zero, then we are done. So assume that $a, b \in A \backslash \{0\}$. From $ab \in Z(A)$, we get $ab(ab) = a(ab)b$; hence $a(ba - ab)b = 0$ and $ab = ba$ follows.

Next let $\{P_i \mid i \in I\}$ be the set of all minimal prime ideals of R. Then R/P_i is a domain by [13, Proposition 1.11]. Note that R is a subdirect product of R/P_i's, i.e., the homomorphism $\sigma : R \to \prod_{i \in I} R/P_i$, with $r \mapsto (r + P_i)$, is injective. Let $ab \in Z(R)$ for $a, b \in R$. Then $\sigma(ab) = (ab + P_i) = (a + P_i)(b + P_i) \in Z(R/P_i)$ for all i; hence $\sigma(ab) = (a + P_i)(b + P_i) = (b + P_i)(a + P_i) = (ba + P_i) = \sigma(ba)$ by the preceding claim. This yields $ab = ba$.

(Another proof) This is done through a direct computation. Let $ab \in Z(R)$ for $a, b \in R$. Then we obtain

$$(ab - ba)^2 = abab - baab - abba + b(ab)a = abab - baab - abba + (ab)ba = abab - baab$$

and

$$a(ab - ba)^2 = a(abab - baab) = aabab - (ab)aab = aabab - aab(ab) = 0.$$

So we get $ab(ab - ba)^2 = 0$ by [2, Theorem I.3], since R is reduced. Moreover $ba(ab - ba)^2 = 0$. Combining these two equalities, we have $(ab - ba)^3 = 0$. Therefore $ab = ba$ also since R is reduced. $\square$

Following [9], a ring R is called *symmetric-over-center* if $abc \in Z(R)$ for $a, b, c \in R$ implies $acb \in Z(R)$. Symmetric-over-center rings are Abelian by [9, Lemma 2.2(1)], and the symmetric-over-center ring property is left-right symmetric by [9, Proposition 2.1].

Based on the above, we define a generalization of symmetric-over-center rings as follows.

Definition 1.2. *A ring R is called reversible-over-center if $ab \in Z(R)$ for $a, b \in R$ implies $ba \in Z(R)$.*

It is obvious that a ring R is reversible-over-center (resp., symmetric-over-center) if and only if $ab \in Z(R)$ for $a, b \in R\backslash\{0\}$ implies $ba \in Z(R)$ (resp., $abc \in Z(R)$ for $a, b, c \in R\backslash\{0\}$ implies $acb \in Z(R)$). Symmetric-over-center rings are clearly reversible-over-center, but not conversely by Remark 1.4(2,5) to follow.

Notice that Theorem 1.1 is not valid for a reversible-over-center ring which is not reduced. Consider $D_3(A)$ over a commutative ring A. Then $D_3(A)$ is reversible-over-center by Proposition 1.8 below. Observe that $E_{12}E_{23} = E_{13} \in Z(D_3(A))$ by [9, Lemma 1.1(1)], but $E_{23}E_{12} = 0$.

The following contains basic properties of reversible-over-center rings which do roles throughout this note.

Proposition 1.3. (1) *Every reversible-over-center ring is Abelian.*

(2) *Every reduced ring is reversible-over-center.*

(3) *Let R be a reduced ring. If $AB \in Z(D_2(R))$ for $A, B \in D_2(R)$, then $AB = BA$ (hence $D_2(R)$ is reversible-over-center).*

(4) *For a ring R, if $D_n(R)$ for $n = 2, 3$ is reversible-over-center (resp., symmetric-over-center), then R is reversible-over-center (resp., symmetric-over -center).*

(5) *Let R be a reversible-over-center ring. If $ab \in Z(R)$ for $a, b \in R$, then $(ab)^k = (ba)^k = a^k b^k = b^k a^k$ for all $k \geq 2$.*

(6) *Let R be a division ring. Suppose that R is symmetric-over-center. Then, for every $(a, b) \in R^2$, $ab = qba$ for some $q \in Z(R)$.*

(7) *Every free algebra over a commutative domain is symmetric-over-center.*

Proof. (1) Let R be a reversible-over-center ring, and assume on the contrary that there exist $e \in I(R)$ and $a \in R$ such that $ea(1-e) \neq 0$. Since R is reversible-over-center, $[ea(1-e)]e = 0$ implies $ea(1-e) = e[ea(1-e)] \in Z(R)$. This yields $0 = (1-e)[ea(1-e)] = [ea(1-e)](1-e) = ea(1-e) \neq 0$, a contradiction. Therefore R is Abelian.

(2) is shown by Theorem 1.1.

To prove (3) and (4), we use the fact that $Z(D_2(R)) = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \mid x, y \in Z(R)\}$ ([9, Lemma 1.1(2)]), and will use this fact without reference.

(3) Let R be a reduced ring and suppose that $AB \in Z(D_2(R))$ for $A = \left(\begin{smallmatrix} a & a_1 \\ 0 & a \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b & b_1 \\ 0 & b \end{smallmatrix}\right) \in D_2(R)\backslash\{0\}$. Then $ab, ab_1 + a_1 b \in Z(R)$. We get $ab = ba$ from $ab \in Z(R)$, by Theorem 1.1. Since $ab_1 + a_1 b \in Z(R)$, we have $c(ab_1 +$

$a_1b) = (ab_1 + a_1b)c$ for all $c \in \{a, b, a_1, b_1\}$. From these, we obtain

$$\begin{aligned} a(ab_1 + a_1b) &= (ab_1 + a_1b)a = ab_1a + a_1(ba) \\ &= ab_1a + (ba)a_1 = ab_1a + (ab)a_1 = a(ba_1 + b_1a) \end{aligned}$$

and

$$\begin{aligned} (ab_1 + a_1b)b &= b(ab_1 + a_1b) = (ba)b_1 + ba_1b \\ &= b_1(ba) + ba_1b = b_1(ab) + ba_1b = (ba_1 + b_1a)b; \end{aligned}$$

and these yield

$$a[(ab_1 + a_1b) - (ba_1 + b_1a)] = 0 \text{ and } [(ab_1 + a_1b) - (ba_1 + b_1a)]b = 0.$$

Moreover, since R is reduced, we get the following equalities by help of [2, Theorem I.3]:

$$[(ab_1 + a_1b) - (ba_1 + b_1a)]d = 0 \text{ for all } d \in \{ab_1, a_1b, ba_1, b_1a\},$$

entailing

$$[(ab_1+a_1b)-(ba_1+b_1a)]^2 = [(ab_1+a_1b)-(ba_1+b_1a)][(ab_1+a_1b)-(ba_1+b_1a)] = 0.$$

Also since R is reduced, we have $(ab_1 + a_1b) - (ba_1 + b_1a) = 0$. This result implies $AB = BA$.

