

SLANT H-TOEPLITZ OPERATORS ON THE HARDY SPACE

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ABSTRACT. The notion of slant H-Toeplitz operator V_ϕ on the Hardy space H^2 is introduced and its characterizations are obtained. It has been shown that an operator on the space H^2 is a slant H-Toeplitz if and only if its matrix is a slant H-Toeplitz matrix. In addition, the conditions under which slant Toeplitz and slant Hankel operators become slant H-Toeplitz operators are also obtained.

1. Introduction

Let μ denote the normalised Lebesgue measure on the unit circle \mathbb{T} and the space L^2 be the space of all complex valued square integrable measurable functions on \mathbb{T} . The space L^2 is a Hilbert space with the norm $\|\cdot\|_2$ induced by the inner product

$$\langle f, g \rangle = \int f \bar{g} d\mu \quad \text{for all } f, g \in L^2.$$

For each integer n , let $e_n(z) = z^n$ for $z \in \mathbb{T}$. Then the collection $\{e_n\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for L^2 , where \mathbb{Z} denote the set of integers. The Hardy space is defined by

$$H^2 = \left\{ f : f \text{ is analytic on } \mathbb{D} \text{ and } \|f\|^2 = \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty \right\},$$

where $d\theta$ is Lebesgue arc-length measure on the unit circle. Alternatively on the unit circle, the Hardy space is given by

$$H^2 = \left\{ f = \sum_{n=-\infty}^{\infty} a_n e_n \in L^2 : a_n = \langle f, e_n \rangle = 0 \text{ for all } n < 0 \right\}.$$

The space H^2 being a closed subspace of L^2 is a Hilbert space under the norm

$$\|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} < \infty \quad \text{where } f = \sum_{n=0}^{\infty} a_n e_n.$$

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The space L^∞ denotes the Banach space of all essentially bounded measurable functions with norm given by $\|\phi\|_\infty = \text{ess sup } \{|\phi(z)| : z \in \mathbb{T}\}$. Let $\mathcal{B}(L^2)$ and $\mathcal{B}(H^2)$ denote the space of all bounded linear operators on the spaces L^2 and H^2 respectively. Let P denote the orthogonal projection from the space L^2 to the space H^2 . For a given $\phi \in L^\infty$, the induced multiplication operator $M_\phi : L^2 \rightarrow L^2$ is defined as $M_\phi f = \phi f$ for each $f \in L^2$ and the Toeplitz operator is the operator $T_\phi \in \mathcal{B}(H^2)$ such that $T_\phi = PM_\phi|_{H^2}$. For the symbol $\phi \in L^\infty$, Hankel operator $H_\phi \in \mathcal{B}(H^2)$ is defined as the operator $H_\phi = PM_\phi J$, where J , the flip operator, is the operator $J : H^2 \rightarrow (H^2)^\perp$ given by $J(e_n) = e_{-n-1}$ for all $n \geq 0$. The slant Toeplitz operator [4, 8, 9] with the symbol ϕ is defined as the operator $A_\phi \in \mathcal{B}(L^2)$ such that $A_\phi = WM_\phi$, where the operator W defined on L^2 is given by

$$W(e_n) = \begin{cases} e_{\frac{n}{2}}, & \text{if } n \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

for each integer n and its adjoint is given by $W^*(e_n) = e_{2n}$. The compression of slant Toeplitz operator [13] to the space H^2 is the operator B_ϕ defined by $B_\phi = PA_\phi|_{H^2}$. The slant Hankel operator [5, 14] on the space H^2 is given by L_ϕ such that $L_\phi = WH_\phi$. For a non-constant analytic function ϕ , the composition operator [2, 3] is the operator C_ϕ defined on H^2 such that $C_\phi(f)(z) = f(\phi(z))$ for each $f \in H^2$ and $z \in \mathbb{T}$.

The study of slant Toeplitz operators has gained voluminous importance due to its multidirectional applications as these classes of operators have played major role in wavelet analysis, dynamical system and in curve and surface modelling [6, 7, 10–12]. The study of Hankel and slant Hankel operators has numerous applications, in interpolation problems, Hamburger's moment problem, rational approximation theory and stationary process. In 2007, Arora et al. [1] introduced and studied the notion of H-Toeplitz operators on the space H^2 . The H-Toeplitz system consists matrix equation of the form $Ax = b$, where A is an $n \times n$ H-Toeplitz matrix and $x, b \in \mathbb{C}^n$. It can be noted that the $n \times n$ H-Toeplitz matrix A has $2n - 1$ degree of the freedom rather than n^2 and therefore in this case, it is easier to solve the system of linear equations. Motivated by these studies, we have introduced the notion of slant H-Toeplitz operator on the Hardy space H^2 and studied its basic properties. The importance of the notion of slant H-Toeplitz operators is that it itself is not a slant Toeplitz or slant Hankel operator but under some conditions it coincides with the classes of slant Toeplitz and slant Hankel operators on the Hardy space.

The article is organized as follows. In Section 2, we have defined the notion of slant H-Toeplitz operators on the Hardy space H^2 and obtained the conditions under which the class of slant H-Toeplitz operators become isometry, compact and hyponormal. In Section 3, we have obtained characterizations for an operator to be slant H-Toeplitz operator on the Hardy space H^2 . In

particular, we have shown that an operator on H^2 is a slant H-Toeplitz operator if and only if its matrix is a slant H-Toeplitz matrix. Questions such as when slant H-Toeplitz operators become slant Toeplitz and slant Hankel are also answered.

2. Slant H-Toeplitz operators

The H-Toeplitz operator [1] with a symbol ϕ is the operator $S_\phi \in \mathcal{B}(H^2)$ defined by $S_\phi(f) = PM_\phi K(f)$ for all $f \in H^2$, where the operator $K : H^2 \rightarrow L^2$ is given by $K(e_{2n}) = e_n$ and $K(e_{2n+1}) = e_{-n-1}$ for all non-negative integers n . The adjoint K^* of the operator K is given by $K^*(e_n) = e_{2n}$, $K^*(e_{-n-1}) = e_{2n+1}$ for $n \geq 0$. Thus, $K^*K = I$ on H^2 and $KK^* = I$ on L^2 . Motivated by the definition of H-Toeplitz operator, we define slant H-Toeplitz operator on the space H^2 as follows:

Definition 2.1. For $\phi \in L^\infty$, the slant H-Toeplitz operator is defined as the operator $V_\phi : H^2 \rightarrow H^2$ such that $V_\phi(f) = WPM_\phi K(f)$ for each f in H^2 .

The operator V_ϕ with symbol $\phi \in L^\infty$ is a bounded linear operator as we have $\|V_\phi\| = \|WS_\phi\| = \|WPM_\phi K\| \leq \|W\|\|\phi\|_\infty\|K\| \leq \|\phi\|_\infty$.

Theorem 2.2. The correspondence $\phi \rightarrow V_\phi$ is one-one.

