

## LIUVILLE THEOREMS FOR GENERALIZED SYMPHONIC MAPS

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ABSTRACT. In this paper, we introduce the notion of the generalized symphonic map with respect to the functional  $\Phi_\varepsilon$ . Then we use the stress-energy tensor to obtain some monotonicity formulas and some Liouville results for these maps. We also obtain some Liouville type results by assuming some conditions on the asymptotic behavior of the maps at infinity.

### 1. Introduction

Liouville type theorems for harmonic maps,  $p$ -harmonic maps,  $F$ -harmonic maps, and  $F$ -stationary maps were investigated by several authors ([1, 6, 10, 16, 18, 28, 29, 31, 32] and the references therein). It is well known that the stress-energy tensor is a useful tool to investigate the energy behavior and some vanishing results of related energy functional. Most Liouville results have established by assuming either the finiteness of the energy of the map or the smallness of whole image of the domain manifold under the map. In [18], Jin proved several interesting Liouville theorems for harmonic maps from complete manifolds, whose assumptions concern the asymptotic behavior of the maps at infinity. One special case of his results is that if  $u : (\mathbb{R}^m, g_0) \rightarrow (N^n, h)$  is a harmonic map, and  $u(x) \rightarrow p_0 \in N^n$  as  $|x| \rightarrow \infty$ , then  $u$  is a constant map. In [9], Dong, Lin and Yang, generalized Jin's method to  $F$ -harmonic maps and obtained some Liouville theorems and their applications.

Let  $\Omega \subset \mathbb{R}^2$  be a smooth bounded simply connected domain. Consider the following functional defined for maps  $u \in H^1(\Omega, \mathbf{C})$ :

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dv + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2 dv.$$

Ginzburg-Landau introduced this functional in study of phase transition problems and it plays an important role ever since, especially in superconductivity, superfluidity and XY-magnetism (see details for [21, 25, 27]). A lot of paper

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devote to the asymptotic behavior of minimizers  $u_\varepsilon$  of  $E_\varepsilon(u, \Omega)$  in  $H^1(\Omega, C)$  as  $\varepsilon \rightarrow 0$ . It was shown in those cases that  $u_\varepsilon$  converges strongly to a harmonic map  $u_0$  on any compact subset away from the zeros. Readers can refer to [3–5, 30] for the progress in this field. For general case, the  $p$ -Ginzburg-Landau functional has been introduced. Hong in [17] and Lei in [22] investigated the convergence of a  $p$ -Ginzburg-Landau type functional when parameter goes to zero. In [7], Chong, Cheng, Dong and Zhang investigated the critical points of the  $p$ -Ginzburg-Landau type functional and obtained some Liouville theorems for these maps.

Let  $(M^m, g)$  and  $(N^n, h)$  be two Riemannian manifolds, and let  $u$  be a smooth map from  $M$  to  $N$ . In [19], S. Kawai, and N. Nakauchi introduced a functional

$$\Phi(u) = \int_M \frac{\|u^*h\|^2}{4} dv_g,$$

where  $dv_g$  is the volume form on  $(M, g)$ ,  $u^*h$  is the symmetric 2-tensor (pullback metric) defined by  $u^*h(X, Y) = h(du(X), du(Y))$  for any vector fields  $X, Y$  on  $M$  and  $\|u^*h\|$ , its norm as  $\|u^*h\|^2 = \sum_{i,j=1}^m [h(du(e_i), du(e_j))]^2$ , with respect to  $\{e_i\}$  which is a local orthonormal frame on  $(M, g)$ . The map  $u$  is a symphonic (or stationary) map if it is a critical point of  $\Phi(u)$  with respect to any compact supported variation of  $u$  and  $u$  is symphonic (or stationary) stable if the second variation for the functional  $\Phi(u)$  is nonnegative. When  $M$  and  $N$  are compact without boundary, the same authors showed the non-existence of non-constant stable symphonic map for  $\Phi$ , if  $M$  (respectively  $N$ ) is a standard sphere  $S^m$  (respectively  $S^n$ ). Readers can refer to [1, 13, 14, 20, 23, 24] for the progress in this field.

In this paper, we can consider a smooth map  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  from a Riemannian manifold to the standard Euclidean space and the following functional

$$(1) \quad \Phi_\varepsilon(u) = \int_M \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g,$$

where  $\varepsilon$  is any small positive number. We call  $u$  a generalized symphonic map for the functional  $\Phi_\varepsilon(u)$ , if

$$\frac{d}{dt} \Phi_\varepsilon(u_t)|_{t=0} = 0$$

for any compactly supported variation  $u_t : (M, g) \rightarrow (\mathbb{R}^n, h)$  with  $u_0 = u$ . To generalized the Liouville type results for harmonic maps to the generalized symphonic maps, we first introduce the stress-energy tensor  $S_{\Phi_\varepsilon}$  associated with the functional  $\Phi_\varepsilon(u)$ . We prove that the generalized symphonic map satisfies the conservation law, that is,  $div S_{\Phi_\varepsilon} = 0$ . By using the stress-energy tensor, we obtain some monotonicity formulas for these maps, then we can prove some Liouville type results from these monotonicity formulas under suitable growth conditions on the functional  $\Phi_\varepsilon(u)$ .

Next we generalize Jin’s method and results to the generalized symphonic maps. The procedure consists of two steps. The first step is to use the stress-energy tensor to establish the monotonicity formula which gives a lower bound for the growth rates of the functional  $\Phi_\varepsilon(u)$ . The second step is to use the asymptotic assumption of the maps at infinity to obtain the upper functional growth rates of the generalized symphonic maps. Under suitable conditions on  $u$  and the Hessian of the distance functions of the domain manifolds, one may show that these two growth rates are contradictory unless the generalized symphonic map is constant. In this way, we establish some Liouville theorems for the generalized maps with asymptotic property at infinity from some complete manifolds.