(4) Suppose that $D_2(R)$ is reversible-over-center and let $ab \in Z(R)$ for $a, b \in R$. Letting $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $\beta = \left(\begin{smallmatrix} b & 0 \\ 0 & b \end{smallmatrix}\right) \in D_2(R)$, we have $\alpha\beta = \left(\begin{smallmatrix} ab & 0 \\ 0 & ab \end{smallmatrix}\right) \in Z(D_2(R))$. Since $D_2(R)$ is reversible-over-center, $\beta\alpha = \left(\begin{smallmatrix} ba & 0 \\ 0 & ba \end{smallmatrix}\right) \in Z(D_2(R))$. This implies that $ba \in Z(R)$, concluding that R is reversible-over-center. A similar argument to this leads to that $D_3(R)$ is reversible-over-center. Moreover, the proof for the case of symmetric-over-center is much the same.

(5) Let $ab \in Z(R)$ for $a, b \in R$. So we obtain $(ab)^2 = ab(ab) = a(ab)b = a^2b^2$. Let $n \geq 2$. Using induction on n, we have

$$(ab)^n = (ab)^{n-1}(ab) = (a^{n-1}b^{n-1})(ab) = a^{n-1}(ab)b^{n-1} = a^nb^n.$$

Next, since R is reversible-over-center, $ba \in Z(R)$ and thus we also obtain $(ba)^n = b^na^n$ by a similar argument to the above.

Observe that $(ab)^2 = a(ba)b = (ba)ab = b(ab)a = (ba)^2$. Using induction on n, we have

$$(ab)^n = (ab)^{n-1}(ab) = (ba)^{n-1}(ab) = (ba)^{n-2}(baab) = (ba)^{n-2}(ba)^2 = (ba)^n.$$

(6) Let R be symmetric-over-center and $a, b \in R$. If $ab \in Z(R)$, then $ab = ba$ by Theorem 1.1. Assume that $ab \notin Z(R)$ and let $c = (ab)^{-1}$. Then $cab = 1$ implies $cba \in Z(R)$ since R is symmetric-over-center. Say $cba = p$. It then follows that $cbap^{-1} = 1$ and $0 = c(ab - bap^{-1})$, entailing $ab = bap^{-1}$. Letting $q = p^{-1}$, the proof is done since $p^{-1} \in Z(R)$.

(7) Let $R = S\langle X\rangle$ be a free algebra generated by a set X over a commutative domain S. If $|X| = 1$, then $R \cong S[x]$, so R is commutative. Consider the case of $|X| \geq 2$. We claim $Z(R) = S$. Let $k \in R$ be non-constant, i.e., $k \notin S$. Then $k \in S\langle X_0\rangle$ for some finite subset X_0 of X, where $S\langle X_0\rangle$ is a free algebra

generated by X_0 over S. Here we can let $X_0 = \{x_1, \dots, x_n\}$ such that every x_i occurs in some monomial of k. Take $m \geq 1$ such that $10^m > n$.

If $X_0 \subsetneq X$, then $ky \neq yk$ for all $y \in X \backslash X_0$.

Suppose $X_0 = X$. Write $k = s_0 + s_1 k_1 + \cdots + s_l k_l$, where $s_j \in S$ and every k_t is a nonempty reduced word in X_0. Next we use the method in [7, Example 14]. Observe that k_t's can be embedded into the set of natural numbers through the corresponding "$x_1 \to 10^m + 1, x_2 \to 10^m + 2, \dots, x_n \to 10^m + n$", σ say. Then they are totally ordered via σ (for example, $x_1 < x_2 < x_1^2 < x_2 x_1 < x_n x_1 < x_n x_2 < x_1^3 \cdots$ because $10^m + 1 < 10^m + 2 < (10^m + 1, 10^m + 1) < (10^m + 2, 10^m + 1) < (10^m + n, 10^m + 1) < (10^m + n, 10^m + 2) < (10^m + 1, (10^m + 1, 10^m + 1)) < \cdots$, where $(\alpha, \beta) = \alpha \cdot 10^{f+1} + \beta$ when $10^{f+1} > \beta \geq 10^f$). We identify k_t with $\sigma(k_t)$, and let $k_1 < x_2 < \cdots < k_l$ after reordering if necessary. Assume first that every x_i occurs in k_l. Then $k_1 x_i < k_2 x_i < \cdots < k_l x_i$ and $x_i k_1 < x_i k_2 < \cdots < x_i k_l$ for all i. Moreover $x_i k_l \neq k_l x_i$ and $x_i k \neq k x_i$. Assume next that some x_i does not occur in k_l, x_s say. Then $k_1 x_s < k_2 x_s < \cdots < k_l x_s$, $x_s k_1 < x_s k_2 < \cdots < x_s k_l$, and $x_s k_l \neq k_l x_s$. This implies $x_s k \neq k x_s$. Consequently $k \notin Z(R)$ and $Z(R) = S$ follows.

Now let $fgh \in Z(R)$ for $f, g, h \in R$. If one or more of f, g, and h are non-constant, then fgh is non-constant and so, by the preceding argument, $fgh \notin Z(R)$. Hence $fgh \in Z(R)$ implies $f, g, h \in S$. Thus R is symmetric-over-center. $\square$

In the following we elaborate upon Proposition 1.3 with related examples.

Remark 1.4. (1) The converse of Proposition 1.3(1) need not hold as follows. Let R be an Abelian ring and consider $D_n(R)$ for $n \geq 4$. Then $D_n(R)$ is Abelian by [6, Lemma 2], but it is not reversible-over-center by Example 1.9(2) to follow.

