

THE ARTINIAN POINT STAR CONFIGURATION QUOTIENT AND THE STRONG LEFSCHETZ PROPERTY

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ABSTRACT. It has been little known when an Artinian point quotient has the strong Lefschetz property. In this paper, we find the Artinian point star configuration quotient having the strong Lefschetz property. We prove that if \mathbb{X} is a star configuration in \mathbb{P}^2 of type s defined by forms (a -quadratic forms and $(s-a)$ -linear forms) and \mathbb{Y} is a star configuration in \mathbb{P}^2 of type t defined by forms (b -quadratic forms and $(t-b)$ -linear forms) for $b = \deg(\mathbb{X})$ or $\deg(\mathbb{X}) - 1$, then the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the strong Lefschetz property. We also show that if \mathbb{X} is a set of $(n+1)$ -general points in \mathbb{P}^n , then the Artinian quotient A of a coordinate ring of \mathbb{X} has the strong Lefschetz property.

1. Introduction

Let $R = \mathbb{k}[x_1, \dots, x_n]$ be an n -variable polynomial ring over a field \mathbb{k} of characteristic 0. A standard graded \mathbb{k} -algebra $A = R/I = \bigoplus_{i \geq 0} A_i$ has the *weak Lefschetz property* (WLP) if there is a linear form ℓ such that the multiplication $\times \ell : A_i \rightarrow A_{i+1}$ has maximal rank for every $i \geq 0$, and A has the *strong Lefschetz property* (SLP) if $\times \ell^d : A_i \rightarrow A_{i+d}$ has maximal rank for every $i \geq 0$ and $d \geq 1$. In this case, ℓ is called a *strong Lefschetz element* of A . If $d = 1$, then ℓ is a *weak Lefschetz element* of A .

The WLP and SLP are strongly connected to many topics in algebraic geometry, commutative algebra, combinatorics, and representation theory ([19–25, 29]). In particular, the manuscript [19] gives an overview of the Lefschetz properties from a different perspective focusing on representation theory and combinatorial connections and provides a wonderfully comprehensive exploration of the Lefschetz properties.

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The Jordan type is another way to characterize if an Artinian ring has the WLP or SLP (see [19, 21, 23]). Here the Jordan type of $\ell \in \mathfrak{m}$ is the partition giving the Jordan blocks of the multiplication map $\times \ell : M \rightarrow M$, where M is a module of A . Recently in 2018, Iarrobino, Marques, and McDaniel [21] provided a wonderful exploration of a general invariant of an Artinian Gorenstein algebra A or A -module M , which is the set of Jordan types of elements of the maximal ideal \mathfrak{m} of A .

In 2006, Geramita, Migliore, and Sabourin [16] introduced the notion of a star configuration set of points in \mathbb{P}^2 , called a linear point star configuration in \mathbb{P}^2 in this article. The name “star configuration” was first suggested by A.V. Geramita because the configuration resembles a star for small set of lines in \mathbb{P}^2 . A more general definition follows [9], where the geometric objects are called hypersurface configurations. The more general definition of star configurations has evolved through a series of papers (see [1, 9, 26]); in particular, the codimension 2 case was studied before the general case [1]. It is known that star configurations have many nice algebraic properties ([4, 8]), and at the same time, can be used to exhibit extremal properties ([2, 3, 16]). Another application of star configurations in \mathbb{P}^n of codimension 2 is to find the dimension of the secant variety to the variety of reducible forms based on Terracini’s Lemma in [31] (see [5, 6, 28] for the definition). In particular, if \mathbb{X} and \mathbb{Y} are the two star configurations in \mathbb{P}^n of the same type $(2, s)$ defined by the forms of the same degree with $s \geq 2$ (see Definition 2.4), then one can use the following exact sequence

$$(1.1) \quad 0 \rightarrow R/I_{\mathbb{X} \cup \mathbb{Y}} \rightarrow R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \rightarrow R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \rightarrow 0$$

to find the dimension of the secant line variety to the variety of reducible forms. In addition, it is unknown if the union of two point star configurations in \mathbb{P}^n of types s and t , respectively, defined by the forms of the same degree with $s, t \geq n \geq 2$ has generic Hilbert function. Especially, the Hilbert function of the union of two star configurations in \mathbb{P}^n has a close relation to know if an Artinian star configuration quotient (not necessary point star configuration) has the WLP or the SLP.

In late 1980’s, Roberts and Roitman [27] introduced special configurations of points in \mathbb{P}^2 which they named \mathbb{k} -configurations. This definition was first extended to \mathbb{P}^3 by Harima [18], and later to all \mathbb{P}^n by Geramita, Harima, and Shin (see [12, 13]). As shown by Roberts and Roitman [27, Theorem 1.2], all \mathbb{k} -configurations of type (d_1, \dots, d_s) have the same Hilbert function, which can be computed from the type. This result was later generalized by Geramita, Harima, and Shin [11, Corollary 3.7] to show that all the graded Betti numbers of the associated graded ideal $I_{\mathbb{X}}$ only depend upon the type. Interestingly, \mathbb{k} -configurations of the same type can have very different geometric properties. Recently, in [7], Galetto, Shin, and Van Tuyl distinguish \mathbb{k} -configurations in \mathbb{P}^2 by counting the number of lines that contain d_s points of \mathbb{X} . In particular, they show that for all integers $m \gg 0$, the number of such lines is precisely the

value of $\Delta \mathbf{H}_{m\mathbb{X}}(md_s - 1)$. Here, $\Delta \mathbf{H}_{m\mathbb{X}}(-)$ is the first difference of the Hilbert function of the fat points of multiplicity m supported on \mathbb{X} .

In this paper, we focus on the following question.

Question 1.1. Let \mathbb{X} and \mathbb{Y} be point star configurations in \mathbb{P}^n of type s and t with $s, t \geq n$, respectively (see Section 2 for the definition).

- (a) Does the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the WLP?
- (b) Does the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the SLP?

Regarding the WLP of the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$, it is known that if \mathbb{X} and \mathbb{Y} are linear point star configurations in \mathbb{P}^n of any types, then the Artinian linear point star configuration $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP. In general, if $\mathbb{X}_1, \dots, \mathbb{X}_r$ are linear point star configurations in \mathbb{P}^n of any types, then the Artinian ring $R/(I_{\mathbb{X}_1} + \dots + I_{\mathbb{X}_r})$ has the WLP (see Remark 2.12). There is another Artinian point star configuration quotient having the WLP. If \mathbb{X} is a point star configuration in \mathbb{P}^n of type s defined by $(s - 1)$ -linear forms and a single quadratic form with $n \geq 3$, then the Artinian quotient of a coordinate ring of \mathbb{X} has the WLP with a certain condition (see Remark 2.11(a)).

In [23], the authors find the Artinian point configuration quotient having the SLP, based on the union of two \mathbb{k} -configurations in \mathbb{P}^2 , which are contained in a basic configuration (see [14, 18] for the definition of a basic configuration). In this case, the Artinian ring is Gorenstein, and thus the Hilbert function of this Artinian ring is symmetric. In [29], the author shows that if \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s defined by general quadratic forms, and \mathbb{Y} is a point star configuration in \mathbb{P}^2 of type $(s + 1)$ defined by s -general quadratic forms G_1, \dots, G_s and a general linear form L , then the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP with a Lefschetz element L (see [29, Theorem 3.7]). In [22] the authors generalize this result with the condition $\deg(F_i) = \deg(G_i) \leq 2$ for every $i = 1, \dots, s$ (see [22, Theorem 4.11]).