**2. The first variation formula**

Let  $\nabla$  and  $\mathbb{R}^n \nabla$  be always denote the Levi-Civita connections of  $(M^m, g)$  and  $(\mathbb{R}^n, h)$  respectively. Let  $\tilde{\nabla}$  be the induced connection on  $u^{-1}T\mathbb{R}^n$  defined by  $\tilde{\nabla}_X W = \mathbb{R}^n \nabla_{du(X)} W$ , where  $X$  is a tangent vector of  $M^m$  and  $W$  is a section of  $u^{-1}T\mathbb{R}^n$ . We can choose a local orthonormal frame field  $\{e_i\}_{i=1}^m$  on  $M^m$ . We define the  $\Phi_\varepsilon$ -tension field  $\tau_{\Phi_\varepsilon}(u)$  by

$$(2) \quad \tau_{\Phi_\varepsilon}(u) = \operatorname{div}_g(\sigma_u) + \frac{1}{\varepsilon^n}(1 - |u|^2)u,$$

where  $\sigma_u(X) = \sum_{i=1}^n h(du(e_i), du(X))du(e_i)$ , for any smooth vector field  $X$  on  $M$ . Let  $u_t : (M^m, g \rightarrow (\mathbb{R}^n, h))$ ,  $|t| < k$  with  $u_0 = u$  and  $v = \frac{\partial u}{\partial t}|_{t=0}$  be a one parameter compactly supported variation, we have the following lemma.

**Lemma 2.1** (The first variation formula). *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  be a  $C^2$  map. Then we have*

$$(3) \quad \frac{d}{dt}|_{t=0} \Phi_\varepsilon(u_t) = - \int_M h(\tau_{\Phi_\varepsilon}(u), v) dv_g.$$

*Proof.* Let  $\{e_i\}$  be a local orthonormal frame of  $TM$ . Since the target manifold is the Standard Euclidean space  $\mathbb{R}^n$ , we can perform the following calculations,

$$\begin{aligned} \frac{d}{dt}|_{t=0} \Phi_\varepsilon(u_t) &= \int_M \left[ \frac{\partial}{\partial t}|_{t=0} \left( \frac{\|u_t^* h\|^2}{4} \right) + \frac{\partial}{\partial t}|_{t=0} \left[ \frac{1}{4\varepsilon^n} (1 - |u_t|^2)^2 \right] \right] dv_g \\ &= \int_M \sum_{i,j=1}^m h(\tilde{\nabla}_{\frac{\partial}{\partial t}} du_t(e_i), du_t(e_j)) h(du_t(e_i), du_t(e_j))|_{t=0} dv_g \\ &\quad - \int_M \frac{1}{\varepsilon^n} (1 - |u|^2) h(v, u) dv_g \\ &= \int_M \sum_{i=1}^m h(\tilde{\nabla}_{e_i} du_t(\frac{\partial}{\partial t}), \sigma_{u_t}(e_i))|_{t=0} dv_g \\ &\quad - \int_M \frac{1}{\varepsilon^n} (1 - |u|^2) h(v, u) dv_g \end{aligned}$$

$$\begin{aligned}
 &= \int_M \sum_{i=1}^m h(\tilde{\nabla}_{e_i} v, \sigma_u(e_i)) dv_g - \int_M \frac{1}{\varepsilon^n} (1 - |u|^2) h(v, u) dv_g \\
 &= - \int_M h(\operatorname{div}_g(\sigma_u) + \frac{1}{\varepsilon^n} (1 - |u|^2) u, v) dv_g \\
 &= - \int_M h(\tau_{\Phi_\varepsilon}(u), v) dv_g.
 \end{aligned}$$

Here we have used the Green’s theorem in the fifth equality. □

The first variation formula allows us to define the notion of generalized symphonic map for the functional  $\Phi_\varepsilon(u)$ .

**Definition 2.2.** A  $C^2$  map  $u$  is called generalized symphonic map for the functional  $\Phi_\varepsilon(u)$  if it is a solution of the Euler-Lagrange equation,

$$(4) \quad \tau_{\Phi_\varepsilon}(u) = \operatorname{div}_g(\sigma_u) + \frac{1}{\varepsilon^n} (1 - |u|^2) u = 0.$$

### 3. Stress-energy tensor

Following Baird [2], for a smooth map  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$ , we associate a symmetric 2-tensor  $S_{\Phi_\varepsilon}$  to the functional  $\Phi_\varepsilon$  called the stress-energy tensor

$$(5) \quad S_{\Phi_\varepsilon}(X, Y) = \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] g(X, Y) - h(du(X), \sigma_u(Y)),$$

where  $X, Y$  are smooth vector field on  $M$ .

**Proposition 3.1.** *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  be a smooth map and  $S_{\Phi_\varepsilon}$  be the associated stress-energy tensor. Then for each vector field  $X$  on  $M$ , we have*

$$(\operatorname{div} S_{\Phi_\varepsilon})(X) = -h(\operatorname{div}_g \sigma_u + \frac{1}{\varepsilon^n} (1 - |u|^2) u, du(X)).$$

*Proof.* Let  $\nabla$  and  $\mathbb{R}^n \nabla$  always denote the Levi-Civita connections of  $M$  and  $\mathbb{R}^n$  respectively. Let  $\tilde{\nabla}$  be the induced connection on  $u^{-1}T\mathbb{R}^n$ . We choose a local orthonormal frame field  $\{e_i\}$  around a point  $P$  on  $M$  with  $\nabla_{e_i} e_j|_P = 0$ .

Let  $X$  be a vector field on  $M$ . At  $P$ , we compute

$$\begin{aligned}
 &(\operatorname{div} S_{\Phi_\varepsilon})(X) \\
 &= \sum_{i=1}^m (\nabla_{e_i} S_{\Phi_\varepsilon})(e_i, X) \\
 &= \sum_{i=1}^m [e_i S_{\Phi_\varepsilon}(e_i, X) - S_{\Phi_\varepsilon}(e_i, \nabla_{e_i} X)] \\
 &= \sum_{i=1}^m \left\{ e_i \left( \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] g(e_i, X) \right) - e_i h(\sigma_u(e_i), du(X)) \right. \\
 &\quad \left. - \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] g(e_i, \nabla_{e_i} X) + h(\sigma_u(e_i), du(\nabla_{e_i} X)) \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{i=1}^m [e_i \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n}(1 - |u|^2)^2 \right] g(e_i, X) - h(\tilde{\nabla}_{e_i}\sigma_u(e_i), du(X)) \\
 &\quad - h(\sigma_u(e_i), \tilde{\nabla}_{e_i} du(X)) + h(\sigma_u(e_i), du(\nabla_{e_i}X))] \\
 &= \sum_{i=1}^m \left[ X \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n}(1 - |u|^2)^2 \right] - h(\sigma_u(e_i), (\nabla_{e_i} du)(X)) \right] \\
 &\quad - h(\operatorname{div}(\sigma_u), du(X)) \\
 &= \sum_{i=1}^m \left[ -\frac{1}{\varepsilon^n}(1 - |u|^2)h(u, du(X)) \right. \\
 &\quad \left. + \sum_{j=1}^m h((\nabla_X du)(e_i), du(e_j))h(du(e_i), du(e_j)) \right. \\
 &\quad \left. - h(\sigma_u(e_i), (\nabla_{e_i} du)(X)) - h(\operatorname{div}(\sigma_u), du(X)) \right] \\
 &= \sum_{i=1}^m [h((\nabla_X du)(e_i), \sigma_u(e_i)) - h(\sigma_u(e_i), (\nabla_{e_i} du)(X))] \\
 &\quad - h(\operatorname{div}(\sigma_u) + \frac{1}{\varepsilon^n}(1 - |u|^2)u, du(X)).
 \end{aligned}$$