(2) We recall that the Hamilton quaternions over $\mathbb{R}$, R say. Then R is a division ring. Let $\alpha = a_0 + a_1 i + a_2 j + a_3 k \neq 0$ with $a_i \in \mathbb{R}$ and $\beta = -a_1 - a_0 i + a_3 j - a_2 k$ in R where $a_0 a_2 \neq a_1 a_3$ or $a_0 a_3 \neq a_1 a_2$. Then $\alpha \beta i = \sum_{i=0}^{3} a_i^2 \in Z(R)$ but $\alpha i \beta \notin Z(R)$, showing that R is not symmetric-over-center. Thus $D_2(R)$ is not symmetric-over-center by Proposition 1.3(4), but it is reversible-over-center by Proposition 1.3(3).

(3) We illuminate Proposition 1.3(3) with examples. (i) The converse need not hold as can be seen by the commutative ring $D_2(\mathbb{Z}_4)$. (ii) The proposition need not hold when R is a reversible-over-center ring but not reduced. Let R_0 the Hamilton quaternions over $\mathbb{R}$. Then $R = D_2(R_0)$ is reversible-over-center by Proposition 1.3(3), and $Z(R) = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid a, b \in \mathbb{R}\}$ by [9, Lemma 1.1(2)], since $Z(R_0) = \mathbb{R}$. Consider $S = D_2(R)$. Then $Z(S) = \{\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right) \mid A, B \in Z(R)\}$. Consider

$$\alpha = \begin{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} j & 0 \\ 0 & j \\ 0 & i \\ 0 & 0 \end{pmatrix} \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

in S. Then $\alpha\beta = 0 \in Z(S)$ but

$$\beta\alpha = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 2j \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \notin Z(S), \text{ since } \begin{pmatrix} 0 & 2j \\ 0 & 0 \end{pmatrix} \notin Z(R).$$

(4) The argument in (1) also illuminates that Proposition 1.3(4) cannot be extended to the case of $D_n(R)$ for $n \geq 4$ over any ring R.

(5) We apply Proposition 1.3(6) to show that the Hamilton quaternions R_1 over $\mathbb{R}$ is not symmetric-over-center. Assume on the contrary that R_1 is symmetric-over-center. Then $(1+i)(j+k) = q(j+k)(1+i)$ for some $0 \neq q \in \mathbb{R}$ by Proposition 1.3(6), and this yields $2k = 2qj$, a contradiction. Thus R_1 is not symmetric-over-center.

As another application of Proposition 1.3(6), let R be the first Weyl algebra $R = K[x][y; \partial/\partial x]$ over a field K of characteristic zero, where each polynomial of R is written in the form $\sum y^i a_i$ with $a_i \in K[x]$. Then R is a domain that is simple right Noetherian by [12, Theorem 1.3.5], and so the right quotient ring R_2 of R exists by [12, Corollary 2.1.14 and Theorem 2.1.15]. In fact, R_2 is a division ring. Assume on the contrary that R_2 is symmetric-over-center. Then $xy = qyx$ for some $0 \neq q \in K$ by Proposition 1.3(6). But $xy - yx = 1$, and so $1 = xy - yx = qyx - yx = yx(q-1)$, a contradiction. Thus R_2 is not symmetric-over-center.

(6) Relating to Proposition 1.3(2), the class of reduced rings and the class of symmetric-over-center rings do not imply each other. For example, the Hamilton quaternions over $\mathbb{R}$ and $D_3(R)$ over a commutative ring R which is symmetric-over-center by Proposition 1.8 below.

(7) Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by ab, and set $R = A/I$. Then R is not reversible-over-center since $\bar{a}\bar{b} = 0 \in Z(R)$ and $\bar{b}\bar{a} \notin Z(R)$ (in fact, $0 \neq \bar{b}\bar{a}\bar{a} \neq \bar{a}\bar{b}\bar{a} = 0$).

The free algebra A is symmetric-over-center by Proposition 1.3(7). So the existence of the above ring R also shows that the classes of reversible-over-center rings and symmetric-over-center rings are not closed under homomorphic images.

The non-reversible-over-center factor ring in Remark 1.4(7) is constructed from a free algebra (hence reversible-over-center by Proposition 1.3(7)). One may compare this fact with the following.

Example 1.5. There exists a non-reversible-over-center ring in which every nontrivial factor ring is reversible-over-center. Consider $R = T_2(D)$ over a reduced ring D. Nontrivial factor rings of R are $R/I \cong D \oplus D$, $R/J \cong D$, and $R/K \cong D$ and they are clearly reversible-over-center, where $I = \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right)$, $J = \left(\begin{smallmatrix} D & D \\ 0 & 0 \end{smallmatrix}\right)$, and $K = \left(\begin{smallmatrix} 0 & D \\ 0 & D \end{smallmatrix}\right)$. But R is non-Abelian and so it is not reversible-over-center by Proposition 1.3(1).

Next we consider another example of reversible-over-center ring. Let R be an algebra (with or without identity) over a commutative ring S. Following Dorroh [4], the *Dorroh extension* of R by S is the Abelian group $R \times S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2+s_1r_2+s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$. Denote it by $R \oplus_{dor} S$.

Proposition 1.6. *Let S be a commutative ring and R be a nilpotent ring that is an algebra over S.*

(1) *If $R^2 = 0$, then the Dorroh extension of R by S is commutative.*

(2) *If $R^3 = 0$, then the Dorroh extension of R by S is symmetric-over-center.*

Proof. Write $D = R \oplus_{dor} S$. Let $(r, s) \in Z(D)$. Then $(r'r + sr', 0) = (r', 0)(r, s) = (r, s)(r', 0) = (rr' + sr', 0)$ for all $r' \in R$. This yields $r'r = rr'$, and so $(r, s) \in Z(R) \oplus_{dor} S$, entailing $Z(D) \subseteq Z(R) \oplus_{dor} S$. The converse inclusion is clear, hence $Z(D) = Z(R) \oplus_D S$. We use this fact freely.

(1) Suppose $R^2 = 0$. Then $Z(R) = R$, so $Z(D) = Z(R)\oplus_D S = R\oplus_D S = D$. Thus D is commutative.