In this paper, we show that if \mathbb{X} is a finite set of points in \mathbb{P}^2 and \mathbb{Y} is a linear point star configuration in \mathbb{P}^2 of type t with $t \geq \deg(\mathbb{X}) - 1$ and $s \geq 3$, then $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function under a certain condition (see Corollary 2.7, Proposition 3.3, and Corollary 3.4). Moreover, if \mathbb{X} and \mathbb{Y} are finite sets of points in \mathbb{P}^n having generic Hilbert function with $\sigma(\mathbb{X}) \leq \sigma(\mathbb{Y})$ and

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{n + (\sigma(\mathbb{Y}) - 1)}{n} \quad \text{or} \quad \binom{n + (\sigma(\mathbb{Y}) - 1)}{n} + 1,$$

then the Artinian point quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP (see Theorem 4.4 (b)). As an application, we show that if \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s defined by forms a -quadratic forms and $(s - a)$ -linear forms and \mathbb{Y} is a point star configuration in \mathbb{P}^2 of type t defined by forms b -quadratic forms and $(t - b)$ -linear forms with $\deg(\mathbb{X}) \leq t$ and $b = \deg(\mathbb{X})$ or $\deg(\mathbb{X}) - 1$, then the Artinian point star configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the SLP (see Corollary 4.5). Furthermore, if \mathbb{X} is a set of $(n + 1)$ -general points in \mathbb{P}^n , then the Artinian quotient of a coordinate ring of \mathbb{X} has the SLP (see Proposition 5.3).

2. The Artinian quotient having the weak Lefschetz property

In this section we will compute the Hilbert functions of the union of some particular types of \mathbb{k} -configurations of points in \mathbb{P}^2 , and introduce certain Artinian point quotients having the WLP. We first recall the definition of a \mathbb{k} -configuration in \mathbb{P}^2 and some known results of the \mathbb{k} -configurations (see [11–14, 17, 27]).

Definition 2.1. A \mathbb{k} -configuration in \mathbb{P}^2 is a finite set \mathbb{X} of points in \mathbb{P}^2 which satisfy the following conditions: there exist integers $1 \leq d_1 < \dots < d_m$, and subsets $\mathbb{X}_1, \dots, \mathbb{X}_m$ of \mathbb{X} , and distinct lines $\mathbb{L}_1, \dots, \mathbb{L}_m \subseteq \mathbb{P}^2$ such that

- (a) $\mathbb{X} = \bigcup_{i=1}^m \mathbb{X}_i$,
- (b) $|\mathbb{X}_i| = d_i$ and $\mathbb{X}_i \subset \mathbb{L}_i$ for each $i = 1, \dots, m$, and
- (c) \mathbb{L}_i ($1 < i \leq m$) does not contain any points of \mathbb{X}_j for all $j < i$.

In this case, the \mathbb{k} -configuration in \mathbb{P}^2 is said to be of type (d_1, \dots, d_m) .

Lemma 2.2. Let \mathbb{X} be a \mathbb{k} -configuration in \mathbb{P}^2 of type $\mathcal{T} = (1, 2, 3, \dots, d - 1, d + 1, d + 2, \dots, s)$ with $s \geq 3$ and $d \geq 1$. Then \mathbb{X} has generic Hilbert function

$$\mathbf{H}_{\mathbb{X}} : 1 \binom{1+2}{2} \dots \binom{2+(s-2)}{2} \binom{2+(s-1)}{2} - d \rightarrow.$$

Proof. It is immediate from [27, Theorem 1.2] (see also [11, Theorem 3.6]). \square

Proposition 2.3 ([29, Proposition 2.6]). If \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s defined by a -quadratic forms and $(s - a)$ -linear forms with $s \geq 2$, then \mathbb{X} has generic Hilbert function as

$$\mathbf{H}_{\mathbb{X}} : 1 \binom{1+2}{2} \dots \binom{2+(d-3)}{2} \deg(\mathbb{X}) \rightarrow,$$

where $d = s + a$.

We now recall the definition of a general star configuration in \mathbb{P}^n and some related results.

Definition 2.4 ([26, Definition 2.1]). Let $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ be a polynomial ring over a field \mathbb{k} . For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose F_1, \dots, F_s are general forms in R of degrees d_1, \dots, d_s , respectively. We call the variety \mathbb{X} defined by the ideal

$$\bigcap_{1 \leq i_1 < \dots < i_r \leq s} (F_{i_1}, \dots, F_{i_r})$$

a star configuration in \mathbb{P}^n of type (r, s) . In particular, if F_1, \dots, F_s are general linear forms in R , then we call \mathbb{X} a linear star configuration in \mathbb{P}^n of type (r, s) .

Notice that each n -forms F_{i_1}, \dots, F_{i_n} among s -general forms F_1, \dots, F_s in R define $d_{i_1} \cdots d_{i_n}$ points in \mathbb{P}^n for each $1 \leq i_1 < \dots < i_n \leq s$. Thus the ideal

$$\bigcap_{1 \leq i_1 < \dots < i_n \leq s} (F_{i_1}, \dots, F_{i_n})$$

defines a finite set \mathbb{X} of points in \mathbb{P}^n with

$$\deg(\mathbb{X}) = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq s} d_{i_1} d_{i_2} \cdots d_{i_n}.$$

In this case, we call \mathbb{X} a *point star configuration in \mathbb{P}^n of type s* instead of *type (n, s)* . In particular, if \mathbb{X} is a point star configuration in \mathbb{P}^n defined by s -linear forms L_1, \dots, L_s in R , then we call \mathbb{X} a *linear point star configuration in \mathbb{P}^n of type s* .

Remark 2.5. (a) In Definition 2.4, the forms F_1, \dots, F_s don't have to be general. Indeed, it is enough to assume that all subsets of size $1 \leq r \leq \min\{n + 1, s\}$ are regular sequences in R , and if $\mathcal{H} = \{F_1, \dots, F_s\}$ is a collection of distinct hypersurfaces in \mathbb{P}^n corresponding to forms F_1, \dots, F_s , respectively, then the hypersurfaces *meet properly*, by which we mean that the intersection of any r of these hypersurfaces with $1 \leq r \leq \min\{n, s\}$ has codimension r .

(b) Moreover, any two \mathbb{k} -configurations in \mathbb{P}^2 of the same type have the same minimal free resolution, and so the same Hilbert function ([11–14, 17, 27]). We recall that if \mathbb{X} is a linear point star configuration in \mathbb{P}^2 of type s with $3 \leq s$, then \mathbb{X} is a \mathbb{k} -configuration in \mathbb{P}^2 of type $\mathcal{T} = (1, 2, \dots, s - 1)$.

Theorem 2.6 ([26, Theorem 2.3]). *Let F_1, \dots, F_s be general forms in $R = \mathbb{k}[x_0, x_1, \dots, x_n]$ with $s \geq 2$ and $n \geq 2$. Then*

$$\bigcap_{1 \leq j_1 < \dots < j_r \leq s} (F_{j_1}, \dots, F_{j_r}) = \sum_{1 \leq i_1 < \dots < i_{r-1} \leq s} \left(\frac{\prod_{\ell=1}^s F_\ell}{F_{i_1} \cdots F_{i_{r-1}}} \right)$$

for $1 \leq r \leq \min\{n, s\}$.

The following corollary is the results of Carlini, Guardo, and Van Tuyl [4, Theorem 2.5], Geramita, Harbourne, and Migliore [8, Proposition 2.9], and Park and Shin [26, Corollary 2.4].

Corollary 2.7. *Let \mathbb{X} be a linear point star configuration in \mathbb{P}^n of type s with $s \geq n \geq 2$. Then \mathbb{X} has generic Hilbert function, i.e.,*

$$\mathbf{H}_{\mathbb{X}}(i) = \min \left\{ \deg(\mathbb{X}), \binom{i+n}{n} \right\}$$

for every $i \geq 0$.