Since  $(\nabla_X du)(e_i) = (\nabla_{e_i} du)(X)$ , we have

$$(\operatorname{div}S_{\Phi_\varepsilon})(X) = -h \left( \operatorname{div}\sigma_u + \frac{1}{\varepsilon^n}(1 - |u|^2)u, du(X) \right).$$

This completes the proof. □

**Definition 3.2.** We say that  $u$  satisfies the conservation law if  $\operatorname{div}S_{\Phi_\varepsilon} = 0$ .

By the above proposition, we can obtain the following result.

**Corollary 3.3.** *If  $u : (M, g) \rightarrow (\mathbb{R}^n, h)$  is a generalized symphonic map, then  $u$  satisfies the conservation law, i.e.,  $\operatorname{div}S_{\Phi_\varepsilon} = 0$ .*

Recall that for two 2-tensors  $T_1, T_2 \in \Gamma(T^*M \otimes T^*M)$ , their inner product is defined as follows;

$$(6) \quad \langle T_1, T_2 \rangle = \sum_{i,j=1}^m T_1(e_i, e_j)T_2(e_i, e_j),$$

where  $\{e_i\}$  is an orthonormal basis with respect to  $g$ . For a vector field  $X \in \Gamma(TM)$ , we denote by  $\theta_X$  its dual one form, i.e.,  $\theta_X(Y) = g(X, Y)$ , where  $Y \in \Gamma(TM)$ . The covariant derivative of  $\theta_X$  gives a 2-tensor field  $\nabla\theta_X$ :

$$(7) \quad (\nabla\theta_X)(Y, Z) = (\nabla_Y\theta_X)(Z) = g(\nabla_Y X, Z).$$

If  $X = \nabla\varphi$  is the gradient field of some  $C^2$  function  $\varphi$  on  $M$ , then  $\theta_X = d\varphi$  and  $\nabla\theta_X = \operatorname{Hess}\varphi$ .

**Lemma 3.4** (cf. [2,10]). *Let  $T$  be a symmetric  $(0, 2)$ -type tensor field and let  $X$  be a vector field. Then*

$$(8) \quad \operatorname{div}(i_X T) = (\operatorname{div} T)(X) + \langle T, \nabla \theta_X \rangle = (\operatorname{div} T)(X) + \frac{1}{2} \langle T, L_X g \rangle,$$

where  $L_X$  is the Lie derivative of the metric  $g$  in the direction of  $X$ . Indeed, let  $\{e_1, \dots, e_m\}$  be a local orthonormal frame field on  $M$ . Then

$$\begin{aligned} \frac{1}{2} \langle T, L_X g \rangle &= \sum_{i,j=1}^m \frac{1}{2} \langle T(e_i, e_j), L_X g(e_i, e_j) \rangle \\ &= \sum_{i,j=1}^m T(e_i, e_j) g(\nabla_{e_i} X, e_j) = \langle T, \nabla \theta_X \rangle. \end{aligned}$$

Let  $D$  be any bounded domain of  $M$  with  $C^1$  boundary. By using the Stokes' theorem, we have the following integral formula:

$$(9) \quad \int_{\partial D} T(X, \nu) ds_g = \int_D [\langle T, \frac{1}{2} L_X g \rangle + (\operatorname{div} T)(X)] dv_g,$$

where  $\nu$  is the unit outward normal vector field along  $\partial D$ . By the definition of generalized symphonic map and (9), we have

$$(10) \quad \int_{\partial D} S_{\Phi_\varepsilon}(X, \nu) ds_g = \int_D \langle S_{\Phi_\varepsilon}, \frac{1}{2} L_X g \rangle dv_g.$$

#### 4. Monotonicity formulas

Let  $(M^m, g)$  be a complete Riemannian manifold with a pole  $x_0$ . A pole  $x_0 \in M$  is a point such that the exponential map from the tangent space to  $M$  at  $x_0$  is a diffeomorphism. Denote by  $r(x)$  the  $g$ -distance function relative to the pole  $x_0$ , that is,  $r(x) = \operatorname{dist}_g(x, x_0)$ . Set  $B(r) = \{x \in M^m : r(x) \leq r\}$ . It is known that  $\frac{\partial}{\partial r}$  is always an eigenvector of  $\operatorname{Hess}_g(r^2)$  associated to eigenvalue 2. Denote by  $\lambda_{\max}$  (resp.  $\lambda_{\min}$ ) the maximum (resp. minimal) eigenvalues of  $\operatorname{Hess}_g(r^2) - 2dr \otimes dr$  at each point of  $M \setminus \{x_0\}$ .

**Theorem 4.1.** *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  be a generalized symphonic map from a complete Riemannian manifold  $(M^m, g)$  to  $(\mathbb{R}^n, h)$ . If there exists a constant  $\sigma > 0$  such that*

$$(11) \quad 1 + \frac{m-1}{2} \lambda_{\min} - 2 \max\{2, \lambda_{\max}\} \geq \sigma,$$

then

$$\frac{\int_{B(\rho_1)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g}{\rho_1^\sigma} \leq \frac{\int_{B(\rho_2)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g}{\rho_2^\sigma}$$

for any  $0 < \rho_1 \leq \rho_2$ . In particular, if  $\int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g = o(R^\sigma)$ , then  $u$  is constant and  $u \in S^{n-1}$ .