Proof. Let $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty$ be such that $V_\phi = V_\psi$. Therefore, $V_\phi - V_\psi = 0$, or equivalently, $WPM_{\phi-\psi}K = 0$. This implies that

$$(2.1) \quad WPM_{\phi-\psi}K(e_m) = 0 \quad \text{for all } m \geq 0.$$

Therefore, in particular, we have $WPM_{\phi-\psi}K(e_{2m}(z)) = 0$ for all $z \in \mathbb{T}$, which gives

$$WP \sum_{n=-\infty}^{\infty} (a_n - b_n) z^{n+m} = 0,$$

that is, $\sum_{n=0}^{\infty} (a_{2n-m} - b_{2n-m}) z^n = 0$. Therefore,

$$\langle WPM_{\phi-\psi}K(e_{2m}(z)), WPM_{\phi-\psi}K(e_{2m}(z)) \rangle = 0$$

which implies that $\langle \sum_{n=0}^{\infty} (a_{2n-m} - b_{2n-m}) z^n, \sum_{n=0}^{\infty} (a_{2n-m} - b_{2n-m}) z^n \rangle = 0$, or equivalently, $\sum_{n=0}^{\infty} |a_{2n-m} - b_{2n-m}|^2 = 0$. Thus, it follows that $a_{2n-m} = b_{2n-m}$ for all $n, m \geq 0$. Similarly, using equation (2.1), we get that

$$WPM_{\phi-\psi}K(e_{2m+1}(z)) = 0$$

which on using the definitions of operators W, P and K shows that

$$\sum_{n=0}^{\infty} (a_{2n+m+1} - b_{2n+m+1}) z^n = 0$$

and therefore it follows that $a_{2n+m+1} = b_{2n+m+1}$ for each $n \geq 0$ and $m \geq 0$. Hence, $a_n = b_n$ for all integers n and this proves $\phi = \psi$ a.e. on \mathbb{T} . \square

Let $\phi = \sum_{n=-\infty}^{\infty} a_n e_n \in L^\infty$ and $(a_{i,j})$ be the matrix of slant H-Toeplitz operator V_ϕ with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$, where the $(i,j)^{th}$ entry, $a_{i,j} = \langle V_\phi e_j, e_i \rangle$ satisfies the following:

$$a_{k,0} = \begin{cases} a_{k+j,4j} & \text{for all } j \geq 0 \text{ and } k \geq 0, \\ a_{k-j,4j-1} & \text{for all } j = 1, 2, 3, \dots, k-j \geq 0 \text{ and } k \geq 1, \end{cases}$$

and

$$a_{0,2k} = a_{i,2k+4i} \quad \text{for all } i \geq 1, k \geq 1.$$

Therefore, the matrix representation of slant H-Toeplitz operator V_ϕ is given by

$$V_\phi = \begin{bmatrix} a_0 & a_1 & a_{-1} & a_2 & a_{-2} & a_3 & a_{-3} & \cdots \\ a_2 & a_3 & a_1 & a_4 & a_0 & a_5 & a_{-1} & \cdots \\ a_4 & a_5 & a_3 & a_6 & a_2 & a_7 & a_1 & \cdots \\ a_6 & a_7 & a_5 & a_8 & a_4 & a_9 & a_3 & \cdots \\ a_8 & a_9 & a_7 & a_{10} & a_6 & a_{11} & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

which is a two way infinite matrix and it is an upper triangular matrix if the symbol ϕ is co-analytic. Also, for each non-negative integer n , it follows that

$$V_\phi(e_{2n}) = WPM_\phi K(e_{2n}) = WPM_\phi(e_n) = WT_\phi(e_n) = B_\phi(e_n)$$

and

$$\begin{aligned} V_\phi(e_{2n+1}) &= WPM_\phi K(e_{2n+1}) = WPM_\phi(e_{-n-1}) = WPM_\phi J(e_n) \\ &= WH_\phi(e_n) = L_\phi(e_n). \end{aligned}$$

This shows that the matrix of slant Toeplitz operator B_ϕ can be obtained by deleting every odd column of the matrix of slant H-Toeplitz operator V_ϕ and the matrix of slant Hankel operator L_ϕ can be obtained by deleting every even column of the matrix of the operator V_ϕ . Hence, the $(i,j)^{th}$ entry of the matrix of V_ϕ is given by:

$$a_{i,j} = \begin{cases} a_{2i-n} & \text{if } j = 2n, \\ a_{2i+n+1} & \text{if } j = 2n+1, \end{cases}$$

where $n \in \mathbb{N} \cup \{0\}$. This motivates us to define the slant H-Toeplitz matrix in the following way:

Definition 2.3. A two way doubly infinite matrix $(a_{i,j})$ is said to be a slant H-Toeplitz matrix if it satisfies the following:

$$(2.2) \quad a_{k,0} = \begin{cases} a_{k+j,4j} & \text{for all } j \geq 0, \text{ and } k \geq 0, \\ a_{k-j,4j-1} & \text{for all } j = 1, 2, 3, \dots, k-j \geq 0 \text{ and } k \geq 1 \end{cases}$$

and

$$(2.3) \quad a_{0,2k} = a_{i,2k+4i} \quad \text{for all } i \geq 1 \text{ and } k \geq 1.$$

It can be observed that an $n \times n$ slant H-Toeplitz matrix has $3n - 2$ degree of freedom rather than n^2 and therefore for large n , it is comparatively easy to solve the system of linear equations where the coefficient matrix is slant H-Toeplitz matrix. The H-Toeplitz matrix give rises to slant Toeplitz and slant Hankel operators that can be seen by the following theorem:

Theorem 2.4. *If the matrix of a bounded linear operator A defined on H^2 is a slant H-Toeplitz matrix, then AC_{z^2} is a slant Toeplitz operator and $AM_zC_{z^2}$ is a slant Hankel operator.*

Proof. Let A be a bounded linear operator on H^2 such that its matrix $(a_{i,j})$ with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ is a slant H-Toeplitz matrix and therefore, it satisfies relations (2.2) and (2.3). Let $(\alpha_{i,j})$ be the matrix of bounded linear operator AC_{z^2} , defined on H^2 with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$. Then using the definition of slant H-Toeplitz matrix, we have

$$\begin{aligned}\alpha_{i+1,j+2} &= \langle AC_{z^2}z^{j+2}, z^{i+1} \rangle = \langle Az^{2j+4}, z^{i+1} \rangle = a_{i+1,2j+4} = a_{i,2j} \\ &= \langle Az^{2j}, z^i \rangle = \langle AC_{z^2}z^j, z^i \rangle = \alpha_{i,j} \text{ for all } i, j \geq 0.\end{aligned}$$

Therefore, $(\alpha_{i,j})$ is a slant Toeplitz matrix and hence the operator AC_{z^2} is a slant Toeplitz operator. Next, let $(\beta_{i,j})$ be the matrix of the bounded linear operator $AM_zC_{z^2}$, defined on H^2 with respect to the basis $\{e_n\}_{n \geq 0}$. Then, by the definition of slant H-Toeplitz matrix, it follows that

$$\begin{aligned}\beta_{i-1,j+2} &= \langle AM_zC_{z^2}z^{j+2}, z^{i-1} \rangle = \langle Az^{2j+5}, z^{i-1} \rangle = a_{i-1,2j+5} = a_{i,2j+1} \\ &= \langle Az^{2j+1}, z^i \rangle = \langle AM_zC_{z^2}z^j, z^i \rangle = \beta_{i,j} \text{ for all } i \geq 1, j \geq 0.\end{aligned}$$

Thus, the matrix $(\beta_{i,j})$ is a slant Hankel matrix and hence the operator $AM_zC_{z^2}$ is a slant Hankel operator. \square

Corollary 2.5. *If the matrix of a bounded linear operator A defined on H^2 is a slant H-Toeplitz matrix, then $AC_{z^2} = WT_\phi$ and $AM_zC_{z^2} = WH_\phi$ for some $\phi \in L^\infty$.*

Proof. Let $A \in \mathcal{B}(H^2)$ be such that its matrix $(a_{i,j})$ with respect to orthonormal basis $\{e_n\}_{n \geq 0}$ is a slant H-Toeplitz matrix. Therefore, by Theorem 2.4, the operators AC_{z^2} and $AM_zC_{z^2}$ are slant Toeplitz operator and slant Hankel operator, respectively. Let $(\alpha_{i,j})$ and $(\beta_{i,j})$ be the matrices of the operators AC_{z^2} and $AM_zC_{z^2}$, respectively, with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ of H^2 . Then by the definition of slant H-Toeplitz matrix, it follows that

$$\alpha_{k,0} = \langle WT_\phi z^0, z^k \rangle = \langle AC_{z^2}z^0, z^k \rangle = \langle Az^0, z^k \rangle = a_{k,0}$$

and

$$\alpha_{0,j} = \langle WT_\phi z^j, z^0 \rangle = \langle AC_{z^2}z^j, z^0 \rangle = \langle Az^{2j}, z^0 \rangle = a_{0,2j}.$$