We start with two propositions on the WLP from [10] and provide a complete answer to Question 1.1(a) for the Artinian linear point star configuration quotient (see Remark 2.12), but it is still unknown for the Artinian general star configuration quotient. Let \mathbb{X} be a finite set of points in \mathbb{P}^n and define

$$\sigma(\mathbb{X}) = \min\{i \mid \mathbf{H}_{\mathbb{X}}(i - 1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

Proposition 2.8 ([30, Proposition 2.6]). *Let \mathbb{X} be a point star configuration in \mathbb{P}^n defined by general forms F_1, \dots, F_s of degrees $1 \leq d_1 \leq \dots \leq d_s$ with $2 \leq n \leq s$. Then*

$$\sigma(\mathbb{X}) = \left[\sum_{i=1}^s d_i \right] - (n - 1).$$

Proposition 2.9 ([10, Proposition 5.15]). *Let \mathbb{X} be a finite set of points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring $R/I_{\mathbb{X}}$. Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for all $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Then A has the WLP.*

We recall the result about the WLP of the Artinian point quotient in \mathbb{P}^2 in [29].

Theorem 2.10 ([29, Theorem 3.3]). *Let \mathbb{X} and \mathbb{Y} be point star configurations in \mathbb{P}^2 defined by forms F_1, \dots, F_s and G_1, \dots, G_t , respectively, with $s \geq t \geq 3$. Assume that $\deg F_i \leq 2$ and $\deg G_j \leq 2$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. If $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then the Artinian point quotient $A = R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP.*

Remark 2.11. (a) Here is another example of a point star configuration in \mathbb{P}^n having generic Hilbert function with $n \geq 3$. Let \mathbb{X} be a point star configuration in \mathbb{P}^n of type s defined by $(s - 1)$ -linear forms and a single quadratic form. By Proposition 2.8,

$$\sigma(\mathbb{X}) = s - n + 2.$$

Moreover, by Theorem 2.6, the initial degree of $I_{\mathbb{X}}$ is

$$s - n + 1 = \sigma(\mathbb{X}) - 1.$$

Hence if \mathbb{Y} is either a finite set of general points in \mathbb{P}^n or a point star configuration in \mathbb{P}^n of type t defined by $(t - 1)$ -linear forms and a single quadratic form such that $\sigma(\mathbb{X}) < \sigma(\mathbb{Y})$, then

$$\mathbf{H}_{\mathbb{Y}}(d) = \mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(d) = \binom{n + d}{d}$$

for $0 \leq d \leq \sigma(\mathbb{X}) - 1$. Using equation (1.1), one can easily obtain that

$$\mathbf{H}_{R/(I_{\mathbb{X}} + I_{\mathbb{Y}})}(d) = \mathbf{H}_{\mathbb{X}}(d) + \mathbf{H}_{\mathbb{Y}}(d) - \mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(d) = \mathbf{H}_{\mathbb{X}}(d)$$

for such d . Applying Proposition 2.9, the Artinian point star configuration quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP.

- (b) Let \mathbb{X} be a point star configuration in \mathbb{P}^n of type s defined by forms of degrees d_1, \dots, d_s . Notice that if either more than or equal to two forms have degree ≥ 2 or one of the forms has degree ≥ 3 , then by Theorem 2.6 \mathbb{X} does not have generic Hilbert function. So the assumption for a point star configuration in \mathbb{P}^n in (a) is necessary to obtain a point star configuration having generic Hilbert function.
- (c) Let \mathbb{X} and \mathbb{Y} be finite sets of points in \mathbb{P}^n and let $\alpha_{\mathbb{Y}}$ be the initial degree of the ideal $I_{\mathbb{Y}}$. If \mathbb{X} has generic Hilbert function and $\sigma(\mathbb{X}) \leq \alpha_{\mathbb{Y}}$, then, by Proposition 2.9, the Artinian quotient $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP.

- (1) In general, let $\mathbb{X}_1, \dots, \mathbb{X}_r$ be finite sets of points in \mathbb{P}^n with $r \geq 2$ and $\alpha_{\mathbb{X}_i}$ be the initial degree of the ideal $I_{\mathbb{X}_i}$ for $2 \leq i \leq r$. If $\sigma(\mathbb{X}_1) \leq \alpha_{\mathbb{X}_i}$ for $2 \leq i \leq r$, then by Proposition 2.9, the Artinian quotient $R/(I_{\mathbb{X}_1} + \dots + I_{\mathbb{X}_r})$ has the WLP.
- (2) Moreover, if \mathbb{X}_i is a finite set of points in \mathbb{P}^n having generic Hilbert function for $1 \leq i \leq r$ with $r \geq 2$ and $\sigma(\mathbb{X}_1) < \sigma(\mathbb{X}_j)$ for $2 \leq j \leq r$, then by Proposition 2.9, the Artinian quotient $R/(I_{\mathbb{X}_1} + \dots + I_{\mathbb{X}_r})$ has the WLP as well.

Remark 2.12. Suppose $\mathbb{X}_1, \dots, \mathbb{X}_\ell$ are linear point star configurations in \mathbb{P}^n of any types with $\ell \geq 2$. Using induction on $\ell \geq 2$, Corollary 2.7, and Proposition 2.9, one can easily show that the Artinian point quotient

$$R/(I_{\mathbb{X}_1} + \dots + I_{\mathbb{X}_\ell})$$

has the WLP. But it is still unknown if the Artinian ring $R/(I_{\mathbb{X}_1} + \dots + I_{\mathbb{X}_\ell})$ has the SLP.

3. The union of two star configurations in \mathbb{P}^2 having generic Hilbert function

In this section, we introduce the union of two star configurations in \mathbb{P}^2 having generic Hilbert function, and then we shall show that certain Artinian star configuration quotients have the SLP in the next section.

Remark 3.1. (a) Let \mathbb{X} be a general star configuration in \mathbb{P}^n of type (r, s) defined by forms of degrees d_1, \dots, d_s and let \mathbb{Y} be a special star configuration in \mathbb{P}^n of type (r, s) defined by forms G_1, \dots, G_s , where G_i 's are products of d_i -linear forms in $R = \mathbb{k}[x_1, \dots, x_n]$. If any r -forms G_{i_1}, \dots, G_{i_r} among s -forms G_1, \dots, G_s are a regular sequence, then by the same argument as in the proof of [26, Theorem 3.4], $R/I_{\mathbb{Y}}$ has the same graded minimal free resolution as $R/I_{\mathbb{X}}$.

- (b) Let \mathbb{X} and \mathbb{Y} be general star configurations in \mathbb{P}^n of type (r, s) and (r, t) defined by s -forms and t -forms, respectively. As long as the defining forms (not necessary general forms) for \mathbb{X} and \mathbb{Y} have two conditions, i.e., (i) any r -forms among s -forms (t -forms, respectively) is a regular sequence, (ii) \mathbb{X} and \mathbb{Y} do not share their defining forms, and (iii) \mathbb{X} and \mathbb{Y} are disjoint, we see that \mathbb{X} , \mathbb{Y} , and $\mathbb{X} \cup \mathbb{Y}$ don't change the degrees of minimal generators and the minimal graded free resolutions of the ideals. This is the way how to define star configurations \mathbb{X} and \mathbb{Y} throughout this paper.

Example 3.2. (a) Let $\mathbb{X} = \{\wp_1, \wp_2, \dots, \wp_5\}$ be a finite set of 5 points in \mathbb{P}^2 and \mathbb{Y} be a linear point star configuration in \mathbb{P}^2 of type 5 defined by linear forms M_1, \dots, M_5 in R . If M_i vanishes on a point \wp_i for $i = 1, \dots, 5$, then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, \dots, 5)$. By Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.

- (b) Let \mathbb{X} be a point star configuration in \mathbb{P}^2 of type 3 defined by forms $L_{1,1}L_{1,2}, L_2, L_3$, and \mathbb{Y} be a linear point star configuration in \mathbb{P}^2 of type 5 defined by forms M_1, \dots, M_5 in R , where $L_{i,j}, L_k, M_\ell$ are all linear forms in $R = \mathbb{k}[x_0, x_1, x_2]$. Let $\mathbb{X} = \{\wp_1, \wp_2, \dots, \wp_5\}$. Without loss of generality, we may assume that M_i vanishes on a point \wp_i for $i = 1, \dots, 5$. Indeed, this is always possible because we can always make 3-forms $L_{1,1}L_{1,2}, L_2, L_3$ and any 3-forms among 5-linear forms M_1, \dots, M_5 regular sequences. Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, \dots, 5)$. By Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.