(2) Suppose $R^3 = 0$. Let $\alpha\beta\gamma \in Z(D)$ for $\alpha = (r_1, s_1), \beta = (r_2, s_2), \gamma = (r_3, s_3) \in D$. Then, from $\alpha\beta\gamma = (r_1r_2r_3 + s_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 + s_1s_3r_2 + s_2s_3r_1 + s_1s_2r_3, s_1s_2s_3) \in Z(D)$, we get $s_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 + s_1s_3r_2 + s_2s_3r_1 + s_1s_2r_3 \in Z(R)$, noting $r_1r_2r_3 = 0$. Observe that $ab \in Z(R)$ for all $a, b \in R$ since $abr = 0 = rab$ for all $r \in R$. So $s_1r_2r_3 + s_2r_1r_3 + s_3r_1r_2 \in Z(R)$; hence we obtain $s_1s_3r_2+s_2s_3r_1+s_1s_2r_3 \in Z(R)$ from $s_1r_2r_3+s_2r_1r_3+s_3r_1r_2+s_1s_3r_2 + s_2s_3r_1 + s_1s_2r_3 \in Z(R)$.

While, $\alpha\gamma\beta = (s_2r_1r_3 + s_3r_1r_2 + s_1r_3r_2 + s_1s_3r_2 + s_2s_3r_1 + s_1s_2r_3, s_1s_2s_3)$, noting $r_1r_3r_2 = 0$. But $s_2r_1r_3+s_3r_1r_2+s_1r_3r_2$ and $s_1s_3r_2+s_2s_3r_1+s_1s_2r_3$ are contained in $Z(R)$ by the arguments above. Consequently $s_2r_1r_3 + s_3r_1r_2 + s_1r_3r_2 + s_1s_3r_2 + s_2s_3r_1 + s_1s_2r_3 \in Z(R)$, and $(r_1r_3r_2 + s_2r_1r_3 + s_3r_1r_2 + s_1r_3r_2+s_1s_3r_2+s_2s_3r_1+s_1s_2r_3, s_1s_2s_3) \in Z(D)$ follows; entailing $\alpha\gamma\beta \in Z(D)$. Therefore D is symmetric-over-center. □

Let R be a ring and $n \geq 2$. Note that both $\mathrm{Mat}_n(R)$ and $T_n(R)$ cannot be reversible-over-center for all $n \geq 2$ over any ring R, by Proposition 1.3(1). So we consider the ring $D_n(R)$. Write $N_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ii} = 0 \text{ for all } i\}$. The center of $D_3(R)$ is $\left\{\left(\begin{smallmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & a \end{smallmatrix}\right) \mid a, b \in Z(R)\right\}$ by [9, Lemma 1.1(1)], where R is a given ring. We extend this result to the general case, i.e., $D_n(R)$ for $n \geq 4$.

Proposition 1.7. *Let R be a ring and $n \geq 2$. The center of $D_n(R)$ is*

$$\left\{ \begin{pmatrix} a & 0 & 0 & \cdots & 0 & b \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix} \mid a, b \in Z(R) \right\}.$$

Proof. We extend the method in the proof of [9, Lemma 1.1(1)] to the case of $n \geq 4$. Let $D = D_n(R)$ and first observe that the subring

$$\left\{ \begin{pmatrix} b & 0 & 0 & \cdots & 0 & c \\ 0 & b & 0 & \cdots & 0 & 0 \\ 0 & 0 & b & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b & 0 \\ 0 & 0 & 0 & \cdots & 0 & b \end{pmatrix} \mid b, c \in Z(R) \right\}$$

of R is contained in $Z(D)$.

Let $M = (a_{ij}) \in Z(D)$ with $a_{ii} = a$ for all i. Consider arbitrary matrix $(r_{ij}) \in R$ with $r = r_{ii}$ for all i. Then r is arbitrary in R, and $(r_{ij})M = M(r_{ij})$ implies $ra = ar$. This implies $a \in Z(R)$ because r is arbitrary in R. Furthermore $N = M - (aI_n)$ is contained in $Z(D) \cap N_n(R)$, where I_n is the identity matrix in R. We use this fact freely.

Let $s \in \{2, 3, \ldots, n-1\}$. Then the $(1, n)$-entry of NE_{sn} is a_{1s}, and the $(1, n)$-entry of $E_{sn}N = 0$ is zero. But $NE_{sn} = E_{sn}N$, forcing $a_{1s} = 0$ for all s.

Let $t \in \{3, 4, \ldots, n-1\}$. Then the $(2, n)$-entry of NE_{tn} is a_{2t}, and the $(2, n)$-entry of $E_{tn}N = 0$ is zero. But $NE_{tn} = E_{tn}N$, forcing $a_{2t} = 0$ for all t.

Proceeding in this manner, we finally obtain that $a_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n-1\}$ with $i \neq j$; and

$$N = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & 0 & 0 & \cdots & 0 & a_{2n} \\ 0 & 0 & 0 & \cdots & 0 & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Next, the $(1, n)$-entry of $E_{12}N$ is a_{2n}, and the $(1, n)$-entry of $NE_{12} = 0$ is zero. But $E_{12}N = NE_{12}$, forcing a_{2n}. Similarly we can obtain $a_{kn} = 0$ for all $k = 3, \ldots, n-1$ by using $E_{1k}N = NE_{1k} = 0$.

Consequently, we get

$$M = \begin{pmatrix} a & 0 & 0 & \cdots & 0 & a_{1n} \\ 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & 0 \\ 0 & 0 & 0 & \cdots & 0 & a \end{pmatrix}.$$

Next we have

$$\begin{aligned} (r_{ij})aI_n + (ra_{1n})E_{1n} &= (r_{ij})aI_n + (r_{ij})a_{1n}E_{1n} = (r_{ij})[aI_n + a_{1n}E_{1n}] = (r_{ij})M \\ &= M(r_{ij}) = [aI_n + a_{1n}E_{1n}](r_{ij}) = aI_n(r_{ij}) + a_{1n}E_{1n}(r_{ij}) \end{aligned}$$

$$= aI_n(r_{ij}) + (a_{1n}r)E_{1n}.$$

Since $a \in Z(R)$, $(r_{ij})aI_n = aI_n(r_{ij})$ and so $(ra_{1n})E_{1n} = (a_{1n}r)E_{1n}$. This yields $a_{1n}r = ra_{1n}$; hence $a_{1n} \in Z(R)$. This completes the proof. $\square$

In fact, the center of $D_n(R)$ is isomorphic to $D_2(Z(R))$ by Proposition 1.7. The following is an extension of [9, Proposition 1.2].