Thus, for all $k \geq 0$, $j \geq 1$, it follows that $\alpha_{k,0} = \langle \phi, z^{2k} \rangle = a_{k,0}$ and $\alpha_{0,j} = \langle \phi, z^{2j} \rangle = a_{0,2j}$. Also by relation (2.2), $\alpha_{i,j} = \langle AC_{z^2}z^j, z^i \rangle = \langle Az^{2j}, z^i \rangle =$

$a_{i,2j} = a_{2i-j,0} = a_{2i-j}$. Now since $AM_zC_{z^2} = WH_\phi$, therefore, for $k \geq 0$ and by relation (2.2) it follows that

$$\beta_{k,0} = \langle WH_\phi z^0, z^k \rangle = \langle AM_zC_{z^2} z^0, z^k \rangle = \langle Az, z^k \rangle = a_{k,1} = a_{k+1,2}.$$

Also, $\langle WH_\phi z^0, z^k \rangle = \langle PM_\phi J z^0, z^{2k} \rangle = \langle M_\phi z^{-1}, z^{2k} \rangle = \langle \phi, z^{2k+1} \rangle$ and so $\langle \phi, z^{2k+1} \rangle = a_{k,1}$. Since, $\langle \phi, z^{2k} \rangle = a_{k,0}$ and $\langle \phi, z^{2k+1} \rangle = a_{k,1}$, therefore we define the function ϕ as follows:

$$(2.4) \quad \langle \phi, z^k \rangle = \begin{cases} a_{k/2,0} & k \geq 0 \text{ and } k \text{ is even,} \\ a_{(k-1)/2,1} & k > 0 \text{ and } k \text{ is odd,} \\ a_{0,-2k} & k \leq -1. \end{cases}$$

Since A is a bounded linear operator on H^2 , therefore, the function $\phi \in L^\infty$. Hence, the operator AC_{z^2} is a slant Toeplitz operator B_ϕ and the operator $AM_zC_{z^2}$ is a slant Hankel operator S_ϕ with ϕ defined by (2.4). Also,

$$\begin{aligned} \beta_{i,j} &= \langle AM_zC_{z^2} z^j, z^i \rangle = \langle Az^{2j+1}, z^i \rangle \\ &= a_{i,2j+1} = a_{i-1,2j+5} = \langle AM_zC_{z^2} z^{j+2}, z^{i-1} \rangle = \beta_{i-1,j+2} \end{aligned}$$

for all $i \geq 1, j \geq 0$. □

Remark 2.6. From the matrix representation, given by the relations (2.2) and (2.3) of slant H-Toeplitz operator V_ϕ with symbol $\phi \in L^\infty$, it can be observed that with respect to a suitable basis on the domain and range spaces for the operator V_ϕ , the matrix of V_ϕ can be represented as the matrix whose columns on the left side are of the matrix of B_ϕ and the columns on the right side are of the matrix of L_ϕ . Therefore, with respect to above representation, we can conclude that any slant H-Toeplitz operator is unitarily equivalent to a direct sum of a slant Toeplitz operator and a slant Hankel operator.

For $\phi \in L^\infty$, the adjoint V_ϕ^* of the operator V_ϕ on H^2 is the operator satisfying

$$V_\phi^* = (WS_\phi)^* = (WPM_\phi K)^* = K^*M_\phi^*P^*W^* = K^*M_{\bar{\phi}}W^*.$$

If $\phi = \sum_{n=-\infty}^{\infty} a_n e_n \in L^\infty$, then $(i, j)^{\text{th}}$ entry of the matrix of V_ϕ^* with respect to orthonormal basis $\{e_n\}_{n \geq 0}$ is given by

$$\begin{aligned} \langle V_\phi^* e_j, e_i \rangle &= \langle K^*M_\phi^*P^*W^* e_j, e_i \rangle = \langle K^*M_{\bar{\phi}} e_{2j}, e_i \rangle \\ &= \left\langle \sum_{n=-\infty}^{\infty} \overline{a_n} e_{2j-n}, K e_i \right\rangle \\ &= \begin{cases} \overline{a_{2j-m}} & \text{if } i = 2m, \\ \overline{a_{2j+m+1}} & \text{if } i = 2m + 1, \end{cases} \end{aligned}$$

where i, j and m are non-negative integers. Hence, the matrix of V_ϕ^* is given by

$$V_\phi^* = \begin{bmatrix} \overline{a_0} & \overline{a_2} & \overline{a_4} & \overline{a_6} & \overline{a_8} & \overline{a_{10}} & \overline{a_{12}} \cdots \\ \overline{a_1} & \overline{a_3} & \overline{a_5} & \overline{a_7} & \overline{a_9} & \overline{a_{11}} & \overline{a_{13}} \cdots \\ \overline{a_{-1}} & \overline{a_1} & \overline{a_3} & \overline{a_5} & \overline{a_7} & \overline{a_9} & \overline{a_{11}} \cdots \\ \overline{a_2} & \overline{a_4} & \overline{a_6} & \overline{a_8} & \overline{a_{10}} & \overline{a_{12}} & \overline{a_{14}} \cdots \\ \overline{a_{-2}} & \overline{a_0} & \overline{a_2} & \overline{a_4} & \overline{a_6} & \overline{a_8} & \overline{a_{10}} \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Moreover, we have $\|V_\phi^* f\|^2 = \|(WT_\phi)^* f\|^2 + \|(WH_\phi)^* f\|^2$ for each $f \in H^2$.

Proposition 2.7. *If $\phi \in L^\infty$ is an inner function, then the operator V_ϕ^* is an isometry.*

Proof. If $\phi \in L^\infty$ is an inner function, then $|\phi| = 1$ a.e. on \mathbb{T} . Then, for each non-negative integer n and by the definition of operators V_ϕ and V_ϕ^* it follows that $V_\phi V_\phi^*(e_n) = (WPM_\phi K)(K^*M_{\bar{\phi}}PW^*)(e_n) = WPM_\phi(KK^*)M_{\bar{\phi}}(e_{2n}) = WPM_{|\phi|^2}(e_{2n}) = WW^*(e_n) = (e_n)$. Thus, $V_\phi V_\phi^*(e_n) = (e_n)$ for all $n \geq 0$. Hence, the operator V_ϕ^* is an isometry. \square

The condition in the above theorem is only necessary but not sufficient as shown in the following example.

Example 2.8. For $\phi(z) = (1+z)/\sqrt{2} \in L^\infty$ such that $z \in \mathbb{T}$, we have

$$\begin{aligned} V_\phi V_\phi^*(e_n(z)) &= WPM_\phi M_{\bar{\phi}}(z^n) = \frac{1}{\sqrt{2}}WP\left(\frac{1+z}{\sqrt{2}}(z^{2n} + z^{2n-1})\right) \\ &= \frac{1}{2}WP(z^{2n} + z^{2n-1} + z^{2n+1} + z^{2n}) = z^n. \end{aligned}$$

Therefore, $V_\phi V_\phi^*(e_n) = e_n$ for each non-negative integer n and hence V_ϕ^* is an isometry, but ϕ is not an inner function.

Theorem 2.9. *If $\phi = \sum_{n=-\infty}^{\infty} a_n e_n \in L^\infty$ and the operator V_ϕ^* is an isometry on H^2 , then $\sum_{n=-\infty}^{\infty} |a_n|^2 = 1$.*

Proof. If the operator V_ϕ^* is an isometry, then $V_\phi V_\phi^* = I$ which on using the definition of V_ϕ implies that $WT_{|\phi|^2}W^* = WW^*$ or equivalently, $W(I - T_{|\phi|^2})W^* = 0$ and therefore we get that $WT_{1-|\phi|^2}W^* = 0$. Thus, for $m \geq 0$, it follows that $\langle WT_{1-|\phi|^2}W^*e_m, e_m \rangle = 0$, that is, $\langle (T_{1-|\phi|^2})e_{2m}, e_{2m} \rangle = 0$ which gives $\langle (1-|\phi|^2)e_{2m}, e_{2m} \rangle = 0$ and so $\langle z^{2m}, z^{2m} \rangle - \langle \phi(z)\bar{\phi}(z)z^{2m}, z^{2m} \rangle = 0$. Therefore, on substituting the value of ϕ , we get that

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^{n+2m}, \sum_{k=-\infty}^{\infty} a_k z^{k+2m} \right\rangle = 1$$

or, equivalently, we have $\sum_{n=-\infty}^{\infty} a_n \sum_{k=-\infty}^{\infty} \overline{a_k} \langle z^{n+2m}, z^{k+2m} \rangle = 1$ for all $m \geq 0$. Hence, it follows that $\sum_{n=-\infty}^{\infty} |a_n|^2 = 1$. \square

If $\phi \in L^\infty$ is an inner function, then the operator V_ϕ being coisometry is also a partial isometry and in the next result we have obtained the necessary condition for V_ϕ to be partial isometry on H^2 .