Example 3.2 motivates the following proposition and corollary.

Proposition 3.3. *Let \mathbb{X} be a finite set of points in \mathbb{P}^2 and \mathbb{Y} be a linear point star configuration in \mathbb{P}^2 of type t defined by linear forms M_1, \dots, M_t with $t \geq \deg(\mathbb{X}) \geq 3$.*

- (a) *If each linear form M_i vanishes on a distinct point in \mathbb{X} for every $1 \leq i \leq \deg(\mathbb{X})$, then $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.*
- (b) *If \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s with $s \geq 3$, then $\mathbb{X} \cup \mathbb{Y}$ has also generic Hilbert function.*

Proof. (a) Let $\mathbb{X} = \{\wp_1, \dots, \wp_{\deg(\mathbb{X})}\}$. By assumption, we suppose that

$$\begin{array}{ll}
 M_1 & \text{vanishes on a point } \wp_1 \text{ in } \mathbb{X} \text{ and } (t-1)\text{-points in } \mathbb{Y}, \\
 M_2 & \text{vanishes on a point } \wp_2 \text{ in } \mathbb{X} \text{ and } (t-2)\text{-points in } \mathbb{Y}, \\
 & \vdots \\
 M_{\deg(\mathbb{X})} & \text{vanishes on a point } \wp_{\deg(\mathbb{X})} \text{ in } \mathbb{X} \text{ and } (t-\deg(\mathbb{X}))\text{-points in } \mathbb{Y}, \\
 M_{\deg(\mathbb{X})+1} & \text{vanishes on } (t-\deg(\mathbb{X})-1) \text{ points,} \\
 & \vdots \\
 M_{t-2} & \text{vanishes on 2 points,} \\
 M_{t-1} & \text{vanishes on 1 point.}
 \end{array}$$

Hence $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type

$$(1, 2, \dots, t - \deg(\mathbb{X}) - 1, t - \deg(\mathbb{X}) + 1, \dots, t).$$

- (b) By an analogous argument as in Example 3.2(b), we may assume that a linear form M_i vanishes on a point \wp_i in \mathbb{X} and $(t-i)$ -points in \mathbb{Y} for $1 \leq i \leq \deg(\mathbb{X})$. Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of the same type as in (a).

Therefore, by Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function, which completes the proof. □

Corollary 3.4. *Let \mathbb{X} be a finite set of points in \mathbb{P}^2 and \mathbb{Y} be a linear point star configuration in \mathbb{P}^2 of type t defined by linear forms M_1, \dots, M_t with $t \geq \deg(\mathbb{X}) - 1 \geq 2$.*

- (a) *If M_1 vanishes on two distinct points in \mathbb{X} and each linear form M_i vanishes on a distinct point in \mathbb{X} for every $2 \leq i \leq \deg(\mathbb{X}) - 1$, then $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.*

- (b) *In particular, if \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s with $s \geq 3$, then $\mathbb{X} \cup \mathbb{Y}$ has also generic Hilbert function.*

Proof. If $t \geq \deg(\mathbb{X})$, then by Proposition 3.3 it holds. Now we assume that $t = \deg(\mathbb{X}) - 1$ and let $\mathbb{X} = \{\wp_1, \dots, \wp_{\deg(\mathbb{X})}\}$.

- (a) Without loss of generality, we assume that
 - (1) M_1 vanishes on $(t + 1)$ points, i.e., two points \wp_1, \wp_2 in \mathbb{X} and $(t - 1)$ -point in \mathbb{Y} ,
 - (2) M_2 vanishes on $(t - 1)$ points, i.e., one point \wp_3 in \mathbb{X} and $(t - 2)$ -points in \mathbb{Y} ,
 - ⋮
 - (3) $M_{\deg(\mathbb{X})-2}$ vanishes on 2 points, i.e., one point $\wp_{\deg(\mathbb{X})-1}$ in \mathbb{X} and 1-point in \mathbb{Y} ,
 - (4) $M_{\deg(\mathbb{X})-1}$ vanishes on 1 point, i.e., one point $\wp_{\deg(\mathbb{X})}$ in \mathbb{X} .

Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, \dots, t - 1, t + 1)$.

- (b) By the same method as above, one can see that $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, \dots, t - 1, t + 1)$.

Therefore, by Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function, which completes the proof. □

Before we prove the main theorem (Theorem 3.10), we introduce the following example to show the idea used in the proof of the theorem.

Example 3.5. Let \mathbb{X} be a point star configuration in \mathbb{P}^2 of type 3 defined by one quadratic form $L_{1,1}L_{1,2}$, and two linear forms L_2, L_3 and \mathbb{Y} be a point star configuration in \mathbb{P}^2 of type 5 defined by one quadratic form $M_{1,1}M_{1,2}$, and four linear forms M_2, \dots, M_5 (see Figure 1). Notice that $\deg(\mathbb{X}) = 5$.

As in Figure 1, we assume that

- $M_{1,1}$ vanishes on 2 points \wp_1, \wp_2 in \mathbb{X} and 4 points in \mathbb{Y} , and so $M_{1,1}$ vanishes on 6 points in $\mathbb{X} \cup \mathbb{Y}$,
- $M_{1,2}$ vanishes on 1 points \wp_3 in \mathbb{X} and 4 points in \mathbb{Y} , and so $M_{1,2}$ vanishes on 5 points in $\mathbb{X} \cup \mathbb{Y}$,
- M_2 vanishes on 1 points \wp_4 in \mathbb{X} and 3 points in \mathbb{Y} , and so M_2 vanishes on 4 points in $\mathbb{X} \cup \mathbb{Y}$,
- M_3 vanishes on 1 point \wp_5 in \mathbb{X} and 2 points in \mathbb{Y} , and so M_3 vanishes on 3 points in $\mathbb{X} \cup \mathbb{Y}$, and
- M_4 vanishes on 1 point in \mathbb{Y} , and so M_4 vanishes on 1 point in $\mathbb{X} \cup \mathbb{Y}$.

Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 3, 4, 5, 6)$. By Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.

Remark 3.6. Let \mathbb{X} be a point star configuration in \mathbb{P}^2 of type s defined by a -quadratic forms and $(s - a)$ -linear forms and let \mathbb{Y} be a \mathbb{k} -configuration in \mathbb{P}^2 of type

$$\mathcal{T} = (1, 2, \dots, a - 1, a + 1, \dots, s + a - 2, s + a - 1).$$

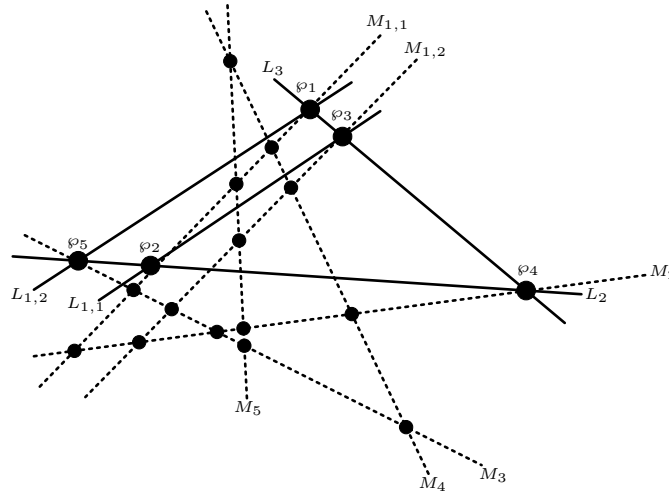


FIGURE 1. The union of two point star configurations in \mathbb{P}^2 of types 3 and 5

Then, by Proposition 2.3 and Lemma 2.2, \mathbb{X} and \mathbb{Y} have the same Hilbert function as

$$\mathbf{H}_{\mathbb{X}} = \mathbf{H}_{\mathbb{Y}} : 1 \binom{1+2}{2} \cdots \binom{2+(d-3)}{2} \deg(\mathbb{X}) \rightarrow,$$

where $d = s + a$. Here

$$\deg(\mathbb{X}) = \deg(\mathbb{Y}) = \frac{(s+a)^2 - s - 3a}{2}.$$

The following theorem is useful to prove the main theorem in this section.