Proposition 1.8. *For a ring R the following conditions are equivalent*:
(1) *R is a commutative ring*;
(2) *$D_3(R)$ is a symmetric-over-center ring*;
(3) *$D_3(R)$ is a reversible-over-center ring.*

Proof. (1) $\Rightarrow$ (2) is proved by Proposition 1.6(2), noting that $T = \left(\begin{smallmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is an algebra over R, $T^3 = 0$, and $D_3(R) \cong T \oplus_{dor} R$. (2) $\Rightarrow$ (3) is obvious. For the proof of (3) $\Rightarrow$ (1), let $D_3(R)$ be reversible-over-center and assume on the contrary that A is noncommutative. Take $a \notin Z(R)$.

Let $M_1 = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $M_2 = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ in $D_3(R)$. Then $M_1M_2 = 0 \in Z(D_3(R))$. But $M_2M_1 = \left(\begin{smallmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \notin Z(D_3(R))$ by Proposition 1.7 because $a \notin Z(R)$. This is contrary to $D_3(R)$ being reversible-over-center. Thus R is commutative. $\square$

The direction (1) $\Rightarrow$ (2) in Proposition 1.8 is also proved by [9, Proposition 1.2]. Considering Proposition 1.8, it is natural to conjecture that if $D_2(R)$ is symmetric-over-center, then R is commutative, and that $D_n(R)$ is also reversible-over-center for $n \geq 4$. However the following provides counterexamples for these.

Example 1.9. (1) Let R be a noncommutative free algebra over a commutative domain S. Then $Z(R) = S$ by the proof of Proposition 1.3(7). Let $ABC \in Z(D_2(R))$ for $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}) \in D_2(R)$. Then $a_{ii}b_{ii}c_{ii} \in Z(R)$ and $a_{11}b_{11}c_{12}+a_{11}b_{12}c_{11}+a_{12}b_{11}c_{11} \in Z(R)$ by Proposition 1.7. But $a_{ii}b_{ii}c_{ii} \in Z(R)$ implies $a_{ii}, b_{ii}, c_{ii} \in S$, since $Z(R) = S$. Then $a_{11}b_{11}c_{12} + a_{11}b_{12}c_{11} + a_{12}b_{11}c_{11} = a_{11}c_{11}b_{12}+a_{11}c_{12}b_{11}+a_{12}c_{11}b_{11}$; hence $ABC = ACB$. Thus $D_2(R)$ is symmetric-over-center.

(2) Let A be any ring. We first consider the case of $D_4(A)$. Let

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

be matrices in $D_4(A)$. Then

$$M_1M_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in Z(D_4(A)), \text{ but } M_2M_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin Z(D_4(A))$$

by Proposition 1.7. Thus $D_4(A)$ is not reversible-over-center. This computation can be applicable to show that $D_n(A)$ is not reversible-over-center for $n \geq 5$.

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor). A multiplicatively closed subset S of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. It is shown, by [12, Theorem 2.1.12], that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring of R with respect to S exists. Notice that let R be a right Ore ring and R_S be the right quotient ring of R. Then every element in R_S is expressed by rs^{-1} with $r \in R$ and $s \in S$. Let $rs^{-1} \in Z(R_S)$. Then $r = (rs^{-1})s = s(rs^{-1})$ yields $rs = sr$ (equivalently, $rs^{-1} = s^{-1}r$).

Given reversible-over-center rings, we can construct other kinds of reversible-over-center rings via right quotient rings.

Theorem 1.10. (1) *Let S be multiplicatively closed subset of a ring R, and suppose that S satisfies the right Ore condition and S consists of regular elements. If R_S is reversible-over-center, then so is R.*

(2) *Let R be a ring and S consists of central regular elements in R. Then R is reversible-over-center if and only if so is R_S.*

Proof. (1) It is easily checked that $Z(R) \subseteq Z(R_S)$. Suppose that R_S is reversible-over-center. Let $ab \in Z(R)$ for $a, b \in R$. Then $ab \in Z(R_S)$. Since R_S is reversible-over-center, $ab \in Z(R_S)$ implies $ba \in Z(R_S)$. But $ba \in R$, so $ba \in Z(R) = R \cap Z(R_S)$. Thus R is reversible-over-center.

(2) Note first $Z(R_S) = S^{-1}Z(R)$. Assume that R is reversible-over-center, and let $\alpha\beta \in Z(R_S)$ for $\alpha = u^{-1}a, \beta = v^{-1}b$ with $u, v \in S$ and $a, b \in R$. Since $\alpha\beta = (uv)^{-1}(ab) \in Z(R_S)$, we have $ab \in Z(R)$ and so $ba \in Z(R)$ by assumption. Thus $\beta\alpha = v^{-1}bu^{-1}a = (uv)^{-1}(ba) \in S^{-1}Z(R) = Z(R_S)$. Therefore R_S is reversible-over-center. The proof of the converse is almost same as one of (1). □

Let R be a ring and consider $R[x]$. Since $S = \{x^n \mid n = 0, 1, 2, \ldots\}$ is a right Ore subset of $R[x]$, there exists the right quotient ring $R[x]_S$. $R[x]_S$ is denoted by $R[x; x^{-1}]$ and said to be the *Laurent polynomial* ring in x over R. Since S is a multiplicatively closed subset of $R[x]$ and $R[x; x^{-1}] = S^{-1}R[x]$, we get the following by Theorem 1.10.

Corollary 1.11. *Let R be a ring. Then $R[x]$ is reversible-over-center if and only if so is $R[x; x^{-1}]$.*

We also obtain other elementary properties for reversible-over-center rings.