Theorem 2.10. *If $\phi \in L^\infty$ and the operator V_ϕ is partial isometry, then*

$$(WT_{1-|\phi|^2}W^*)WT_\phi K = 0.$$

Proof. Let the operator V_ϕ be a partial isometry on H^2 . Therefore, $V_\phi = V_\phi V_\phi^* V_\phi$, that is, $V_\phi = (WT_{|\phi|^2}W^*)V_\phi$ which further implies that

$$(I - WT_{|\phi|^2}W^*)V_\phi = 0.$$

So, $(WPM_1W^* - WT_{|\phi|^2}W^*)WPM_\phi K = 0$, or equivalently,

$$(WT_{1-|\phi|^2}W^*)WPM_\phi K = 0$$

and hence it follows that $(WT_{1-|\phi|^2}W^*)WT_\phi K = 0$. \square

Theorem 2.11. *For $\phi \in L^\infty$, the operator V_ϕ is a Hilbert-Schmidt operator if and only if $\phi \equiv 0$.*

Proof. Clearly if $\phi \equiv 0$, then the operator V_ϕ is a Hilbert-Schmidt operator. Conversely, take $\phi = \sum_{n=-\infty}^{\infty} a_n e_n \in L^\infty$ and assume that the operator V_ϕ is a Hilbert-Schmidt operator. From the definitions of the operators W and K , we have

$$\begin{aligned} & \sum_{m=0}^{\infty} \langle V_\phi e_m, V_\phi e_m \rangle \\ &= \sum_{m=0}^{\infty} \langle V_\phi e_{2m}, V_\phi e_{2m} \rangle + \sum_{m=0}^{\infty} \langle V_\phi e_{2m+1}, V_\phi e_{2m+1} \rangle \\ &= \sum_{m=0}^{\infty} \langle WPM_\phi e_m, WPM_\phi e_m \rangle + \sum_{m=0}^{\infty} \langle WPM_\phi e_{-m-1}, WPM_\phi e_{-m-1} \rangle \\ &= \sum_{m=0}^{\infty} \left\langle WP \sum_{n=-\infty}^{\infty} a_n e_{n+m}, WP \sum_{n=-\infty}^{\infty} a_n e_{n+m} \right\rangle \\ & \quad + \sum_{m=0}^{\infty} \left\langle WP \sum_{n=-\infty}^{\infty} a_n e_{n-m-1}, WP \sum_{n=-\infty}^{\infty} a_n e_{n-m-1} \right\rangle \\ &= \sum_{m=0}^{\infty} \left\langle \sum_{n=0}^{\infty} a_{2n-m} e_n, \sum_{j=0}^{\infty} a_{2j-m} e_j \right\rangle + \sum_{m=0}^{\infty} \left\langle \sum_{n=0}^{\infty} a_{2n+m+1} e_n, \sum_{j=0}^{\infty} a_{2j+m+1} e_j \right\rangle \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{2n-m}|^2 \right) + \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{2n+m+1}|^2 \right). \end{aligned}$$

Since the operator V_ϕ is Hilbert-Schmidt, therefore it follows that

$$\sum_{m=0}^{\infty} \|V_\phi e_m\|^2 = \sum_{m=0}^{\infty} \langle V_\phi e_m, V_\phi e_m \rangle < \infty.$$

Hence, this implies that

$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{2n-m}|^2 \right) + \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{2n+m+1}|^2 \right) < \infty$$

which holds if and only if $|a_n| = 0$ for each n . Thus, $\phi \equiv 0$. \square

It is known that the only compact slant Toeplitz operators on the Hardy space is the zero operator [13]. Following theorem shows that the same holds true for slant H-Toeplitz operators.

Theorem 2.12. *The slant H-Toeplitz operator V_ϕ is compact if and only if $\phi \equiv 0$.*

Proof. Let ϕ be a bounded measurable function and V_ϕ be a slant H-Toeplitz operator with the matrix $(a_{i,j})$ with respect to orthonormal basis $\{e_n\}_{n \geq 0}$ satisfying relations (2.2) and (2.3). Let V_ϕ be a compact operator. Since e_n converges to 0 weakly and therefore $\|V_\phi e_n\| \rightarrow 0$. This implies that $\|WT_\phi e_n\| \rightarrow 0$ and $\|WH_\phi e_n\| \rightarrow 0$. This further implies that $\langle \phi, e_n \rangle = 0$ for each $n \in \mathbb{Z}$. Hence, $\phi \equiv 0$, that is, $V_\phi = 0$. Thus, the only compact slant H-Toeplitz operator is the zero operator. \square

The slant H-Toeplitz operator V_ϕ is a non-normal operator, that is, $V_\phi V_\phi^* \neq V_\phi^* V_\phi$. Moreover, in the following theorem we prove that zero operator is the only hyponormal slant H-Toeplitz operator.

Theorem 2.13. *For $\phi \in L^\infty$, the operator V_ϕ is hyponormal if and only if $\phi \equiv 0$.*

Proof. Clearly for $\phi \equiv 0$, the operator V_ϕ is hyponormal on H^2 . Conversely, assume that the operator V_ϕ is hyponormal on H^2 with the symbol $\phi = \sum_{n=-\infty}^{\infty} a_n e_n$. Then from hyponormality of V_ϕ , it follows that

$$(2.5) \quad \|V_\phi^* f\|^2 \leq \|V_\phi f\|^2 \quad \text{for all } f \in H^2.$$

In particular for $f(z) = e_0(z)$ in (2.5), we have $\|V_\phi^* e_0\|^2 \leq \|V_\phi e_0\|^2$, that is, $\|K^* (\sum_{n=-\infty}^{\infty} \overline{a_{-n}} e_n)\|^2 \leq \|W \sum_{n=0}^{\infty} a_n e_n\|^2$ which further implies that

$$\left\| \sum_{n=0}^{\infty} \overline{a_{-n}} e_{2n} + \sum_{n=0}^{\infty} \overline{a_{n+1}} e_{2n+1} \right\|^2 \leq \left\| \sum_{n=0}^{\infty} a_{2n} e_n \right\|^2.$$

Therefore, on expanding it follows that

$$\sum_{n=0}^{\infty} |a_{-n}|^2 + \sum_{n=0}^{\infty} |a_{n+1}|^2 \leq \sum_{n=0}^{\infty} |a_{2n}|^2$$

which implies that $\sum_{n=1}^{\infty} |a_{-n}|^2 + \sum_{n=0}^{\infty} |a_{2n+1}|^2 \leq 0$. So, this gives that $a_{-n} = 0$ for $n \geq 1$ and $a_{2n+1} = 0$ for $n \geq 0$. Similarly on taking $f(z) = e_1(z) = z$ in (2.5), it follows that $\|V_{\phi}^* e_1\|^2 \leq \|V_{\phi} e_1\|^2$. Now from the definitions of operators V_{ϕ} and V_{ϕ}^* we have $\left\| \sum_{n=-\infty}^{\infty} \overline{a_{n+3}} e_{2n+1} \right\|^2 \leq \left\| \sum_{n=0}^{\infty} \overline{a_{2n+1}} e_n \right\|^2$, or equivalently, $\sum_{n=-\infty}^{\infty} |a_{n+3}|^2 \leq \sum_{n=0}^{\infty} |a_{2n+1}|^2$, that is, $\sum_{n=-\infty}^{\infty} |a_{2n}|^2 \leq 0$. Therefore, $|a_{2n}| = 0$ for each n and hence we have that $a_n = 0$ for each $n \in \mathbb{Z}$. Thus, it follows that $\phi \equiv 0$. \square

It is evident that every isometry operator is hyponormal, therefore it follows that a slant H-Toeplitz operator can not be isometry.