Theorem 3.7 ([15, Corollary 2.8]). *Let \mathbb{X} be a variety of \mathbb{P}^n and let \mathbb{H} be a hyperplane of \mathbb{P}^n not containing any irreducible component of \mathbb{X} . Let \mathbb{V} be a subvariety of \mathbb{H} such that*

$$\mathbf{H}_{\mathbb{V}}(i) = \binom{i+(n-1)}{n-1} \text{ for } i \leq s.$$

If $\mathbf{H}_{\mathbb{X}}(s-1) = \mathbf{H}_{\mathbb{X}}(s)$ (that is if \mathbb{k} is algebraically closed then \mathbb{X} consists of $\mathbf{H}_{\mathbb{X}}(s-1)$ points), then

$$\mathbf{H}_{\mathbb{X} \cup \mathbb{V}}(i) = \mathbf{H}_{\mathbb{V}}(i) + \mathbf{H}_{\mathbb{X}}(i-1) \text{ for every } i \geq 0.$$

We now introduce some other types of star configurations in \mathbb{P}^2 whose union has generic Hilbert function. For the rest of this paper, we assume that $L_{i,j}, L_j, M_{i,j}, M_j$ are all linear forms in $R = \mathbb{k}[x_0, x_1, x_2]$. The following two lemmas are helpful.

Lemma 3.8. *Let $\mathbb{X} = \bigcup_{i=1}^m \mathbb{X}_i$ be a k -configuration in \mathbb{P}^2 of type (d_1, \dots, d_m) and let \mathbb{L}_i be a line defined by a linear form L_i for $1 \leq i \leq m$ such that $\mathbb{X}_i \subseteq \mathbb{L}_i$. Let \mathbb{Y} be a finite set of points in \mathbb{P}^2 . If the linear form L_i does not vanish on any point in \mathbb{Y} for every $1 \leq i \leq m$ and $\sigma(\mathbb{Y}) < d_1$, then the Hilbert function of $\mathbb{X} \cup \mathbb{Y}$ is*

$$\mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(i) = \mathbf{H}_{\mathbb{X}}(i) + \mathbf{H}_{\mathbb{Y}}(i - m)$$

for $i \geq 0$.

Proof. We use induction on m . Let $m = 1$. The Hilbert function of a set of d_1 -collinear points is $1 \ 2 \ \dots \ d_1 \ \rightarrow \ .$ Since $\sigma(\mathbb{Y}) < d_1$, we see that

$$\mathbf{H}_{\mathbb{Y}}(d_1 - 1) = \mathbf{H}_{\mathbb{Y}}(d_1),$$

and thus, by Theorem 3.7,

$$\mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(i) = \mathbf{H}_{\mathbb{X}}(i) + \mathbf{H}_{\mathbb{Y}}(i - 1) \quad \text{for every } i \geq 0.$$

Let $m > 1$. Notice that

$$\sigma(\mathbb{X}_1 \cup \mathbb{Y}) = d_1.$$

Define

$$\mathbb{Z} := \mathbb{X}_1 \cup \dots \cup \mathbb{X}_{m-1}.$$

By the inductive assumption, we have that

$$\mathbf{H}_{\mathbb{Z} \cup \mathbb{Y}}(i) = \mathbf{H}_{\mathbb{Z}}(i) + \mathbf{H}_{\mathbb{Y}}(i - (m - 1)) \quad \text{for every } i \geq 0, \quad \text{and}$$

$$\sigma(\mathbb{Z} \cup \mathbb{Y}) = d_{m-1}.$$

Moreover, since $\sigma(\mathbb{Z} \cup \mathbb{Y}) = d_{m-1} < d_m$, we see that, by Theorem 3.7, the Hilbert function of $\mathbb{X} \cup \mathbb{Y}$ is

$$\begin{aligned} \mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(i) &= \mathbf{H}_{\mathbb{X}_m \cup (\mathbb{Z} \cup \mathbb{Y})}(i) \\ &= \mathbf{H}_{\mathbb{X}_m}(i) + \mathbf{H}_{\mathbb{Z} \cup \mathbb{Y}}(i - 1) \\ &= \mathbf{H}_{\mathbb{X}_m}(i) + (\mathbf{H}_{\mathbb{Z}}(i - 1) + \mathbf{H}_{\mathbb{Y}}((i - 1) - (m - 1))) \\ &= \mathbf{H}_{\mathbb{X}}(i) + \mathbf{H}_{\mathbb{Y}}(i - m) \end{aligned}$$

for every $i \geq 0$. This completes the proof. □

Lemma 3.9. *Assume \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s defined by a -quadratic forms and $(s - a)$ -linear forms $L_{1,1}L_{1,2}, L_{2,1}L_{2,2}, \dots, L_{a,1}L_{a,2}, L_{a+1}, \dots, L_s$ with $0 \leq a \leq s$, and \mathbb{Y} is a point star configuration in \mathbb{P}^2 of type t defined by b -quadratic forms and $(t - b)$ -linear forms $M_{1,1}M_{1,2}, M_{2,1}M_{2,2}, M_{3,1}M_{3,2}, \dots, M_{b,1}M_{b,2}, M_{b+1}, \dots, M_{t-1}, M_t$ with $0 \leq b \leq t$. If $u := \deg(\mathbb{X}) \leq b \leq t$, then $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.*

Proof. Let $u = b$. We may assume that

- $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $1 \leq i \leq u$, and
- $M_{i,2}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} for $1 \leq i \leq u$ (see Table 1).

TABLE 1. A \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t + b - 2, t + b - 1, t + b)$

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$
$M_{1,1}$	1	$t + b - 2$	$t + b$
$M_{1,2}$	0	$t + b - 2$	$t + b - 1$
$M_{2,1}$	1	$t + b - 4$	$t + b - 2$
$M_{2,2}$	0	$t + b - 4$	$t + b - 3$
\vdots	\vdots	\vdots	\vdots
$M_{b,1}$	1	$t - b$	$t - b + 1$
$M_{b,2}$	0	$t - b$	$t - b$
M_{b+1}	0	$t - b - 1$	$t - b - 1$
\vdots	\vdots	\vdots	\vdots
M_{t-2}	0	2	2
M_{t-1}	0	1	1
M_t	0	0	0

Hence $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t + b - 2, t + b - 1, t + b)$, and so by Lemma 2.2 $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.

Now suppose that $1 \leq u < b \leq t$. We may assume that

- $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $1 \leq i \leq u$,
- $M_{i,2}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} for $1 \leq i \leq u$ (see Table 2).

We now split $\mathbb{X} \cup \mathbb{Y}$ into two disjoint sets \mathbb{U} and \mathbb{V} as follows. First, \mathbb{U} is the set of all points on conics defined by the following u -quadratic forms

$$M_{1,1}M_{1,2}, \dots, M_{u,1}M_{u,2},$$

and

$$\mathbb{V} := \mathbb{X} \cup \mathbb{Y} - \mathbb{U},$$

respectively (see Tables 3 and 4).