*Proof.* We take  $D = B(R)$  and  $X = r \frac{\partial}{\partial r} = \frac{1}{2} \nabla r^2$  in (10), where  $\nabla$  denotes the covariant derivative determined by  $g$ , we have

$$(12) \quad \int_{\partial B(R)} S_{\Phi_\varepsilon} \left( r \frac{\partial}{\partial r}, \nu \right) ds_g = \int_{B(R)} \left\langle S_{\Phi_\varepsilon}, \frac{1}{2} L_{r \frac{\partial}{\partial r}} g \right\rangle dv_g.$$

Let  $\{e_i\}_{i=1}^m$  be an orthonormal basis with respect to  $g$  and  $e_m = \nu = \frac{\partial}{\partial r}$ . We may assume that  $Hess_g(r^2)$  becomes a diagonal matrix with respect to  $\{e_i\}_{i=1}^m$ . Now we compute,

$$\begin{aligned} \langle S_{\Phi_\varepsilon}, L_{r \frac{\partial}{\partial r}} g \rangle &= \sum_{i,j=1}^m S_{\Phi_\varepsilon}(e_i, e_j) (L_{r \frac{\partial}{\partial r}} g)(e_i, e_j) \\ &= \sum_{i,j=1}^m \left[ \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] g(e_i, e_j) (L_{r \frac{\partial}{\partial r}} g)(e_i, e_j) \right. \\ &\quad \left. - h(du(e_i), \sigma_u(e_j)) (L_{r \frac{\partial}{\partial r}} g)(e_i, e_j) \right] \\ &= \sum_{i=1}^m \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] Hess_g(r^2)(e_i, e_i) \\ &\quad - \sum_{i,j=1}^m h(du(e_i), \sigma_u(e_j)) Hess_g(r^2)(e_i, e_j) \\ &\geq \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] [2 + (m - 1)\lambda_{\min}] \\ &\quad - \max\{2, \lambda_{\max}\} \sum_{i=1}^m h(du(e_i), \sigma_u(e_i)) \\ &= \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] [2 + (m - 1)\lambda_{\min}] \\ &\quad - 4 \max\{2, \lambda_{\max}\} \frac{\|u^* h\|^2}{4} \\ (13) \quad &\geq \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] [2 + (m - 1)\lambda_{\min} - 4 \max\{2, \lambda_{\max}\}]. \end{aligned}$$

From (11) and (13), we have

$$(14) \quad \langle S_{\Phi_\varepsilon}, \frac{1}{2} L_{r \frac{\partial}{\partial r}} g \rangle \geq \sigma \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right].$$

On the other hand, by the coarea formula, we have

$$\begin{aligned} &\int_{\partial B(r)} S_{\Phi_\varepsilon} \left( r \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) ds_g \\ &= r \int_{\partial B(r)} \left[ \frac{\|u^* h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] ds_g \end{aligned}$$

$$\begin{aligned}
& -r \int_{\partial B(r)} h(du(\frac{\partial}{\partial r}), \sigma_u(\frac{\partial}{\partial r})) \\
& = r \int_{\partial B(r)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] ds_g \\
& \quad - r \int_{\partial B(r)} \left[ \sum_{j=1}^m h^2(du(\frac{\partial}{\partial r}), du(e_j)) \right] \\
& \leq r \int_{\partial B(r)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] ds_g \\
(15) \quad & = r \frac{d}{dr} \int_{B(r)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g.
\end{aligned}$$

From (12), (14) and (15), we have

$$\begin{aligned}
& \sigma \int_{B(r)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g \\
& \leq r \frac{d}{dr} \int_{B(r)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g,
\end{aligned}$$

so we have

$$\frac{\int_{B(\rho_1)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g}{\rho_1^\sigma} \leq \frac{\int_{B(\rho_2)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g}{\rho_2^\sigma}$$

for any  $0 < \rho_1 \leq \rho_2$ .  $\square$

**Lemma 4.2** ([8, 10–13, 15]). *Let  $(M^m, g)$  be a complete Riemannian manifold with a pole  $x_0$ . Denote by  $K_r$  the radial curvature of  $M^m$ .*

(1) *If  $-\alpha^2 \leq K_r \leq -\beta^2$  with  $\alpha \geq \beta \geq 0$  and  $(m-1)\beta - 4\alpha > 0$ , then*

$$1 + \frac{m-1}{2} \lambda_{\min} - 2 \max\{2, \lambda_{\max}\} \geq m - \frac{4\alpha}{\beta}.$$

(2) *If  $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$  with  $\epsilon > 0$ ,  $A \geq 0$  and  $0 \leq B \leq 2\epsilon$ , then*

$$1 + \frac{m-1}{2} \lambda_{\min} - 2 \max\{2, \lambda_{\max}\} \geq 1 + (m-1) \left(1 - \frac{B}{2\epsilon}\right) - 4e^{\frac{A}{2\epsilon}}.$$

(3) *If  $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$  with  $a \geq 0$ ,  $b^2 \in [0, \frac{1}{4}]$  and  $c^2 \geq 0$ , then*

$$\begin{aligned}
& 1 + \frac{m-1}{2} \lambda_{\min} - 2 \max\{2, \lambda_{\max}\} \\
& \geq 1 + (m-1) \frac{1 + \sqrt{1-4b^2}}{2} - 4 \frac{1 + \sqrt{1-4a^2}}{2}.
\end{aligned}$$

From Theorem 4.1 and Lemma 4.2, we have the following result.



**Corollary 4.3.** *Let  $(M^m, g)$  be an  $m$ -dimensional complete manifold with a pole  $x_0$ . Assume that the radial curvature  $K_r$  of  $M$  satisfies one of the following three conditions:*

- (1) *If  $-\alpha^2 \leq K_r \leq -\beta^2$  with  $\alpha \geq \beta \geq 0$  and  $(m-1)\beta - 4\alpha > 0$ .*
- (2) *If  $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$  with  $\epsilon > 0$ ,  $A \geq 0$ ,  $0 \leq B \leq 2\epsilon$  and  $1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}} > 0$ .*
- (3) *If  $-\frac{a^2}{c^2+r^2} \leq K_r \leq \frac{b^2}{c^2+r^2}$  with  $a \geq 0$ ,  $b^2 \in [0, \frac{1}{4}]$  and  $1+(m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2} > 0$ . If  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  is a generalized symphonic map, then*

$$\frac{\int_{B(\rho_1)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g}{\rho_1^\Lambda} \leq \frac{\int_{B(\rho_2)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g}{\rho_2^\Lambda}$$

for  $0 < \rho_1 \leq \rho_2$ , where  $\Lambda$  is given as follows,

$$\Lambda = \begin{cases} m - \frac{4\alpha}{\beta} & \text{if } K_r \text{ satisfies (1),} \\ 1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}} & \text{if } K_r \text{ satisfies (2),} \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2} & \text{if } K_r \text{ satisfies (3).} \end{cases}$$

In particular, if  $\int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g = o(R^\Lambda)$ , then  $u$  is constant and  $u \in S^{n-1}$ .

### 5. Liouville theorems

We first give the lower  $\Phi_\epsilon$  functional growth rates for generalized symphonic map.