3. Characterizations of slant H-Toeplitz operator

Let $S \in \mathcal{B}(H^2)$ be a forward shift operator, that is, $S(f(z)) = z(f(z))$ for all $f \in H^2$, $z \in \mathbb{T}$ and the operator S^* denotes the adjoint of S . Let the operator $U \in \mathcal{B}(L^2)$ be the multiplication operator with symbol z .

Theorem 3.1. *If A is a bounded linear operator on H^2 whose matrix with respect to orthonormal basis $\{e_n\}_{n \geq 0}$ is a slant H-Toeplitz matrix, then for each non-negative integer m , there exist a bounded linear operator A_m defined from H^2 to L^2 which satisfies the following:*

- (a) $A_m C_{z^2} = U^{*m} A C_{z^2} S^{2m}$.
- (b) $S^* A_m M_{z^3} C_{z^4} = A M_{z^3} C_{z^4} S$.
- (c) $S^* A_m z^0 = A M_{z^3} z^0$.

Proof. Let $A \in \mathcal{B}(H^2)$ has a slant H-Toeplitz matrix $(\alpha_{i,j})$ with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$. For each non-negative integer m , define a bounded linear operator A_m from H^2 to $H^2 \cup \text{span}\{e_{-1}, e_{-2}, e_{-3}, \dots, e_{-m}\} \subset L^2$ such that its matrix satisfies the following relation:

$$(3.1) \quad \alpha_{k,0} = \begin{cases} \alpha_{k+j,4j} & \text{for } j \geq 0 \text{ and } k \geq -m, \\ \alpha_{k-j,4j-1} & \text{for } j = 1, 2, 3, \dots \text{ and } k - j \geq -m \end{cases}$$

and

$$(3.2) \quad \alpha_{-m,2k} = a_{i,2k+4i} \quad \text{for } i \geq -m \text{ and } k \geq 1.$$

Using the matrix representation of operators A_m , it follows that

$$(3.3) \quad \alpha_{q,2p} = \alpha_{q+j,2p+4j} \quad \text{for all } p, q, j \geq 0.$$

Therefore, in particular for $j = m \geq 1$ by equation (3.3), it follows that

$$(3.4) \quad \alpha_{q,2p} = \alpha_{q+m,2p+4m} \quad \text{for all } p, q \geq 0.$$

Then, by equation (3.4), we obtain

$$\langle A_m e_{2p}, e_q \rangle = \langle A e_{2p+2m}, e_{q+m} \rangle,$$

or equivalently,

$$\langle A_m C_{z^2} e_p, e_q \rangle = \langle U^* A C_{z^2} S^{2m} e_p, e_q \rangle$$

for all $p \geq 0$, $m \geq 1$ and $q \geq -m$ and hence $A_m C_{z^2} = U^{*m} A C_{z^2} S^{2m}$. Since, $\alpha_{k,0} = \alpha_{k-j,4j-1}$ for all $j \geq 1$ and $k - j \geq -m$, therefore it follows that $\alpha_{k+r+2,0} = \alpha_{k+r+2-j,4j-1}$ for $j \geq 1, k \geq j - m$ and $r \geq 0$. In particular, for $j = r + 1, r + 2$, we see that

$$\alpha_{k+1,4r+3} = \alpha_{k,4r+7} \quad \text{for all } r \geq 0, k \geq 0.$$

Therefore, $\langle A_m e_{4r+3}, e_{k+1} \rangle = \langle A e_{4r+7}, e_k \rangle$, or equivalently, we get that

$$\langle S^* A_m M_{z^3} C_{z^4} e_r, e_k \rangle = \langle A M_{z^3} C_{z^4} S e_r, e_k \rangle \quad \text{for each } r, k \geq 0.$$

Hence, it follows that $S^* A_m M_{z^3} C_{z^4} = A M_{z^3} C_{z^4} S$. Again by the definition of matrix $(\alpha_{i,j})$, it follows that $\alpha_{k+1,0} = \alpha_{k,3}$ for all $k \geq 0$. This implies that $\langle A_m e_0, e_{k+1} \rangle = \langle A e_3, e_k \rangle$, or equivalently, $\langle S^* A_m e_0, e_k \rangle = \langle A M_{z^3} e_0, e_k \rangle$ for all $k \geq 0$. Thus, it gives $S^* A_m z^0 = A M_{z^3} z^0$. \square

In the above theorem, for each fixed non-negative integer m , the $(i, j)^{\text{th}}$ entry of the matrix of the operator A_m is independent of m , which is shown in the following lemma:

Lemma 3.2. *Let m be a fixed non-negative integer. Then for all $j \geq 0$ and $i \geq -m$, $\langle A_m e_j, e_i \rangle$ is independent of m .*

Proof. Let $(\alpha_{i,j})$ be the matrix of A_m with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ satisfying relations (3.1) and (3.2). For $0 > i > -m$, using the matrix definition of operator A_m , we get that

$$\langle A_m e_{4p}, e_i \rangle = \alpha_{0,4p+4k} = \langle A e_{4(p+k)}, e_0 \rangle \quad \text{where } i + k = 0$$

and this is true for each non-negative integer p . Also, for $p \geq 1$, we have $\langle A_m e_{4p-1}, e_i \rangle = \alpha_{i+p,0} = \langle A e_0, e_{i+p} \rangle$. For non-negative integers i and j and by using the relation $A_m C_{z^2} = U^{*m} A C_{z^2} S^{2m}$, it follows that

$$\begin{aligned} \langle A_m C_{z^2} e_j, e_i \rangle &= \langle U^{*m} A C_{z^2} S^{2m} e_j, e_i \rangle \\ &= \langle A C_{z^2} S^{2m}, S^m e_i \rangle \\ &= \langle S^{*m} A C_{z^2} S^{2m} e_j, e_i \rangle \\ &= \langle A C_{z^2} e_j, e_i \rangle. \end{aligned}$$

Also, the relation $S^* A_m M_{z^3} C_{z^4} = A M_{z^3} C_{z^4} S$ implies that

$$\langle S^* A_m M_{z^3} C_{z^4} e_j, e_i \rangle = \langle A M_{z^3} C_{z^4} S e_j, e_i \rangle$$

which gives that $\langle A e_{4j+7}, e_i \rangle = \langle A e_{4j+3}, e_{i+1} \rangle$. Moreover, from the definition of A_m , we have $\langle S^* A_m M_{z^3} C_{z^4} e_j, e_i \rangle = \langle A_m e_{4j+3}, e_{i+m} \rangle = \langle A e_{4j+3}, e_{i+1} \rangle$. Again the relation $S^* A_m z^0 = A M_{z^3} z^0$ implies that $\langle S^* A_m e_0, e_i \rangle = \langle A M_{z^3} e_0, e_i \rangle = \alpha_{i,3} = \alpha_{i+1,0} = \langle A e_0, e_{i+1} \rangle$ and also $\langle S^* A_m e_0, e_i \rangle = \langle A_m e_0, e_{i+1} \rangle$. Therefore, for all non-negative integers i and j we get that $\langle A_m e_j, e_i \rangle = \langle A e_j, e_i \rangle$. Hence, for all $j \geq 0$ and $i \geq -m$, $\langle A_m e_j, e_i \rangle$ is independent of m . \square