Then \mathbb{U} is a \mathbb{k} -configuration in \mathbb{P}^2 of type

$$\mathcal{T}_1 = (t + b - 2u, t + b - 2u + 1, \dots, t + b - 2, t + b - 1).$$

Furthermore, \mathbb{V} is a point star configuration in \mathbb{P}^2 of type $(t - u)$ defined by $(b - u)$ -quadratic forms and $(t - b)$ -linear forms. Let $d = 2(b - u) + (t - b) = t + b - 2u$. By Proposition 2.3 and Remark 3.6, the Hilbert function of \mathbb{V} is

$$\mathbf{H}_{\mathbb{V}} : \binom{1+2}{2} \cdots \binom{2+(d-3)}{2} \deg(\mathbb{V}) \rightarrow,$$

which is the same as the Hilbert function of a \mathbb{k} -configuration in \mathbb{P}^2 of type

$$\mathcal{T}_2 = (1, 2, \dots, b - u - 1, b - u + 1, \dots, t + b - 2u - 1).$$

TABLE 2. A set $\mathbb{X} \cup \mathbb{Y} = \mathbb{U} \cup \mathbb{V}$

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$		
$\mathbb{U} =$	$M_{1,1}$	1	$t + b - 2$	$t + b - 1$	
	$M_{1,2}$	0	$t + b - 2$	$t + b - 2$	
	$M_{2,1}$	1	$t + b - 4$	$t + b - 3$	
	$M_{2,2}$	0	$t + b - 4$	$t + b - 4$	
	\vdots	\vdots	\vdots	\vdots	
	$M_{u,1}$	1	$t + b - 2u$	$t + b - 2u + 1$	
	$M_{u,2}$	0	$t + b - 2u$	$t + b - 2u$	
	$\mathbb{V} =$	$M_{u+1,1}$	0	$t + b - 2u - 2$	$t + b - 2u - 2$
		$M_{u+1,2}$	0	$t + b - 2u - 2$	$t + b - 2u - 2$
		\vdots	\vdots	\vdots	\vdots
$M_{b,1}$		0	$t - b$	$t - b$	
$M_{b,2}$		0	$t - b$	$t - b$	
M_{b+1}		0	$t - b - 1$	$t - b - 1$	
M_{b+2}		0	$t - b - 2$	$t - b - 2$	
\vdots		\vdots	\vdots	\vdots	
M_{t-2}		0	2	2	
M_{t-1}		0	1	1	
M_t	0	0	0		

TABLE 3. A \mathbb{k} -configuration \mathbb{U} in \mathbb{P}^2 of type $(t + b - 2u, \dots, t + b - 2, t + b - 1)$

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$
$M_{1,1}$	1	$t + b - 2$	$t + b - 1$
$M_{1,2}$	0	$t + b - 2$	$t + b - 2$
$M_{2,1}$	1	$t + b - 4$	$t + b - 3$
$M_{2,2}$	0	$t + b - 4$	$t + b - 4$
\vdots	\vdots	\vdots	\vdots
$M_{u,1}$	1	$t + b - 2u$	$t + b - 2u + 1$
$M_{u,2}$	0	$t + b - 2u$	$t + b - 2u$

Recall that the first component of \mathcal{T}_1 is $t + b - 2u$ and

$$\sigma(\mathbb{V}) = t + b - 2u - 1 < t + b - 2u.$$

By Lemma 3.8, the Hilbert function of $\mathbb{X} \cup \mathbb{Y} = \mathbb{U} \cup \mathbb{V}$ is

$$\mathbf{H}_{\mathbb{X} \cup \mathbb{Y}}(i) = \mathbf{H}_{\mathbb{U} \cup \mathbb{V}}(i) = \mathbf{H}_{\mathbb{U}}(i) + \mathbf{H}_{\mathbb{V}}(i - 2u) \quad \text{for } i \geq 0.$$

TABLE 4. A point star configuration \mathbb{V} in \mathbb{P}^2 of type $(t - u)$ defined by $(b - u)$ -quadratic forms and $(t - b)$ -linear forms

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$
$M_{u+1,1}$	0	$t + b - 2u - 2$	$t + b - 2u - 2$
$M_{u+1,2}$	0	$t + b - 2u - 2$	$t + b - 2u - 2$
\vdots	\vdots	\vdots	\vdots
$M_{b,1}$	0	$t - b$	$t - b$
$M_{b,2}$	0	$t - b$	$t - b$
M_{b+1}	0	$t - b - 1$	$t - b - 1$
\vdots	\vdots	\vdots	\vdots
M_{t-2}	0	2	2
M_{t-1}	0	1	1
M_t	0	0	0

Since the Hilbert function of \mathbb{V} is the same as the Hilbert function of a \mathbb{k} -configuration in \mathbb{P}^2 of type

$$(1, 2, \dots, b - u - 1, b - u + 1, \dots, t + b - 2u - 1),$$

we get that the Hilbert function of $\mathbb{X} \cup \mathbb{Y}$ is the same as the Hilbert function of a \mathbb{k} -configuration in \mathbb{P}^2 of type

$$(1, 2, \dots, \underline{b - u - 1}, \underline{b - u + 1}, \dots, t + b - 2u - 1, t + b - 2u, \dots, t + b - 1).$$

Therefore, by Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function, as we wished. \square

Theorem 3.10. *With notation as in Lemma 3.9, if $t \geq \deg(\mathbb{X})$ and $s \geq 3$, then $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function.*

Proof. If $b = 0$, then \mathbb{Y} is a linear point star configuration in \mathbb{P}^2 of type t , and so by Proposition 3.3 $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function. Now consider that \mathbb{Y} is not a linear star configuration in \mathbb{P}^2 of type t , i.e., $1 \leq b \leq t$.

Recall that $u := \deg(\mathbb{X})$. If $u \leq b \leq t$, then by Lemma 3.9 it holds. Now suppose that $b < u \leq t$.

- (a) Let $3b \leq u$. We may assume that (see Table 5)
 - $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 2-points in \mathbb{X} for $1 \leq i \leq b$, and
 - $M_{i,2}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $1 \leq i \leq b$.
 - Moreover, M_{b+j} vanishes on one point in \mathbb{X} for $1 \leq j \leq u - 3b$.
 Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, \underline{t - u + 2b - 1}, \underline{t - u + 2b + 1}, \dots, t + b)$.

TABLE 5. A \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t - u + 2b - 1, t - u + 2b + 1, \dots, t + b)$

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$
$M_{1,1}$	2	$t + b - 2$	$t + b$
$M_{1,2}$	1	$t + b - 2$	$t + b - 1$
$M_{2,1}$	2	$t + b - 4$	$t + b - 2$
$M_{2,2}$	1	$t + b - 4$	$t + b - 3$
\vdots	\vdots	\vdots	\vdots
$M_{b,1}$	2	$t - b$	$t - b + 2$
$M_{b,2}$	1	$t - b$	$t - b + 1$
M_{b+1}	1	$t - b - 1$	$t - b$
M_{b+2}	1	$t - b - 2$	$t - b - 1$
\vdots	\vdots	\vdots	\vdots
$M_{b+(u-3b)}$	1	$t - u + 2b$	$t - u + 2b + 1$
$M_{b+(u-3b)+1}$	0	$t - u + 2b - 1$	$t - u + 2b - 1$
\vdots	\vdots	\vdots	\vdots
M_{t-2}	0	2	2
M_{t-1}	0	1	1
M_t	0	0	0

(b) Let $b < u < 3b$ and $u = b + 2\ell$ with $1 \leq \ell < b$. We may assume that (see Table 6)

- $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 2-points in \mathbb{X} for $1 \leq i \leq \ell$,
- $M_{i,2}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $1 \leq i \leq \ell$, and
- $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $\ell + 1 \leq i \leq b$.

Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t + b - 2\ell - 1, t + b - 2\ell + 1, \dots, t + b)$.

(c) Let $b < u < 3b$ and $u = b + 2\ell + 1$ with $0 \leq \ell < b$. We may assume that (see Table 7)

- $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 2-points in \mathbb{X} for $1 \leq i \leq \ell + 1$,
- $M_{i,2}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $1 \leq i \leq \ell$, and
- $M_{i,1}$ vanishes on $(t + b - 2i)$ -points in \mathbb{Y} and 1-point in \mathbb{X} for $\ell + 2 \leq i \leq b$.