**Proposition 5.1.** *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  be a generalized symphonic map from a Riemannian manifold with a pole  $x_0$  to a standard Euclidean space. If  $u(M)$  is not contained in  $S^{n-1}$  and  $r(x)$  satisfies the condition (11), then*

$$\int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g \geq C(u)R^\sigma \quad \text{as } R \rightarrow \infty,$$

where  $C(u)$  is a positive constant only depending on  $u$ .

*Proof.* Since  $u$  satisfies the condition in Theorem 4.1, we have

$$\frac{\int_{B(\rho)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g}{\rho^\sigma} \leq \frac{\int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g}{R^\sigma}$$

for any  $0 < \rho < R$ . Note that  $u(M)$  is not contained in  $S^{n-1}$ , there exists some  $\rho > 0$  such that

$$\int_{B(\rho)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right] dv_g > 0.$$

Set  $C(u) = \frac{\int_{B(\rho)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2 \right] dv_g}{\rho^\sigma}$ , then

$$\int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2 \right] dv_g \geq C(u)R^\sigma.$$

This completes the proof of the proposition. □

**Corollary 5.2.** *Let  $(M^m, g)$  be an  $m$ -dimensional complete manifold with a pole  $x_0$ . Assume that the radial curvature  $K_r$  satisfies the condition in Corollary 4.3. If  $u : (M, g) \rightarrow (\mathbb{R}^n, h)$  is a generalized symphonic map and  $u(M)$  is not contained in  $S^{n-1}$ , then we have*

$$\int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2 \right] dv_g \geq C(u)R^\Lambda \quad \text{as } R \rightarrow \infty,$$

where  $C(u)$  is a positive constant only depending on  $u$  and  $\Lambda$  is given in Corollary 4.3.

Next we will use the assumption for the map at infinity to derivative an upper bound for the growth rate. The condition that we will assume for  $u$  is as follow:

(P<sub>1</sub>) There exists a positive constant  $\tilde{\sigma}$  strictly less than  $\sigma$  in (11) such that

$$\left[ \max_{r(x)=r} h^2(u(x), P_0) \right]^{\frac{2}{3}} \leq r^{\frac{\tilde{\sigma}}{3}} \int_r^\infty \frac{ds}{[vol(\partial B(s))]^{\frac{1}{3}}} \quad \text{for } r(x) \gg 1,$$

where  $P_0$  is a fixed point in  $S^{p-1}$ .

**Theorem 5.3.** *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$  be a generalized symphonic map. Suppose that  $r(x)$  satisfies the condition (11). If  $u(x) \rightarrow P_0 \in S^{n-1}$  as  $r(x) \rightarrow \infty$  and  $u$  satisfies the condition (P<sub>1</sub>), then  $u$  must be a constant map.*

*Proof.* Suppose  $u$  is not constant, then by Proposition 5.1, the functional  $\Phi_\varepsilon$  of  $u$  must be infinite, that is,  $\Phi_\varepsilon^R(u) = \int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n}(1-|u|^2)^2 \right] dv_g \rightarrow \infty$ , as  $R \rightarrow \infty$ .

Since  $P_0 = (c_1, \dots, c_n) \in S^{n-1}$ , then  $\sum_{\alpha=1}^n c_\alpha^2 = 1$ . It is clear that we can choose an orthogonal matrix  $A$  such that  $AP_0 = \tilde{P}_0 = (\tilde{c}_1, \dots, \tilde{c}_n)$ ,  $\tilde{c}_\alpha \neq 0$ , for  $\alpha = 1, \dots, n$ . Clearly if  $u$  is a generalized map, then  $Au$  is also the generalized map. Hence without loss of generality, we may assume that  $u(x) \rightarrow P_0$  as  $r(x) \rightarrow \infty$ , where  $P_0 = (c_1, \dots, c_n)$ , where  $c_\alpha \neq 0$ , for  $\alpha = 1, \dots, n$ .

Now the assumption that  $u(x) \rightarrow P_0$  as  $r(x) \rightarrow \infty$  implies that there exist a  $R_1 > 0$  and a neighborhood  $U$  of  $P_0$  such that for  $r(x) > R_1$ ,  $u(x) \in U$  and  $u_\alpha \neq 0$  for  $\alpha = 1, \dots, n$ .

For  $\omega \in C_0^2(M \setminus B(R_1), U)$ , we consider the variation  $u + t\omega : M \rightarrow \mathbb{R}^n$  defined as follows:

$$(u + t\omega)(q) = \begin{cases} u(q) & q \in B(R_1), \\ (u + t\omega)(q) & q \in M \setminus B(R_1) \end{cases}$$

for sufficiently small  $t$ . Since  $u$  is a generalized symphonic map, we have

$$\frac{d}{dt}\Big|_{t=0}\Phi_\varepsilon(u + t\omega) = 0,$$

that is,

$$\int_{M \setminus B(R_1)} \left[ \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \omega_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} - \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \sum_{\alpha=1}^n u_\alpha \omega_\alpha \right] dv_g = 0.$$

Choosing  $\omega = \phi(r(x))\tilde{u}$  in the above equation for  $\phi(t) \in C_0^\infty(R_1, \infty)$ ,  $\tilde{u}_\alpha = \frac{u_\alpha^2 - c_\alpha^2}{u_\alpha}$ , we obtain

$$\begin{aligned} & \int_{M \setminus B(R_1)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \phi(r(x)) dv_g \\ & - \int_{M \setminus B(R_1)} \frac{1}{\varepsilon^n} \left(1 - \sum_{\alpha=1}^n u_\alpha^2\right) \phi(r(x)) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha dv_g \\ (16) \quad & = - \int_{M \setminus B(R_1)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g. \end{aligned}$$

By a standard approximation argument, (16) holds for any Lipschitz function  $\phi$  with compact support.