Example 3.3. Let A be a bounded linear operator on H^2 whose matrix $(a_{i,j})$ with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ is a slant H-Toeplitz matrix and satisfies the relations (2.2) and (2.3). Then the matrices of the operators A_1 and A_2 defined in Theorem 3.1, are given by

$$A_1 = \begin{bmatrix} a_{-2} & a_{-1} & a_{-3} & a_0 & a_{-4} & a_1 & a_{-5} & \cdots \\ a_0 & a_1 & a_{-1} & a_2 & a_{-2} & a_3 & a_{-3} & \cdots \\ a_2 & a_3 & a_1 & a_4 & a_0 & a_5 & a_{-1} & \cdots \\ a_4 & a_5 & a_3 & a_6 & a_2 & a_7 & a_1 & \cdots \\ a_6 & a_7 & a_5 & a_8 & a_4 & a_9 & a_3 & \cdots \\ a_8 & a_9 & a_7 & a_{10} & a_6 & a_{11} & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} a_{-4} & a_{-3} & a_{-5} & a_{-2} & a_{-6} & a_{-1} & a_{-7} & \cdots \\ a_{-2} & a_{-1} & a_{-3} & a_0 & a_{-4} & a_1 & a_{-5} & \cdots \\ a_0 & a_1 & a_{-1} & a_2 & a_{-2} & a_3 & a_{-3} & \cdots \\ a_2 & a_3 & a_1 & a_4 & a_0 & a_5 & a_{-1} & \cdots \\ a_4 & a_5 & a_3 & a_6 & a_2 & a_7 & a_1 & \cdots \\ a_6 & a_7 & a_5 & a_8 & a_4 & a_9 & a_3 & \cdots \\ a_8 & a_9 & a_7 & a_{10} & a_6 & a_{11} & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Similarly, we can obtain the matrix representation for other operators A_m for $m > 2$ and clearly all these operators satisfying the conditions given in Theorem 3.1.

In the following theorem we give the characterization for slant H-Toeplitz operators.

Theorem 3.4. *A necessary and sufficient condition for an operator $A \in \mathcal{B}(H^2)$ to be a slant H-Toeplitz operator is that its matrix with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ is the slant H-Toeplitz matrix.*

Proof. It is clear that every slant H-Toeplitz operator defined on H^2 has a slant H-Toeplitz matrix with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ of H^2 . Conversely, assume that A is a bounded linear operator on H^2 whose matrix with respect to the orthonormal basis $\{e_n\}_{n \geq 0}$ is a slant H-Toeplitz matrix. So, we claim that A is a slant H-Toeplitz operator. For each non-negative integer m , consider a bounded linear operator A_m defined on H^2 to L^2 such that its matrix satisfies the relations (3.1) and (3.2) given in the Theorem 3.1. Then, the operators A_m satisfies the following:

- (a) $A_m C_{z^2} = U^{*m} A C_{z^2} S^{2m},$
- (b) $S^* A_m M_{z^3} C_{z^4} = A M_{z^3} C_{z^4} S,$
- (c) $S^* A_m z^0 = A M_{z^3} z^0.$

Moreover, for non-negative integers i and j , we have $\langle A_m z^j, z^i \rangle = \langle A z^j, z^i \rangle$. If p and q are finite linear combinations of z^i for $i \geq 0$, then the sequence $\{\langle A_m p, q \rangle\}$ is convergent. Therefore, the sequence $\{A_m\}$ of operator on H^2 is weakly convergent to a bounded linear operator say, B , on H^2 . Then for all $i, j \geq 0$, it follows that

$$\langle PBz^j, z^i \rangle = \langle Bz^j, z^i \rangle = \lim_{m \rightarrow \infty} \langle A_m z^j, z^i \rangle = \langle Az^j, z^i \rangle = a_{i,j}$$

and if f and g are in H^2 , then we have

$$\langle PBf, g \rangle = \lim_{m \rightarrow \infty} \langle A_m f, g \rangle = \lim_{m \rightarrow \infty} \langle Af, g \rangle = \langle Af, g \rangle.$$

Therefore, $PBf = Af$ for each $f \in H^2$. Hence, it follows that operator A is a slant H-Toeplitz operator on H^2 . Fourier coefficients of ϕ that induces the operator A from its matrix are given by

$$\langle \phi, z^k \rangle = \begin{cases} a_{k/2,0} & k \geq 0 \text{ and } k \text{ is even,} \\ a_{(k-1)/2,1} & k > 0 \text{ and } k \text{ is odd,} \\ a_{0,-2k} & k \leq -1. \end{cases}$$

For $f(z) = z^n \in H^2$, $AC_{z^2}f(z) = Az^{2n}$ and $AM_zC_{z^2} = Az^{2n+1}$. Since the operators AC_{z^2} and $AM_zC_{z^2}$ are slant Toeplitz and slant Hankel operators, respectively, therefore $AC_{z^2} = WT_\phi$ and $AM_zC_{z^2} = WH_\phi$. Then for each function $f_1(z^2) \in H^2$, we have

$$AC_{z^2}(f_1(z)) = WT_\phi f_1(z) = WPM_\phi K(f_1(z^2)) = V_\phi(f_1(z^2))$$

and for each $f_2(z^2) \in H^2$, we obtain

$$\begin{aligned} AM_zC_{z^2}(f_2(z)) &= WH_\phi(f_2(z)) = WPM_\phi J(f_2(z)) = WPM_\phi(z^{-1}f_2(z^{-1})) \\ &= WPM_\phi K(zf_2(z^2)) = V_\phi(zf_2(z^2)). \end{aligned}$$

If $h(z) \in H^2$, then $h(z) = h_1(z^2) + zh_2(z^2)$. Moreover, we have

$$\begin{aligned} A(h(z)) &= A(h_1(z^2) + zh_2(z^2)) = A(h_1(z^2)) + A(zh_2(z^2)) \\ &= AC_{z^2}(h_1(z)) + AM_zC_{z^2}(h_2(z)) \\ &= V_\phi(h_1(z^2)) + V_\phi(zh_2(z^2)) \\ &= V_\phi(h_1(z^2) + zh_2(z^2)) = V_\phi(h(z)) \end{aligned}$$

which is true for every $h(z) \in H^2$. Hence, the operator A is a slant H-Toeplitz operator with symbol ϕ . \square

In the next theorem, we give another characterization for slant H-Toeplitz operators.

Theorem 3.5. *A bounded linear operator A on H^2 is a slant H-Toeplitz operator if and only if satisfies*

- (a) $AC_{z^2} = S^*AC_{z^2}S^2$,
- (b) $S^*AM_{z^3}C_{z^4} = AM_{z^3}C_{z^4}S$,

$$(c) \quad S^*Az^0 = AM_{z^3}z^0.$$

Proof. Let the operator $A \in \mathcal{B}(H^2)$ satisfies the conditions (a), (b) and (c). Then from (a) and (b), it follows that AC_{z^2} is a slant Toeplitz operator and $AM_zC_{z^2}$ is a slant Hankel operator. Also if $f \in H^2$, then

$$S^*AC_{z^2}S^2(f(z)) = AC_{z^2}(f(z))$$

and

$$S^*AM_{z^3}C_{z^4}(f(z)) = AM_{z^3}C_{z^4}S(f(z)).$$

This gives that,

$$(3.5) \quad S^*A(z^4f(z^2)) = A(f(z^2)) \text{ and } S^*A(z^3f(z^4)) = A(z^7f(z^4)).$$

This is true for each functions $f(z^2), f(z^4) \in H^2$. Therefore, in particular for $f(z^2) = z^0, z^2, z^4, z^6, \dots$ and $f(z^4) = z^0, z^4, z^8, z^{12}, \dots$ using equation (3.5), we obtain the following relations:

$$(3.6) \quad S^*A(z^{2n+4}) = A(z^{2n}) \text{ and } S^*A(z^{4n+3}) = A(z^{4n+7}) \text{ for } n \geq 0.$$