TABLE 6. A \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t + b - 2\ell - 1, t + b - 2\ell + 1, \dots, t + b)$

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$
$M_{1,1}$	2	$t + b - 2$	$t + b$
$M_{1,2}$	1	$t + b - 2$	$t + b - 1$
$M_{2,1}$	2	$t + b - 4$	$t + b - 2$
$M_{2,2}$	1	$t + b - 4$	$t + b - 3$
\vdots	\vdots	\vdots	\vdots
$M_{\ell,1}$	2	$t + b - 2\ell$	$t + b - 2\ell + 2$
$M_{\ell,2}$	1	$t + b - 2\ell$	$t + b - 2\ell + 1$
$M_{\ell+1,1}$	1	$t + b - 2\ell - 2$	$t + b - 2\ell - 1$
$M_{\ell+1,2}$	0	$t + b - 2\ell - 2$	$t + b - 2\ell - 2$
\vdots	\vdots	\vdots	\vdots
$M_b,1$	1	$t - b$	$t - b + 1$
$M_b,2$	0	$t - b$	$t - b$
M_{b+1}	0	$t - b - 1$	$t - b - 1$
M_{b+2}	0	$t - b - 2$	$t - b - 2$
\vdots	\vdots	\vdots	\vdots
M_{t-2}	0	2	2
M_{t-1}	0	1	1
M_t	0	0	0

Then $\mathbb{X} \cup \mathbb{Y}$ is a \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t + b - 2\ell - 2, t + b - 2\ell, \dots, t + b)$.

Therefore, by Lemma 2.2, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function. This completes the proof. \square

Remark 3.11. In Theorem 3.10, it is not necessary that \mathbb{X} is a point star configuration in \mathbb{P}^2 defined by quadratic and linear forms. Indeed, it is enough if \mathbb{X} satisfies the condition in the proof of Theorem 3.10.

4. The Artinian quotient having the strong Lefschetz property

There is a useful numerical characterization of Lefschetz elements, for which we need some notations.

TABLE 7. A \mathbb{k} -configuration in \mathbb{P}^2 of type $(1, 2, 3, \dots, t + b - 2\ell - 2, t + b - 2\ell, \dots, t + b)$

	\mathbb{X}	\mathbb{Y}	$\mathbb{X} \cup \mathbb{Y}$
$M_{1,1}$	2	$t + b - 2$	$t + b$
$M_{1,2}$	1	$t + b - 2$	$t + b - 1$
$M_{2,1}$	2	$t + b - 4$	$t + b - 2$
$M_{2,2}$	1	$t + b - 4$	$t + b - 3$
\vdots	\vdots	\vdots	\vdots
$M_{\ell,1}$	2	$t + b - 2\ell$	$t + b - 2\ell + 2$
$M_{\ell,2}$	1	$t + b - 2\ell$	$t + b - 2\ell + 1$
$M_{\ell+1,1}$	2	$t + b - 2\ell - 2$	$t + b - 2\ell$
$M_{\ell+1,2}$	0	$t + b - 2\ell - 2$	$t + b - 2\ell - 2$
$M_{\ell+2,1}$	1	$t + b - 2\ell - 4$	$t + b - 2\ell - 3$
$M_{\ell+2,2}$	0	$t + b - 2\ell - 4$	$t + b - 2\ell - 4$
\vdots	\vdots	\vdots	\vdots
$M_b,1$	1	$t - b$	$t - b + 1$
$M_b,2$	0	$t - b$	$t - b$
M_{b+1}	0	$t - b - 1$	$t - b - 1$
M_{b+2}	0	$t - b - 2$	$t - b - 2$
\vdots	\vdots	\vdots	\vdots
M_{t-2}	0	2	2
M_{t-1}	0	1	1
M_t	0	0	0

Definition 4.1. Let $\sum_{i \geq 0} a_i t^i$ be a formal power series, where $a_i \in \mathbb{Z}$. Then we define an associated power series with non-negative coefficients by

$$\left| \sum_{i \geq 0} a_i t^i \right|^+ = \sum_{i \geq 0} b_i t^i,$$

where

$$b_i = \begin{cases} a_i, & \text{if } a_j > 0 \text{ for all } j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 4.2. Let A be a standard artinian graded algebra, and let $L \in A$ be a linear form. Then the following conditions are equivalent:

- (a) L is a Lefschetz element of A .
- (b) The Hilbert function of A/LA is given by

$$\dim_{\mathbb{k}}[A/LA]_i = \max\{0, \dim_{\mathbb{k}}[A]_i - \dim_{\mathbb{k}}[A]_{i-1}\} \quad \text{for all integers } i.$$

(c) The Hilbert series of A/LA is

$$\mathbf{HS}(A/LA) = |(1 - t) \cdot \mathbf{HS}(A)|^+.$$

Lemma 4.3 ([23, Proposition 2.7]). *Let \mathbb{X} be a finite set of points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring of \mathbb{X} . Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for every $0 \leq i \leq s - 2$ with $A_s = 0$, and the Hilbert function of A is of the form*

$$\mathbf{H}_A : h_0 \quad h_1 \quad \cdots \quad h_{\sigma-1} \quad h_{\sigma} \quad \cdots \quad \overset{(s-2)\text{-nd}}{h_{\sigma}} \quad h_{s-1} \quad 0$$

where $h_{\sigma-2} < h_{\sigma-1} = h_{\sigma}$, and $h_{s-1} = 0$ or 1 . Then the Artinian ring A has the SLP.

Theorem 4.4. *Let \mathbb{X} and \mathbb{Y} be finite sets of points in \mathbb{P}^n having generic Hilbert function.*

(a) *If $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then the Artinian point quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP.*

(b) *If $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function such that $\sigma(\mathbb{X}) \leq \sigma(\mathbb{Y})$ and*

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{n + (\sigma(\mathbb{Y}) - 1)}{n} \quad \text{or} \quad \binom{n + (\sigma(\mathbb{Y}) - 1)}{n} + 1,$$

then the Artinian point quotient A has the SLP.

Proof. (a) By Remark 2.11(c), it holds.

(b) Define

$$\begin{aligned} \sigma_1 &:= \sigma(\mathbb{X}), \quad \text{and} \\ \sigma_2 &:= \sigma(\mathbb{Y}). \end{aligned}$$

(i) First we assume that

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{n + \sigma_2 - 1}{n}.$$

Since \mathbb{X} , \mathbb{Y} , and $\mathbb{X} \cup \mathbb{Y}$ all have generic Hilbert function, we obtain that the Hilbert functions of \mathbb{X} , \mathbb{Y} , and $\mathbb{X} \cup \mathbb{Y}$ are

$$\begin{aligned} \mathbf{H}_{\mathbb{X}} &: \cdots \quad \deg(\mathbb{X}) \quad \overset{\sigma_1\text{-st}}{\deg(\mathbb{X})} \quad \cdots \quad \deg(\mathbb{X}) \quad \overset{(\sigma_2-1)\text{-st}}{\deg(\mathbb{X})} \quad \deg(\mathbb{X}) \quad \rightarrow, \\ \mathbf{H}_{\mathbb{Y}} &: \cdots \quad \binom{n + (\sigma_1 - 1)}{n} \quad \binom{n + \sigma_1}{n} \quad \cdots \quad \binom{n + (\sigma_2 - 2)}{n} \quad \deg(\mathbb{Y}) \quad \deg(\mathbb{Y}) \quad \rightarrow, \\ \mathbf{H}_{\mathbb{X} \cup \mathbb{Y}} &: \cdots \quad \binom{n + (\sigma_1 - 1)}{n} \quad \binom{n + \sigma_1}{n} \quad \cdots \quad \binom{n + (\sigma_2 - 2)}{n} \quad \binom{n + (\sigma_2 - 1)}{n} \quad \deg(\mathbb{X}) + \deg(\mathbb{Y}) \quad \rightarrow, \end{aligned}$$

respectively. Using the exact sequence

$$0 \rightarrow R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \rightarrow R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \rightarrow R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \rightarrow 0,$$

the Hilbert function of A is

$$\mathbf{H}_A : 1 \quad \binom{n + 1}{1} \quad \cdots \quad \binom{n + (\sigma_1 - 2)}{n} \quad \deg(\mathbb{X}) \quad \deg(\mathbb{X}) \quad \cdots \quad \overset{(\sigma_2-2)\text{-nd}}{\deg(\mathbb{X})} \quad 0.$$

(ii) Now assume that

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{n + (\sigma_2 - 1)}{n} + 1.$$

By an analogous argument as above, we obtain the Hilbert function of A is

$$\mathbf{H}_A : 1 \quad \binom{n+1}{1} \quad \cdots \quad \binom{n + (\sigma_1 - 2)}{n} \quad \deg(\mathbb{X}) \quad \deg(\mathbb{X}) \quad \cdots \quad \overset{(\sigma_2 - 2)\text{-nd}}{\deg(\mathbb{X})} \quad 1 \quad 0.$$

Therefore, by Lemma 4.3, the Artinian point quotient A has the SLP, which completes the proof. \square

If we couple Theorem 4.4 with Theorem 3.10, we obtain the following corollary.