Proposition 1.12. *Let $\{R_\gamma \mid \gamma \in \Gamma\}$ be a family of rings. Then R_γ is reversible-over-center for every $\gamma \in \Gamma$ if and only if the direct product $R = \prod_{\gamma\in\Gamma} R_\gamma$ of R_γ's is reversible-over-center.*

Proof. Note first $Z(R) = \prod_{\gamma\in\Gamma} Z(R_\gamma)$. Suppose that $R = \prod_{\gamma\in\Gamma} R_\gamma$ is reversible-over-center, and let $ab \in Z(R_\gamma)$ for $a, b \in R_\gamma$. Consider two sequences $\alpha = (a_\gamma)_{\gamma\in\Gamma}, \beta = (b_\gamma)_{\gamma\in\Gamma} \in R$ such that $a_\gamma = a$, $b_\gamma = b$, and $a_\delta = 0$, $b_\delta = 0$ for all $\delta \neq \gamma$. Then $\alpha\beta \in Z(R)$. Since R is reversible-over-center, $\beta\alpha \in Z(R)$. This implies $ba \in Z(R_\gamma)$.

Conversely, suppose that every R_γ is reversible-over-center, and let $\alpha = (a_\gamma)_{\gamma\in\Gamma}, \beta = (b_\gamma)_{\gamma\in\Gamma} \in R$ with $\alpha\beta \in Z(R)$. Then $a_\gamma b_\gamma \in Z(R_\gamma)$ for all $\gamma \in \Gamma$. Since R_γ is reversible-over-center, we get $b_\gamma a_\gamma \in Z(R_\gamma)$. This implies that $\beta\alpha \in Z(R)$, concluding that $R = \prod_{\gamma\in\Gamma} R_\gamma$ is reversible-over-center. □

The following comes from Proposition 1.12, since $R = eR \oplus (1-e)R$ for $e^2 = e \in Z(R)$.

Corollary 1.13. *Let R be a ring and $e^2 = e \in Z(R)$. Then R is reversible-over-center if and only if both eR and $(1-e)R$ are reversible-over-center.*

2. Properties related to radicals

In this section we study the structure of nilradicals in symmetric-over-center rings and reversible-over-center rings. By help of these results we are able to provide useful information to the structure of radicals of polynomial rings.

Lemma 2.1 ([9, Proposition 2.1])**.** *A ring R is symmetric-over-center if and only if $a_1a_2\cdots a_n \in Z(R)$ for $a_1, \ldots, a_n \in R$ implies $a_{\theta(1)}a_{\theta(2)}\cdots a_{\theta(n)} \in Z(R)$ for any permutation θ of the set $\{1, 2, \ldots, n\}$, where n is any positive integer.*

We extend the facts in [9, Proposition 2.4] to the general cases by help of Lemma 2.1.

Theorem 2.2. *Let R be a symmetric-over-center ring. Then we have the following*:

(1) *If $a^n = 0$ for $a \in R$ and $n \geq 2$, then $r_0ar_1ar_2a\cdots ar_{n-1}ar_n \in Z(R)$ for all $r_0, r_1, \ldots, r_n \in R$.*

(2) *If $a \in N(R)$, then RaR is nilpotent.*

(3) $W(R) = N_*(R) = N^*(R) = N(R)$.

(4) $J(R[x]) = W(R[x]) = N_*(R[x]) = N^*(R[x]) = N(R[x]) = N(R)[x] = W(R)[x]$, *and especially $R[x]/J(R[x])$ is a reduced ring.*

Proof. (1) Let $a^n = 0$ for $a \in R$ and $n \geq 2$. Then $a^nr_0r_1\cdots r_n = 0 \in Z(R)$ for all $r_0, r_1, \ldots, r_n \in R$. Since R is symmetric-over-center, $r_0ar_1ar_2a\cdots r_{n-1}ar_n \in Z(R)$ by Lemma 2.1.

(2) Suppose that $a^n = 0$ for $a \in R$ and $n \geq 2$. Since R is symmetric-over-center, we have

$$r_{i0}ar_{i1}ar_{i2}a\cdots ar_{i(n-1)}ar_{in} \in Z(R)$$

for all $r_{i0}, r_{i1}, \ldots, r_{in} \in R$ by (1), where i is any in $\{1, 2, \ldots\}$. Let

$$b_i = r_{i0}ar_{i1}ar_{i2}a\cdots ar_{i(n-1)}ar_{in}$$

for $i = 1, 2, \ldots, n-1$. Then $b_i \in Z(R)$ and we obtain

$$\begin{aligned}
0 &= s_1 a^n (b_1 b_2 \cdots b_{n-1}) s_2 = s_1 a b_1 a b_2 a \cdots a b_{n-1} a s_2 \\
&= s_1 a (r_{10} a r_{11} a r_{12} a \cdots a r_{1n-1} a r_{1n}) a (r_{20} a r_{21} a r_{22} a \cdots a r_{2(n-1)} a r_{2n}) a \\
&\quad \cdots a (r_{(n-1)0} a r_{(n-1)1} a r_{(n-1)2} a \cdots a r_{(n-1)(n-1)} a r_{(n-1)n}) a s_2 \\
&= [s_1 a r_{10} a r_{11} a r_{12} a \cdots a r_{1(n-1)}][a r_{1n}][a r_{20} a r_{21} a r_{22} a \cdots a r_{2(n-1)}][a r_{2n}] \\
&\quad \cdots [a r_{(n-1)0} a r_{(n-1)1} a r_{(n-1)2} a \cdots a r_{(n-1)(n-1)}][a r_{(n-1)n}][a s_2],
\end{aligned}$$

where $s_1, s_2 \in R$. Any element in

$$\begin{aligned}
(RaR)^n (RaR)(RaR)^n (RaR) \cdots (RaR)^n (RaR)(RaR) &= [(RaR)^n]^{n-1} (RaR)^n \\
&= (RaR)^{n^2}
\end{aligned}$$

is a finite sum of elements of the form in the preceding equality. This yields $(RaR)^{n^2} = 0$.

(3) Let $a \in N(R)$. Then RaR is nilpotent by (2), and $a \in W(R)$ follows. This completes the proof, considering the inclusion $W(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$.

(4) By (3), we have $W(R) = N_*(R) = N^*(R) = N(R)$. The remainder of the proof is similar to one of [9, Proposition 2.4]. □

Let $R = D_3(A)$ over a commutative ring A. Then R is a symmetric-over-center ring by Proposition 1.8, and note

$$W(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a \in N(A) \text{ and } b, c, d \in A \right\} = N(R).$$

This provides us an example of Theorem 2.2(3).