Let $(a_{i,j})$ be the matrix of the bounded linear operator A with respect to orthonormal basis $\{e_n\}_{n \geq 0}$. Then, for all $k \geq 0$ and by the relation (3.6), we have

$$\begin{aligned} a_{k,0} &= \langle Az^0, z^k \rangle = \langle S^*Az^4, z^k \rangle = \langle Az^4, z^{k+1} \rangle = a_{k+1,4} \\ &= \langle S^*Az^8, z^{k+1} \rangle = \langle Az^8, z^{k+2} \rangle = a_{k+2,8} \\ &= \langle S^*Az^{12}, z^{k+2} \rangle = \langle Az^{12}, z^{k+3} \rangle = a_{k+3,12} \end{aligned}$$

and so on. On continuing in this manner, for $j \geq 1$ and $k \geq 0$, we obtain that $a_{k,0} = a_{k+j,4j}$. Again for each $k \geq 1$ and by the relation (3.6), it follows that

$$\begin{aligned} a_{k,0} &= \langle Az^0, z^k \rangle = \langle S^*Az^0, z^{k-1} \rangle = \langle AM_{z^3}z^0, z^{k-1} \rangle = a_{k-1,3} \\ &= \langle S^*Az^3, z^{k-2} \rangle = \langle Az^7, z^{k-2} \rangle = a_{k-2,7} \\ &= \langle S^*Az^7, z^{k-3} \rangle = \langle Az^{11}, z^{k-3} \rangle = a_{k-3,11} \end{aligned}$$

and so on. Again on continuing the same manner it follows that for all $k \geq 1$, $a_{k,0} = a_{k-1,3} = a_{k-2,7} = a_{k-3,11} = \dots = a_{0,4k-1}$. Therefore, for each $k \geq 1$ and for $j = 1, 2, 3, \dots, k-j \geq 0$, it follows that $a_{k,0} = a_{k-j,4j-1}$. Again for $k \geq 0$ and from the relation (3.6), it follows that

$$\begin{aligned} a_{0,2k} &= \langle Az^{2k}, z^0 \rangle = \langle S^*Az^{2k+4}, z^0 \rangle = \langle Az^{2k+4}, z^1 \rangle = a_{1,2k+4} \\ &= \langle S^*Az^{2k+8}, z^1 \rangle = \langle Az^{2k+8}, z^2 \rangle = a_{2,2k+8} \\ &= \langle S^*Az^{2k+12}, z^2 \rangle = \langle Az^{2k+12}, z^3 \rangle = a_{3,2k+12} \end{aligned}$$

and so on. Therefore, on continuing the same process, for all $k \geq 0$ and $i \geq 1$ it follows that $a_{0,2k} = a_{i,2k+4i}$. Since the matrix $(a_{i,j})$ satisfies the relations (2.2) and (2.3), therefore the matrix $(a_{i,j})$ is a slant H-Toeplitz matrix. Thus, the

operator A is a slant H-Toeplitz operator on H^2 with symbol ϕ whose Fourier coefficients are given by

$$\langle \phi, z^k \rangle = \begin{cases} a_{k/2,0} & k \geq 0 \text{ and } k \text{ is even,} \\ a_{(k-1)/2,1} & k > 0 \text{ and } k \text{ is odd,} \\ a_{0,-2k} & k \leq -1. \end{cases}$$

Conversely, assume that the operator A is a slant H-Toeplitz operator on H^2 . Then, $A = V_\phi$ for some non-zero $\phi \in L^\infty$ and for each $f \in H^2$, we have

$$AC_{z^2}(f(z)) = V_\phi C_{z^2}(f(z)) = WPM_\phi K(f(z^2)) = WT_\phi(f(z)).$$

Hence, AC_{z^2} is a slant Toeplitz operator and therefore we get that $S^*AC_{z^2}S^2 = AC_{z^2}$. Also for each $f \in H^2$, it follows that

$$\begin{aligned} AM_z C_{z^2}(f(z)) &= V_\phi(zf(z^2)) = WPM_\phi K(zf(z^2)) = WPM_\phi(z^{-1}f(z^{-1})) \\ &= WPM_\phi J(f(z)) = WH_\phi(f(z)). \end{aligned}$$

Therefore, the operator $AM_z C_{z^2}$ is a slant Hankel operator and hence $S^*AM_{z^3}C_{z^4} = AM_{z^3}C_{z^4}S$. Again if $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, then the operator A satisfies the following:

$$S^*A(z^0) = S^*V_\phi(z^0) = S^*WP\phi(z) = S^*\sum_{n=0}^{\infty} a_{2n}z^n = \sum_{n=0}^{\infty} a_{2n+2}z^n$$

and

$$AM_{z^3}(z^0) = V_\phi(z^3) = WPM_\phi K(z^3) = WP\phi(z^{-2}) = \sum_{n=0}^{\infty} a_{2n+2}z^n.$$

Therefore, $AM_{z^3}(z^0) = S^*A(z^0)$. Thus, every slant H-Toeplitz operator satisfies the above three conditions of the theorem. \square

In the following theorem, we have shown that there does not exist any non-zero self-adjoint slant H-Toeplitz operator on H^2 .

Theorem 3.6. *The slant H-Toeplitz operator V_ϕ with the symbol ϕ is self adjoint if and only if $\phi \equiv 0$.*

Proof. If $\phi \equiv 0$, then result is obvious. Conversely, suppose that the operator V_ϕ for some $\phi \in L^\infty$, is self-adjoint on H^2 . Since, $V_\phi = V_\phi^*$, therefore by Theorem 3.5, the operator V_ϕ^* satisfies the following:

- (a) $S^*V_\phi^*C_{z^2}S^2 = V_\phi^*C_{z^2}$.
- (b) $S^*V_\phi^*M_{z^3}C_{z^4} = V_\phi^*M_{z^3}C_{z^4}S$.
- (c) $S^*V_\phi^*z^0 = V_\phi^*M_{z^3}z^0$.

For $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, the relation (c) implies that $(S^*K^*M_{\bar{\phi}})W^*(1) = K^*M_{\bar{\phi}}W^*(z^3)$, or equivalently, $S^*K^*\left(\sum_{n=-\infty}^{\infty} \bar{a}_n z^n\right) = K^*\left(\sum_{n=-\infty}^{\infty} \bar{a}_n z^{n+6}\right)$

which gives that

$$S^* \left(\sum_{n=0}^{\infty} \overline{a_{-n}} z^{2n} + \sum_{n=0}^{\infty} \overline{a_{n+1}} z^{2n+1} \right) = K^* \left(\sum_{n=0}^{\infty} \overline{a_{-n-6}} z^n + \sum_{n=0}^{\infty} \overline{a_{n-5}} z^{-n-1} \right)$$

and this gives

$$\sum_{n=1}^{\infty} \overline{a_{-n}} z^{2n} + \sum_{n=1}^{\infty} \overline{a_{n+1}} z^{2n+1} = \sum_{n=0}^{\infty} \overline{a_{-n-6}} z^{2n} + \sum_{n=0}^{\infty} \overline{a_{n-5}} z^{2n+1}.$$

Therefore, on comparing the coefficients we get $\overline{a_{-6}} = 0$, $\overline{a_{-5}} = 0$ and $\overline{a_{-n}} = \overline{a_{-n-6}}$, $\overline{a_{n+1}} = \overline{a_{n-5}}$ for $n \geq 1$. Now since $a_n \rightarrow 0$ as $n \rightarrow \infty$, therefore this implies that $a_n = 0$ for each n and hence $\phi \equiv 0$. \square

Next we show that a non-zero slant Toeplitz operator can not be a slant H-Toeplitz operator on H^2 .

Theorem 3.7. *A slant Toeplitz operator B_ϕ is a slant H-Toeplitz operator if and only if $\phi \equiv 0$.*

Proof. Clearly if $\phi \equiv 0$, then the result is obvious. Conversely, assume that the slant Toeplitz operator B_ϕ is a slant H-Toeplitz operator on H^2 for some $\phi \in L^\infty$. Then, by using Theorem 3.5, the operator B_ϕ satisfies the following:

- (a) $S^* B_\phi C_{z^2} S^2 = B_\phi C_{z^2}$.
- (b) $S^* B_\phi M_{z^3} C_{z^4} = B_\phi M_{z^3} C_{z^4} S$.
- (c) $S^* B_\phi z^0 = B_\phi M_{z^3} z^0$.