Corollary 4.5. *Assume \mathbb{X} is a point star configuration in \mathbb{P}^2 of type s defined by a -quadratic forms and $(s - a)$ -linear forms $L_{1,1}L_{1,2}, L_{2,1}L_{2,2}, \dots, L_{a,1}L_{a,2}, L_{a+1}, \dots, L_s$ with $s \geq 3$ and $0 \leq a \leq s$, and \mathbb{Y} is a point star configuration in \mathbb{P}^2 of type t defined by b -quadratic forms and $(t - b)$ -linear forms $M_{1,1}M_{1,2}, M_{2,1}M_{2,2}, M_{3,1}M_{3,2}, \dots, M_{b,1}M_{b,2}, M_{b+1}, \dots, M_{t-2}, M_{t-1}, M_t$ with $0 \leq b \leq t$ such that $\mathbb{X} \cap \mathbb{Y} = \emptyset$.*

- (a) *If $\sigma(\mathbb{X}) \neq \sigma(\mathbb{Y})$, then the Artinian point star configuration quotient $A := R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the WLP.*
- (b) *If $\deg(\mathbb{X}) \leq t$ and either $b = \deg(\mathbb{X})$ or $\deg(\mathbb{X}) - 1$, then the Artinian point star configuration quotient A has the SLP.*

Proof. (a) By Theorem 2.10, it holds.

(b) Recall that, by [29, Proposition 2.6] and Theorem 3.10, \mathbb{X} , \mathbb{Y} , and $\mathbb{X} \cup \mathbb{Y}$ all have generic Hilbert function.

Let $b = \deg(\mathbb{X})$. By Proposition 2.8,

$$\begin{aligned} \sigma_1 &:= \sigma(\mathbb{X}) = s + a - 1, \quad \text{and} \\ \sigma_2 &:= \sigma(\mathbb{Y}) = t + b - 1. \end{aligned}$$

Moreover, by Remark 3.6

$$\begin{aligned} \deg(\mathbb{X}) &= \frac{(s + a)^2 - s - 3a}{2}, \quad \text{and} \\ \deg(\mathbb{Y}) &= \frac{(t + b)^2 - t - 3b}{2}. \end{aligned}$$

Thus

$$\begin{aligned} \deg(\mathbb{X}) + \deg(\mathbb{Y}) &= b + \frac{(t + b)^2 - t - 3b}{2} \\ &= \frac{(t + b)^2 - (t + b)}{2} \\ &= \binom{t + b}{2} = \binom{\sigma_2 + 1}{2}. \end{aligned}$$

Hence by Theorem 4.4(b), the Artinian point star configuration quotient A has the SLP.

Now assume that $b = \deg(\mathbb{X}) - 1$. Recall that, by Theorem 3.10, $\mathbb{X} \cup \mathbb{Y}$ has generic Hilbert function. By an analogous argument as above,

$$\deg(\mathbb{X}) + \deg(\mathbb{Y}) = \binom{\sigma_2 + 1}{2} + 1.$$

Thus by Theorem 4.4(b), the Artinian point star configuration quotient A has the SLP as well. This completes the proof. \square

5. The Jordan type for a special Artinian point quotient

In this section, we shall prove that the Artinian quotient of a coordinate ring of a general $(n + 1)$ -points in \mathbb{P}^n has the SLP using the Jordan type argument. We first recall the results in [19] about the WLP and the SLP based on Jordan type.

Lemma 5.1 ([19]). *Assume A is graded and \mathbf{H}_A is unimodal. Then*

- (a) *A has the WLP if and only if the number of parts of the Jordan type $J_{A,\ell} = \max\{\mathbf{H}_A(i)\}$. (The Sperner number of A);*
- (b) *ℓ is a strong Lefschetz element of A if and only if $J_{A,\ell} = \mathbf{H}_A^\vee$ the conjugate of \mathbf{H}_A (exchange rows and columns in the Ferrers diagram of \mathbf{H}_A).*

Lemma 5.2. *Let \mathbb{X} be a finite set of $(n + 1)$ -points in \mathbb{P}^n and let A be an Artinian quotient of the coordinate ring $R/I_{\mathbb{X}}$. Assume that the Hilbert function of A is*

$$\mathbf{H}_A : 1 \quad n + 1 \quad \cdots \quad n + 1 \quad h_s,$$

where $s \geq 2$ and $0 \leq h_s \leq n + 1$. Then A has the SLP.

Proof. We shall prove this by induction on h_s . Recall, that by Lemma 4.3, if $h_s = 0, 1$, then A has the SLP. So we now suppose that $1 < h_s \leq n + 1$ and B is an Artinian quotient of A having Hilbert function

$$\mathbf{H}_B : 1 \quad n + 1 \quad \cdots \quad n + 1 \quad h_s - 1.$$

By induction on h_s , B has the SLP, and thus, for a general linear form ℓ , the Jordan type $J_{B,\ell}$ for B is

$$J_{B,\ell} = \mathbf{H}_B^\vee = (p_1, p_2, \dots, p_n, p_{n+1}),$$

where

$$p_i = \begin{cases} s + 1, & i = 1, \\ s, & i = 2, \dots, h_s - 1, \\ s - 1, & i = h_s, \dots, n + 1. \end{cases}$$

By Proposition 2.9, A has the WLP. Hence, by Lemma 5.1(a), the Jordan type $J_{A,\ell}$ for A is of the form

$$J_{A,\ell} = (p'_1, \dots, p'_{n+1}).$$

Notice that since B is an Artinian quotient of A , we get that

$$p'_i = p_i \quad \text{for } 1 \leq i \leq h_s - 1.$$

Moreover, since

$$\dim_{\mathbb{k}} A = \sum_{i=1}^{n+1} p'_i = \left[\sum_{i=1}^{n+1} p_i \right] + 1,$$

we have

$$\sum_{i=h_s}^{n+1} p'_i = \left[\sum_{i=h_s}^{n+1} p_i \right] + 1.$$

Recall that A_d and B_d agree in degrees $\leq s-1$ and $p_i = s-1$ for $h_s \leq i \leq n+1$. Thus

$$p'_i = p_i + 1 = s$$

for some $h_s \leq i \leq n+1$, say, $i = h_s$. In other words,

$$J_{A,\ell} = \mathbf{H}_A^\vee.$$

Therefore, by Lemma 5.1(b), A has the SLP, as we wished. \square

The following proposition is immediate from double induction on t and h_t with Lemma 5.2. So we omit the proof.

Proposition 5.3. *Let \mathbb{X} be a set of $(n+1)$ -general points in \mathbb{P}^n , and let A be an Artinian quotient of the coordinate ring of \mathbb{X} having Hilbert function of the form*

$$\mathbf{H}_A : 1 \quad n+1 \quad \cdots \quad n+1 \quad h_s \quad \cdots \quad h_t,$$

where $t \geq s \geq 2$. Then A has the SLP.

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