In the following we see similar results to Theorem 2.2 for reversible-over-center rings. For a ring R and $k \geq 2$, write $N_k(R) = \{a \in N(R) \mid a^k = 0\}$.

Theorem 2.3. *For a reversible-over-center ring R, we have the following results.*

(1) *If $ab = 0$ for $a, b \in R$, then $bRa \subseteq Z(R)$.*

(2) *If $ab = 0$ for $a, b \in R$, then $(RaRbR)^2 = 0$ and $(RbRaR)^2 = 0$. Especially, if $a^2 = 0$, then $(RaR)^3 = 0$.*

(3) *Suppose that $aRa \subseteq Z(R)$ for all $a \in N(R)$. Then*

$$W(R) = N_*(R) = N^*(R) = N(R).$$

(4) *Suppose that $aRa \subseteq Z(R)$ for all $a \in N(R)$. Then*

$$J(R[x]) = W(R[x]) = N_*(R[x]) = N^*(R[x]) = W(R)[x] = N(R)[x] = N(R[x]),$$

and $R[x]/J(R[x])$ is a reduced ring.

(5) *Suppose $N(R) = N_2(R)$. Then*

$$W(R) = N_*(R) = N^*(R) = N(R)$$

and

$$J(R[x]) = W(R[x]) = N_*(R[x]) = N^*(R[x]) = W(R)[x] = N(R)[x] = N(R[x]).$$

Proof. (1) Let $ab = 0$ for $a, b \in R$. Then $abr = 0 \in Z(R)$ for all $r \in R$. Since R is reversible-over-center, $bra \in Z(R)$ and so $bRa \subseteq Z(R)$.

(2) Let $ab = 0$ for $a, b \in R$. Then $bRa \subseteq Z(R)$ by (1). This yields that $ras(bta)ubv = ra(bta)subv = 0$ for any $r, s, t, u, v \in R$. Thus $(RaRbR)^2 = 0$. Similarly we have $(RbRaR)^2 = 0$. Next let $a^2 = 0$, and use a in place of b in the preceding argument. Then $aRa \subseteq Z(R)$ and $ras(ata)u = ra(ata)su = 0$ for any $r, s, t, u \in R$, entailing $(RaR)^3 = 0$.

(3) Assume that $aRa \subseteq Z(R)$ for all $a \in N(R)$. Let $a \in N(R)$ with $a^n = 0$ for $n \geq 2$. By assumption, we obtain that for any $r_1, r_2, \ldots, a_{2n-1}, a_{2n} \in R$,

$$\begin{aligned}
&r_1ar_2ar_3a \cdots r_{2n-2}ar_{2n-1}ar_{2n} \\
&= r_1a(ar_{2n-1}a)r_2ar_3a \cdots r_{2n-3}ar_{2n-2}r_{2n} \\
&= r_1a^2r_{2n-1}ar_2ar_3a \cdots r_{2n-4}(ar_{2n-3}a)r_{2n-2}r_{2n} \\
&= r_1a^3r_{2n-3}ar_{2n-1}ar_2ar_3a \cdots r_{2n-6}(ar_{2n-5}a)r_{2n-4}r_{2n-2}r_{2n} \\
&\vdots \\
&= r_1a^nr_3ar_5 \cdots r_{2n-1}ar_2r_4 \cdots r_{2n-2}r_{2n} = 0,
\end{aligned}$$

entailing $(RaR)^{2n-1} = 0$. Therefore $a \in W(R)$, and this completes the proof.

(4) Let $aRa \subseteq Z(R)$ for all $a \in N(R)$. Then we have $W(R) = N_*(R) = N^*(R) = N(R)$ by (3). The remainder of the proof is the same as the proof of Theorem 2.2(4).

(5) The proof is done by (1), (3), and (4). □

We illustrate Theorem 2.3 with examples.

Example 2.4. (1) Related to Theorem 2.3(1,3), there exists a reversible-over-center ring R such that $aRa \subseteq Z(R)$ for $a \in N(R)$ with $a^2 \neq 0$. Consider $R = D_3(A)$ over a commutative ring A. Then R is a reversible-over-center ring with

$$Z(R) = \left\{ \begin{pmatrix} x & 0 & y \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x, y \in A \right\}$$

by Propositions 1.7 and 1.8. If we take $a = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in R$, then $aRa = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in Z(R)$ for any $c \in A$, but $a^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0$ and $a^3 = 0$.

(2) There exists a symmetric-over-center (hence reversible-over-center) ring R such that $aRa \nsubseteq Z(R)$ for some $a \in N(R)$. Let $A = \mathbb{Z}_8$ and consider the ring $R = D_3(A)$. Then R is symmetric-over-center by Theorem 1.8. Let $a = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \in N(R)$ and $r = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in R$. Then $ara = \begin{pmatrix} 4 & 4 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{pmatrix} \in aRa$ but $ara \notin Z(R)$.

Notice that a reversible-over-center ring R, such that $aRa \subseteq Z(R)$ for all $a \in N(R)$, need not be symmetric-over-center as can be seen by the Hamilton quaternions over $\mathbb{R}$ which is reversible-over-center by Proposition 1.3(4) but not symmetric-over-center by Remark 1.4(2).

Recall that a ring R is said to be *semiprime* (resp., *semiprimitive*) if $N_*(R) = 0$ (resp., $J(R) = 0$). Reduced rings are clearly semiprime. Following [5], a ring R is said to be *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Every von Neumann regular ring is clearly semiprimitive (hence semiprime). Observe that for a given ring R, $N(R) = 0$ is equivalent to $N_2(R) = 0$.

Corollary 2.5. (1) *Let R be a reversible-over-center ring. If $N(R) \neq 0$, then $W(R) \neq 0$.*

(2) *For a semiprime ring R, the following conditions are equivalent*: (i) *R is reduced*; (ii) *R is symmetric*; (iii) *R is reversible*; (iv) *R is reversible-over-center.*

(3) *For a von Neumann regular ring R, the following conditions are equivalent*: (i) *R is reduced*; (ii) *R is symmetric*; (iii) *R is reversible*; (iv) *R is reversible-over-center*; (v) *R is Abelian.*

(4) *If a ring R is semiprime and reversible-over-center, then $R[x]$ is semiprimitive.*

(5) *Let R be a semiprime ring. Then the following conditions are equivalent*: (i) *R is reversible-over-center*; (ii) *$R[x]$ is reduced*; (iii) *$R[x]$ is symmetric*; (iv) *$R[x]$ is reversible*; (v) *$R[x]$ is reversible-over-center.*

Proof. (1) Let $N(R) \neq 0$. Then $N_2(R) \neq 0$, and $(RaR)^3 = 0$ for all $0 \neq a \in N_2(R)$ by Theorem 2.3(2). This implies $W(R) \neq 0$.