Take $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$. Since, $S^* B_\phi C_{z^2} S^2(z^m) = S^* B_\phi(z^{2m+4})$ for all $m \geq 0$. Therefore, on using condition (a), it follows that $\langle S^* B_\phi C_{z^2} S^2 z^m, z^j \rangle = \langle B_\phi C_{z^2} z^m, z^j \rangle$. This gives $\langle B_\phi z^{2m+4}, z^{j+1} \rangle = \langle B_\phi z^{2m}, z^j \rangle$ which further implies that

$$\left\langle WP \left(\sum_{n=-\infty}^{\infty} a_n z^{2m+n+4} \right), z^{j+1} \right\rangle = \left\langle WP \left(\sum_{n=-\infty}^{\infty} a_n z^{n+2m} \right), z^j \right\rangle$$

$$\text{or, equivalently, } \left\langle \sum_{n=-2m-4}^{\infty} a_n z^{2m+n+4}, z^{2j+2} \right\rangle = \left\langle \sum_{n=-2m}^{\infty} a_n z^{n+2m}, z^{2j} \right\rangle.$$

Therefore, $a_{2j-2m-2} = a_{2j-2m}$ for all $m, j \geq 0$. Now on substituting $m, j = 0, 1, 2, 3, \dots$, we get that $a_0 = a_{2n}$ for all integer n . Since $a_n \rightarrow 0$ as $n \rightarrow \infty$, therefore for each integer n , we get that $a_{2n} = 0$. Now for $m \geq 0$, we have $S^* B_\phi M_{z^3} C_{z^4}(z^m) = S^* B_\phi M_{z^3}(z^{4m}) = S^* B_\phi(z^{4m+3})$ and $B_\phi M_{z^3} C_{z^4} S(z^m) = B_\phi M_{z^3}(z^{4m+4}) = B_\phi(z^{4m+7})$. Then, from the relation (b) it follows

$$\langle S^* B_\phi M_{z^3} C_{z^4} z^m, z^j \rangle = \langle B_\phi M_{z^3} C_{z^4} S z^m, z^j \rangle,$$

that is, $\langle S^* B_\phi z^{4m+3}, z^j \rangle = \langle B_\phi z^{4m+7}, z^j \rangle$. This further implies that

$$\left\langle WP \left(\sum_{n=-\infty}^{\infty} a_n z^{4m+n+3} \right), z^{j+1} \right\rangle = \left\langle WP \left(\sum_{n=-\infty}^{\infty} a_n z^{n+4m+7} \right), z^j \right\rangle \text{ or,}$$

equivalently, $\left\langle \sum_{n=-4m-3}^{\infty} a_n z^{n+4m+3}, z^{2j+2} \right\rangle = \left\langle \sum_{n=-4m-7}^{\infty} a_n z^{n+4m+7}, z^{2j} \right\rangle$.

Thus, it gives that $a_{2j-4m-1} = a_{2j-4m-7}$ for all $j, m \geq 0$ and this implies that $a_1 = a_{2n+1}$ for all integers n . Since $a_n \rightarrow 0$ as $n \rightarrow \infty$, therefore it follows that $a_{2n+1} = 0$ for all integers n and hence $\phi \equiv 0$. \square

Theorem 3.8. *If a slant Hankel operator L_ϕ is a slant H -Toeplitz operator on H^2 , then $\phi \in (z + z^3 H^\infty)^\perp$, where $(z + z^3 H^\infty) = \{z + z^3 \psi : \psi \in H^\infty\}$.*

Proof. Let the operator $L_\phi = WH_\phi$ be a slant H -Toeplitz operator on H^2 . Then, by Theorem 3.5, the operator WH_ϕ satisfies the following:

- (a) $S^*WH_\phi C_{z^2} S^2 = WH_\phi C_{z^2}$.
- (b) $S^*WH_\phi M_{z^3} C_{z^4} = WH_\phi M_{z^3} C_{z^4} S$.
- (c) $S^*WH_\phi z^0 = WH_\phi M_{z^3} z^0$.

Take $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty$. On using the relation (a), for each non-negative integers m and j , we obtain $\langle S^*WH_\phi C_{z^2} S^2 z^m, z^j \rangle = \langle WH_\phi C_{z^2} z^m, z^j \rangle$ which implies that $\langle H_\phi z^{2(m+2)}, z^{2(j+1)} \rangle = \langle H_\phi z^{2m}, z^{2j} \rangle$. So, from the matrix representation of the operator H_ϕ it follows that

$$(3.7) \quad a_{2m+2j+7} = a_{2m+2j+1} \quad \text{for all } m, j \geq 0.$$

Again on using the relation (b), it follows that

$$\begin{aligned} \langle S^*WH_\phi M_{z^3} C_{z^4} z^m, z^j \rangle &= \langle WH_\phi M_{z^3} C_{z^4} S z^m, z^j \rangle \text{ which implies that} \\ \langle H_\phi z^{4m+3}, z^{2(j+1)} \rangle &= \langle H_\phi z^{4m+7}, z^{2j} \rangle. \end{aligned}$$

Using matrix representation of the operator H_ϕ , the above condition is equivalent to following:

$$(3.8) \quad a_{4m+2j+6} = a_{4m+2j+8} \quad \text{for all } m, j \geq 0.$$

Moreover, from the relation (c), it follows that

$$\langle WH_\phi M_{z^3} z^0, z^j \rangle = \langle S^*WH_\phi z^0, z^j \rangle$$

and then $\langle H_\phi z^3, z^{2j} \rangle = \langle H_\phi z^0, z^{2(j+1)} \rangle$. Therefore, using the matrix representation of the operator H_ϕ , we obtain following relation:

$$(3.9) \quad a_{2j+4} = a_{2j+3} \quad \text{for all } j \geq 0.$$

On substituting $m, j = 0, 1, 2, \dots$ in equations (3.7), (3.8) and (3.9), we obtain

$$a_{2k-1} = a_{2k+5}, \quad a_{2k+1} = a_{2k+2} \quad \text{and} \quad a_{2k+4} = a_{2k+6}, \quad k \in \mathbb{N}.$$

This implies that $a_1 = a_n$ for each $n \geq 3$. Since $a_n \rightarrow 0$ as $n \rightarrow \infty$, we get that $\phi(z) = \sum_{n=-\infty}^0 a_n z^n + a_2 z^2$. Hence, $\phi \in (z + z^3 H^\infty)^\perp$. \square

We can extend the notion of slant H-Toeplitz operator to the space L^2 by defining the operator $\check{V}_\phi : L^2 \rightarrow L^2$ such that $\check{V}_\phi = WM_\phi K$, where $K : L^2 \rightarrow L^2$ defined as $Ke_{2n} = e_n$, $Ke_{2n+1} = e_{-n-1}$ and $W : L^2 \rightarrow L^2$ as $We_{2n} = e_n$, $We_{2n+1} = 0$ for each integer n . The same techniques can be applied to prove the results for \check{V}_ϕ .

The notion of slant H-Toeplitz operator on H^2 can be further extended to generalised slant H-Toeplitz operators, which can be defined as the operator $V_\phi^k \in \mathcal{B}(H^2)$ with the symbol $\phi \in L^\infty$ by $V_\phi^k(f) = W_k PM_\phi K(f)$ for each f in H^2 and $k \geq 2$, where the operator $W_k \in \mathcal{B}(L^2)$ is given by

$$W_k(e_n) = \begin{cases} e_{\frac{n}{k}}, & \text{if } k \text{ divides } n, \\ 0, & \text{otherwise} \end{cases}$$

for each integer n . Clearly, for $k = 2$, the operator V_ϕ^k is same as the slant H-Toeplitz operator V_ϕ . Moreover, results for the operator V_ϕ can be extended for the operator V_ϕ^k .

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