For  $0 \leq \epsilon \leq 1$ , define

$$\varphi_\epsilon(t) = \begin{cases} 1 & t \leq 1, \\ 1 + \frac{1-t}{\epsilon} & 1 < t < 1 + \epsilon, \\ 0 & t \geq 1 + \epsilon \end{cases}$$

and choose the Lipschitz function  $\phi(r(x))$  to be

$$\phi(r(x)) = \varphi_\epsilon \left( \frac{r(x)}{R} \right) \left( 1 - \varphi_1 \left( \frac{r(x)}{R_1} \right) \right), \quad R > 2R_1.$$

Then the first term on the left hand side of (16) becomes

$$\begin{aligned} & \int_{M \setminus B(R_1)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \phi(r(x)) dv_g \\ & = \int_{B(R_2) \setminus B(R_1)} \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \left( 1 - \varphi_1 \left( \frac{r(x)}{R_1} \right) \right) dv_g \\ & \quad + \int_{B(R) \setminus B(R_2)} \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \end{aligned}$$

$$(17) \quad + \int_{B((1+\epsilon)R) \setminus B(R)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \varphi_\epsilon \left( \frac{r(x)}{R} \right) dv_g,$$

where  $R_2 = 2R_1$ . The second term on the left hand side of (16) becomes

$$\begin{aligned} & - \int_{M^m \setminus B(R_1)} \frac{1}{\epsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right) \phi(r(x)) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha dv_g \\ & = - \int_{B(R_2) \setminus B(R_1)} \frac{1}{\epsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right) \left( 1 - \varphi_1 \left( \frac{r(x)}{R_1} \right) \right) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha dv_g \\ & \quad - \int_{B(R) \setminus B(R_2)} \frac{1}{\epsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha dv_g \\ (18) \quad & - \int_{B((1+\epsilon)R) \setminus B(R)} \frac{1}{\epsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right) \varphi_\epsilon \left( \frac{r(x)}{R} \right) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha dv_g. \end{aligned}$$

The term on the right hand of (16) becomes,

$$\begin{aligned} & - \int_{M^m \setminus B(R_1)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \phi(r(x))}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\ & = \int_{B(R_2) \setminus B(R_1)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_1 \left( \frac{r(x)}{R_1} \right)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\ & \quad - \int_{B((1+\epsilon)R) \setminus B(R)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_\epsilon \left( \frac{r(x)}{R} \right)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\ & = \int_{B(R_2) \setminus B(R_1)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_1 \left( \frac{r(x)}{R_1} \right)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g \\ (19) \quad & + \frac{1}{R\epsilon} \int_{B((1+\epsilon)R) \setminus B(R)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g. \end{aligned}$$

From (16)-(19) and let  $\epsilon \rightarrow 0$ , we have

$$\begin{aligned} & \int_{B(R) \setminus B(R_2)} \left[ \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \right. \\ & \quad \left. - \frac{1}{\epsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha \right] dv_g + D(R_1) \\ (20) \quad & = \int_{\partial B(R)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g, \end{aligned}$$

where  $D(R_1)$  is defined as follows:

$$\begin{aligned}
 D(R_1) &= \int_{B(R_2) \setminus B(R_1)} \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \tilde{u}_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \left( 1 - \varphi_1 \left( \frac{r(x)}{R_1} \right) \right) dv_g \\
 &\quad - \int_{B(R) \setminus B(R_2)} \frac{1}{\varepsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right) \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha dv_g \\
 &\quad - \int_{B(R_2) \setminus B(R_1)} \sum_{i, j, k, l=1}^m \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial \varphi_1 \left( \frac{r(x)}{R_1} \right)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g.
 \end{aligned}$$

Note that  $\tilde{u}_\alpha = \frac{u_\alpha^2 - c_\alpha^2}{u_\alpha}$ . Thus we have

$$(21) \quad \frac{\partial \tilde{u}_\alpha}{\partial x_j} = \left( 1 + \frac{c_\alpha^2}{u_\alpha^2} \right) \frac{\partial u_\alpha}{\partial x_j}$$

and

$$(22) \quad \sum_{\alpha=1}^n u_\alpha \tilde{u}_\alpha = \sum_{\alpha=1}^n [u_\alpha^2 - c_\alpha^2] = - \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right).$$

From (20), (21) and (22), we have

$$\begin{aligned}
 &\int_{B(R) \setminus B(R_2)} \left[ \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \left( 1 + \frac{c_\alpha^2}{u_\alpha^2} \right) \right. \\
 &\quad \left. + \frac{1}{\varepsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right)^2 \right] dv_g + D(R_1) \\
 (23) \quad &= \int_{\partial B(R)} \sum_{i, j, k, l=1}^m \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} dv_g.
 \end{aligned}$$

Now we estimate the term on the right hand of (23). Take any point  $p \in \partial B(R)$ . Since the term  $\sum_{i, j, k, l=1}^m \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l}$  does not depend on the coordinate system on  $M$ . At the point  $p$ , we can take the adapt coordinate system, such that  $g_{ij}(p) = \delta_{ij}$  and  $g^{ij}(p) = \delta^{ij}$ . We compute at  $p$ .

$$\begin{aligned}
 &\sum_{i, j, k, l=1}^m \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_\alpha \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \\
 &= \sum_{i, j=1}^m \left[ \sum_{\alpha}^n \frac{\partial u_\alpha}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_\alpha \right] \left[ \sum_{\beta=1}^n \frac{\partial u_\beta}{\partial x_i} \frac{\partial u_\beta}{\partial x_j} \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq \left[ \sum_{i,j=1}^m \left( \sum_{\alpha}^n \frac{\partial u_{\alpha}}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_{\alpha} \right)^2 \right]^{\frac{1}{2}} \left[ \sum_{i,j=1}^m \left( \sum_{\beta=1}^n \frac{\partial u_{\beta}}{\partial x_i} \frac{\partial u_{\beta}}{\partial x_j} \right)^2 \right]^{\frac{1}{2}} \\
&= \left[ \sum_{i=1}^m \left( \sum_{\alpha}^n \frac{\partial u_{\alpha}}{\partial x_i} \tilde{u}_{\alpha} \right)^2 \right]^{\frac{1}{2}} \left[ \sum_{i,j=1}^m \left( \sum_{\beta=1}^n \frac{\partial u_{\beta}}{\partial x_i} \frac{\partial u_{\beta}}{\partial x_j} \right)^2 \right]^{\frac{1}{2}} \\
&= \left[ \sum_{i=1}^m \left( \sum_{\alpha}^n \frac{\partial u_{\alpha}}{\partial x_i} \tilde{u}_{\alpha} \right)^2 \right]^{\frac{1}{2}} \left[ \sum_{i,j=1}^m \left( h\left( du\left(\frac{\partial}{\partial x_i}\right), du\left(\frac{\partial}{\partial x_j}\right) \right) \right)^2 \right]^{\frac{1}{2}} \\
&= \left[ \sum_{i=1}^m \left( \sum_{\alpha}^n \frac{\partial u_{\alpha}}{\partial x_i} \tilde{u}_{\alpha} \right)^2 \right]^{\frac{1}{2}} \|u^* h\| \\
&\leq \left[ \sum_{i=1}^m \left( \sum_{\alpha}^n \frac{\partial u_{\alpha}}{\partial x_i} \frac{\partial u_{\alpha}}{\partial x_i} \right) \left( \sum_{\alpha=1}^m \tilde{u}_{\alpha}^2 \right) \right]^{\frac{1}{2}} \|u^* h\| \\
&\leq \sqrt[4]{m} \left[ \left[ \sum_{i=1}^m \left( h\left( du\left(\frac{\partial}{\partial x_i}\right), du\left(\frac{\partial}{\partial x_i}\right) \right) \right)^2 \right]^{\frac{1}{2}} \left( \sum_{\alpha=1}^m \tilde{u}_{\alpha}^2 \right) \right]^{\frac{1}{2}} \|u^* h\| \\
&\leq \sqrt[4]{m} \left[ \left[ \sum_{i,j=1}^m \left( h\left( du\left(\frac{\partial}{\partial x_i}\right), du\left(\frac{\partial}{\partial x_j}\right) \right) \right)^2 \right]^{\frac{1}{2}} \left( \sum_{\alpha=1}^m \tilde{u}_{\alpha}^2 \right) \right]^{\frac{1}{2}} \|u^* h\| \\
&= \sqrt[4]{m} \|u^* h\|^{\frac{3}{2}} \left( \sum_{\alpha=1}^m \tilde{u}_{\alpha}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Here we have used  $|\nabla r|^2 = 1$  in the second equality and