(2) Recall that $N_*(R) = N^*(R) = N(R)$ when R is a reversible ring. So the conditions (i), (ii), and (iii) are equivalent since R is semiprime. (i) $\Rightarrow$ (iv) is proved by Proposition 1.3(2). Let R be reversible-over-center, and assume on the contrary that $N(R) \neq 0$ (equivalently, $N_2(R) \neq 0$). Then $N_*(R) \neq 0$ by (1), contrary to $N_*(R) = 0$. Thus $N(R) = 0$, proving (iv) $\Rightarrow$ (i).

(3) The equivalent relations among the conditions (i), (ii), (iii), and (iv) are proved by (2) since a von Neumann regular ring is semiprime. (iv) $\Rightarrow$(v) comes from Proposition 1.3(1), and (v) $\Rightarrow$ (i) is shown by [5, Theorem 3.2].

(4) Let R be semiprime and reversible-over-center. Then R is reduced by (2). Hence $R[x]$ is semiprimitive by [1, Theorem 1].

(5) A ring R is semiprime if and only if $R[x]$ is semiprime by [1, Theorem 3]. So the result is obtained from (2), noting that R is reduced if and only if so is $R[x]$. $\square$

The condition "R is Abelian" need not be contained in Corollary 2.5(2). We use the argument in [8, Theorem 2.2(2)]. Let S be a reduced ring and define a map $\sigma : D_n(S) \to D_{n+1}(S)$ by $B \mapsto \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, where $n \geq 1$. Then $D_n(S)$ can be considered as a subring of $D_{n+1}(S)$ via σ (i.e., $B = \sigma(B)$ for

$B \in D_n(S)$). Set $R = \cup_{n=1}^{\infty} D_n(S)$. Then R is a semiprime ring by [8, Theorem 2.2(2)], and moreover R is Abelian by applying [6, Lemma 2]. However R is not reversible-over-center by Example 1.9(2). Observe next that the Hamilton quaternions over $\mathbb{R}$ is both von Neumann regular and reversible-over-center, but not symmetric-over-center as noted above. Corollary 2.5(3) is illuminated with this argument, when the condition of symmetric-over-center is considered.

Considering Corollary 2.5(4), one may ask whether a reversible-over-center ring is semiprimitive when it is semiprime. The answer is negative as can be seen by the power series ring $R[[x]]$ over a commutative domain R, noting that $R[[x]]$ is commutative (hence symmetric-over-center) and $J(R[[x]]) = J(R) + xR[[x]]$.

We do not know any example of symmetric-over-center (or reversible-over-center) ring R over which $R[x]$ is not reversible-over-center. But we provide an information which may be helpful to think of this problem. For a ring R, it is easily checked that $Z(R[x]) = Z(R)[x]$.

Remark 2.6. Let R be a symmetric-over-center ring. We claim that if $(a_0 + a_1x)(b_0 + b_1x) \in Z(R[x])$ for $a_0 + a_1x, b_0 + b_1x \in R[x]$, then $(a_0^{2^k} + a_1^{2^k}x)(b_0^{2^k} + b_1^{2^k}x) \in Z(R[x])$ for any $k \geq 0$.

To see this suppose $(a_0+a_1x)(b_0+b_1x) \in Z(R[x])$ for $a_0+a_1x, b_0+b_1x \in R[x]$. Then $a_0b_0, a_1b_1, a_0b_1 + a_1b_0 \in Z(R)$. Write $a_0b_1 + a_1b_0 = q$ with $q \in Z(R)$. Multiplying $a_0b_1 + a_1b_0 = q$ by a_0 on the left and b_1 on the right (resp., by a_1 on the left and b_0 on the right), we have $a_0a_0b_1b_1 + a_0a_1b_0b_1 = qa_0b_1$ (resp., $a_1a_0b_1b_0 + a_1a_1b_0b_0 = qa_1b_0$). Adding these, we obtain

$$(*) \qquad a_0^2b_1^2 + a_1^2b_0^2 = q(a_0b_1 + a_1b_0) - a_0a_1b_0b_1 - a_1a_0b_1b_0.$$

Since $a_0b_0, a_1b_1 \in Z(R)$, we get $a_0a_1b_0b_1, a_1a_0b_1b_0 \in Z(R)$ by Lemma 2.1. Then we obtain $a_0^2b_1^2+a_1^2b_0^2 \in Z(R)$ from the equality $(*)$, since $q(a_0b_1+a_1b_0) \in Z(R)$. Moreover, by Lemma 2.1, we get $a_0^2b_0^2, a_1^2b_1^2 \in Z(R)$ from $a_0b_0, a_1b_1 \in Z(R)$. Thus $(a_0^2 + a_1^2x)(b_0^2 + b_1^2x) \in Z(R[x])$.

Next we repeat the preceding argument by using a_i^2 and b_j^2 in place of a_i and b_j, respectively. Then we obtain $(a_0^{2^2} + a_1^{2^2}x)(b_0^{2^2} + b_1^{2^2}x) \in Z(R[x])$. Proceeding inductively we finally have $(a_0^{2^k} + a_1^{2^k}x)(b_0^{2^k} + b_1^{2^k}x) \in Z(R[x])$ for any $k \geq 0$.

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Kwang-Jin Choi
Smith Liberal Arts College
Sahmyook University
Seoul 01795, Korea
Email address: `kjchoi@syu.ac.kr`

Tai Keun Kwak
Department of Mathematics
Daejin University
Pocheon 11159, Korea
Email address: `tkkwak@daejin.ac.kr`

Yang Lee
Department of Mathematics
Yanbian University
Yanji 133002, P. R. China
and
Institute of Basic Science
Daejin University
Pocheon 11159, Korea
Email address: `ylee@pusan.ac.kr`