$$\|u^* h\|^2 = \sum_{i,j=1}^m \left( h\left( du\left(\frac{\partial}{\partial x_i}\right), du\left(\frac{\partial}{\partial x_j}\right) \right) \right)^2$$

at  $p$ . So we have

$$\sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_{\alpha}}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_{\alpha} \frac{\partial u_{\beta}}{\partial x_k} \frac{\partial u_{\beta}}{\partial x_l} \leq \sqrt[4]{m} \|u^* h\|^{\frac{3}{2}} \left( \sum_{\alpha=1}^m \tilde{u}_{\alpha}^2 \right)^{\frac{1}{2}}$$

and

(24)

$$\int_{\partial B(R)} \sum_{i,j,k,l=1}^m \sum_{\alpha,\beta=1}^n g^{ik} g^{jl} \frac{\partial u_{\alpha}}{\partial x_i} \frac{\partial r(x)}{\partial x_j} \tilde{u}_{\alpha} \frac{\partial u_{\beta}}{\partial x_k} \frac{\partial u_{\beta}}{\partial x_l} dv_g$$

$$\begin{aligned} &\leq \int_{\partial B(R)} \sqrt[4]{m} \|u^*h\|^{\frac{3}{2}} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^{\frac{1}{2}} dv_g \\ &\leq \sqrt[4]{m} \left[ \int_{\partial B(R)} \|u^*h\|^2 dv_g \right]^{\frac{3}{4}} \left[ \int_{\partial B(R)} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2 dv_g \right]^{\frac{1}{4}} \\ &\leq \sqrt[4]{m} \left[ \int_{\partial B(R)} \left[ \|u^*h\|^2 + \frac{1}{\varepsilon^n} (1 - |u|^2)^2 \right] dv_g \right]^{\frac{3}{4}} \left[ \int_{\partial B(R)} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2 dv_g \right]^{\frac{1}{4}}. \end{aligned}$$

Set

$$\begin{aligned} Z(R) &= \int_{B(R) \setminus B(R_2)} \left[ \sum_{\alpha, \beta=1}^n g^{ik} g^{jl} \frac{\partial u_\alpha}{\partial x_i} \frac{\partial u_\alpha}{\partial x_j} \frac{\partial u_\beta}{\partial x_k} \frac{\partial u_\beta}{\partial x_l} \right. \\ &\quad \left. + \frac{1}{\varepsilon^n} \left( 1 - \sum_{\alpha=1}^n u_\alpha^2 \right)^2 \right] dv_g + D(R_1) \\ &= \int_{B(R) \setminus B(R_2)} \left[ \|u^*h\|^2 + \frac{1}{\varepsilon^n} (1 - |u|^2)^2 \right] dv_g + D(R_1). \end{aligned}$$

Then

$$(25) \quad \frac{\partial}{\partial R} Z(R) = \int_{\partial B(R)} \left[ \|u^*h\|^2 + \frac{1}{\varepsilon^n} (1 - |u|^2)^2 \right] ds_g.$$

From (23), (24) and the fact that  $1 + \frac{c_\alpha^2}{u_\alpha^2} \geq 1$  for any  $\alpha = 1, \dots, n$ , we have

$$Z(R) \leq \sqrt[4]{m} [Z'(R)]^{\frac{3}{4}} \left[ \int_{\partial B(R)} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2 dv_g \right]^{\frac{1}{4}},$$

that is

$$(26) \quad Z(R)^{\frac{4}{3}} \leq m^{\frac{1}{3}} Z'(R) \left[ \int_{\partial B(R)} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2 dv_g \right]^{\frac{1}{3}}.$$

On the other hand, we have

$$(27) \quad Z(R) - D(R_1) = \int_{B(R) \setminus B(R_2)} \left[ \|u^*h\|^2 + \frac{1}{\varepsilon^n} (1 - |u|^2)^2 \right] dv_g.$$

Since  $\Phi_\varepsilon(u)$  is infinity, there is an  $R_3 \geq R_2$ , such that  $Z(R) > 0$  for any  $R > R_3$ . Denote

$$M(R) = \left[ \int_{\partial B(R)} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2 dv_g \right]^{\frac{1}{3}}.$$

Then we have

$$Z(R)^{\frac{4}{3}} \leq m^{\frac{1}{3}} Z'(R)M(R).$$

For any  $R_4 > R > R_3$ , we have

$$\int_R^{R_4} \frac{Z'(s)}{Z^{\frac{4}{3}}(s)} ds \geq \frac{1}{\sqrt[3]{m}} \int_R^{R_4} \frac{1}{M(s)} ds.$$

Let  $R_4 \rightarrow \infty$  and notice that  $Z(R) > 0$ , we have

$$\frac{1}{Z^{\frac{1}{3}}(R)} \geq \frac{1}{3\sqrt[3]{m}} \int_R^\infty \frac{ds}{M(s)}$$

which implies that

$$(28) \quad Z(R) \leq [3\sqrt[3]{m}]^3 \left( \int_R^\infty \frac{ds}{M(s)} \right)^3 \quad \text{for } R > R_3.$$

Using the condition  $(P_1)$  and  $u(x) \rightarrow P_0$  as  $r(x) \rightarrow \infty$ , we obtain

$$\begin{aligned} M(r) &= \left[ \int_{\partial B(r)} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2 dv_g \right]^{\frac{1}{3}} \\ &\leq \left[ \int_{\partial B(r)} \eta(r) dv_g \right]^{\frac{1}{3}} \\ &= \eta^{\frac{1}{3}}(r) [vol(\partial B(r))]^{\frac{1}{3}}, \end{aligned}$$

where  $\eta(r)$  is chosen in such a way that

- (i)  $\eta(r)$  is nonincreasing on  $(R_3, \infty)$  and  $\eta(r) \rightarrow 0$  as  $r \rightarrow \infty$ ,
- (ii)  $\eta(r) \geq \max_{r(x)=r} \left( \sum_{\alpha=1}^m \tilde{u}_\alpha^2 \right)^2$ ,
- (iii)  $\eta^{\frac{1}{3}}(r) \leq Cr^{\frac{\sigma}{3}} \int_r^\infty \frac{ds}{[vol(\partial B(s))]^{\frac{1}{3}}}$ ,

where  $C$  is a constant only depending on  $P_0$ . Then we have

$$(29) \quad \int_R^\infty \frac{dr}{M(r)} \geq \frac{1}{\eta^{\frac{1}{3}}(R)} \int_R^\infty \frac{dr}{(vol(\partial B(r)))^{\frac{1}{3}}} \geq \frac{1}{CR^{\frac{\sigma}{3}}}.$$

Hence we have  $Z(R) \leq C_1 R^{\tilde{\sigma}}$  for any  $R > R_3$ . Therefore, by the definition of  $Z(R)$  and (28), we obtain

$$\begin{aligned} \Phi_\varepsilon^R(u) &= \int_{B(R)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g \\ &\leq (3\sqrt[3]{m}C)^3 R^{\tilde{\sigma}} - \frac{D(R_1)}{4} + \int_{B(R_2)} \left[ \frac{\|u^*h\|^2}{4} + \frac{1}{4\varepsilon^n} (1 - |u|^2)^2 \right] dv_g \\ (30) \quad &\leq (CR^{\tilde{\sigma}-\sigma} + \frac{C(u)}{R^\sigma}) R^\sigma, \end{aligned}$$



where  $C$  is positive constant only depending on  $m, P_0$ , and  $C(u)$  is a positive constant only depending on  $u$ . Since  $\tilde{\sigma} < \sigma$ , it contradicts with Proposition 5.1.  $\square$

**Corollary 5.4.** *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$ , ( $m > 4$ ) be a generalized symphonic map. Assume that the radial curvature  $K_r$  of  $M$  satisfies the following condition*

$$-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$$

with  $\epsilon > 0$ ,  $A \geq 0$ ,  $0 \leq B \leq 2\epsilon$  and  $1 + (m - 1)(1 - \frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}} > 0$ . If  $u(x) \rightarrow P_0 \in S^{n-1}$  as  $r(x) \rightarrow \infty$ , and

$$\left[ \max_{r(x)=r} h^2(u(x), P_0) \right]^2 \leq \left[ \frac{3}{m-4} \right]^3 \frac{r^{\tilde{\sigma}-(m-4)}}{\omega_m e^{\frac{(m-1)A}{2\epsilon}}},$$

then  $u$  must be a constant map. Here  $\omega_m$  is the  $(m - 1)$ -volume of the unit sphere in  $\mathbb{R}^m$  and  $\tilde{\sigma}$  is a positive constant such that  $\tilde{\sigma} < 1 + (m - 1)(1 - \frac{B}{2\epsilon}) - 4e^{\frac{A}{2\epsilon}}$ .

*Proof.* By the assumption, we have

$$Ric^M(x) \geq -\frac{(m-1)A}{(1+r^2(x))^{1+\epsilon}}, \quad \forall x \in M^m.$$

Since

$$\int_0^\infty \frac{Ar}{(1+r^2)^{1+\epsilon}} dr = \frac{A}{2\epsilon}.$$

Then the volume comparison theorem (cf. [26]) implies that

$$vol(\partial B(R)) \leq \omega_m e^{\frac{(m-1)A}{2\epsilon}} R^{m-1},$$

where  $\omega_m$  is the  $(m - 1)$ -volume of the unit sphere in  $\mathbb{R}^m$ , and thus

$$\left[ \int_R^\infty \frac{dr}{(vol(\partial B(r)))^{\frac{1}{3}}} \right]^{-3} \leq \left[ \frac{m-4}{3} \right]^3 \omega_m e^{\frac{(m-1)A}{2\epsilon}} R^{m-4}$$

for  $R \gg 1$ . By using Corollary 5.2 and Theorem 5.3, we can obtain the result.  $\square$

**Corollary 5.5.** *Let  $u : (M^m, g) \rightarrow (\mathbb{R}^n, h)$ , ( $m > 4$ ) be a generalized symphonic map. Assume that the radial curvature  $K_r$  of  $M$  satisfies the following condition*

$$-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$$

with  $a \geq 0$ ,  $b^2 \in [0, \frac{1}{4}]$  and  $1 + (m - 1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2} > 0$ . If  $u(x) \rightarrow P_0 \in S^{n-1}$  as  $r(x) \rightarrow \infty$ , and

$$\left[ \max_{r(x)=r} h^2(u(x), P_0) \right]^2 \leq C^{-1} \left[ \frac{3}{(m-1)\frac{1+\sqrt{1+4a^2}}{2}} \right]^3 R^{\tilde{\sigma}-(m-1)\frac{1+\sqrt{1+4a^2}}{2}+3},$$

then  $u$  must be a constant map. Here  $\tilde{\sigma}$  is a positive constant such that  $\tilde{\sigma} < 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 4\frac{1+\sqrt{1-4a^2}}{2}$  and  $C$  is a positive constant only depending on  $m$ .

*Proof.* By the assumption, we have

$$\text{Ric}^M(x) \geq -\frac{(m-1)a^2}{1+r^2(x)}, \quad \forall x \in M^m.$$

Then the volume comparison theorem (cf. [26]) implies that

$$\text{vol}(\partial B(R)) \leq CR^{(m-1)\frac{1+\sqrt{1+4a^2}}{2}},$$

where  $C$  is a suitable constant. Thus we have

$$\left[ \int_R^\infty \frac{dr}{(\text{vol}(\partial B(r)))^{\frac{1}{3}}} \right]^{-3} \leq C \left[ \frac{(m-1)\frac{1+\sqrt{1+4a^2}}{2}}{3} \right]^3 R^{(m-1)\frac{1+\sqrt{1+4a^2}}{2}-3}$$

for  $R \gg 1$ . By using Corollary 5.2 and Theorem 5.3, we can obtain the result.  $\square$

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