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THE HOMOTOPY CATEGORIES OF N-COMPLEXES OF INJECTIVES AND PROJECTIVES

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Abstract. We investigate the homotopy category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ of N-complexes of injectives in a Grothendieck abelian category $\mathscr{A}$ not necessarily locally noetherian, and prove that the inclusion $\mathcal{K}(\mathrm{Inj}\mathscr{A}) \to \mathcal{K}(\mathscr{A})$ has a left adjoint and $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ is well generated. We also show that the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$ of N-complexes of projectives is compactly generated whenever R is right coherent.

1. Introduction

Homological algebra mostly studies complexes, having a differential d satisfying $d^2 = 0$. It is natural to ask why $d^2 = 0$ and not, say, $d^3 = 0$. The idea to investigate complexes with a differential d such that $d^N = 0$ where $N \geqslant 3$ was introduced by Kapranov [6] and there he hinted to their possible connections to quantum theories. Since then many papers have appeared on the subject, many of them studying their interesting homology and indicating some possible applications of N-complexes for certain nonassociative algebras. In 2015, Yang and Ding [15] provided an effective construction of left and right triangles, and proved that the homotopy category and the derived category of N-complexes over an abelian category are pretriangulated categories. Iyama, Kato and Miyachi [3] proved that the homotopy category $\mathcal{K}_N(\mathscr{B})$ of N-complexes of an additive category $\mathscr{B}$ is a triangulated category.

Krause [9] studied the homotopy category $\mathcal{K}(\mathrm{Inj}\mathscr{A})$ of complexes ($N = 2$) of injectives in a locally noetherian Grothendieck abelian category $\mathscr{A}$. Because $\mathscr{A}$ is locally noetherian, arbitrary direct sums of injectives are injective, and hence the category $\mathcal{K}(\mathrm{Inj}\mathscr{A})$ has coproducts. It turns out that $\mathcal{K}(\mathrm{Inj}\mathscr{A})$ is compactly generated. Neeman [12] studied this further in the nonnoetherian case. He proved that for a Grothendieck abelian category $\mathscr{A}$, the category $\mathcal{K}(\mathrm{Inj}\mathscr{A})$ has coproducts and is well generated. The current paper considers the homotopy category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ of N-complexes of injectives, we prove that:

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Theorem A. *Let $\mathscr{A}$ be a Grothendieck abelian category not necessarily locally noetherian.*

(1) *The inclusion $i : \mathcal{K}_N(\mathrm{Inj}\mathscr{A}) \to \mathcal{K}_N(\mathscr{A})$ has a left adjoint i_ℓ. Therefore, the homotopy category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ has arbitrary coproducts.*

(2) *There is a regular cardinal μ for which the category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ is μ-compactly generated.*

Jøgensen [5] studied the analogue where injectives are replaced by projectives. Since Grothendieck abelian categories rarely have enough projectives, Jøgensen's results all assumed that he was working over a ring. Under suitable noetherian hypotheses, he proved an analogue of Krause's theorem: the homotopy category $\mathcal{K}(\mathrm{Prj}R)$ is compactly generated. Afterwards, the homotopy category of flat R-modules came into play by Neeman [11]. Among other things, he showed that $\mathcal{K}(\mathrm{Prj}R)$ is compactly generated if R is right coherent which gives a generalization of result of [5]. We have the following result for the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$ of N-complexes of projectives.

Theorem B. *Let R be a right coherent ring. Then the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$ of N-complexes of projectives is compactly generated.*

2. Preliminaries and basic facts

This section is devoted to recalling some notions and basic consequences for use throughout this paper. For terminology we shall follow [2] and [3].

N-complexes. An N-complex X in an abelian category $\mathscr{A}$ is a sequence of objects in $\mathscr{A}$

$$\cdots \xrightarrow{d_{n+2}^X} X_{n+1} \xrightarrow{d_{n+1}^X} X_n \xrightarrow{d_n^X} X_{n-1} \xrightarrow{d_{n-1}^X} \cdots$$

satisfying $d^N = 0$. That is, composing any N-consecutive morphisms gives 0. So a 2-complex is a chain complex in the usual sense. A chain map or simply map $f : X \to Y$ of N-complexes is a collection of morphisms $f_n : X_n \to Y_n$ making all the rectangles commute. In this way we get a category of N-complexes, denoted by $\mathcal{C}_N(\mathscr{A})$.

For an N-complex X, there are $N-1$ choices for homology. Indeed for $t = 1, \ldots, N$, we define

$$\mathrm{Z}_n^t(X) = \ker(d_{n-(t-1)} \cdots d_{n-1} d_n) \text{ and } \mathrm{B}_n^t(X) = \mathrm{Im}(d_{n+1} d_{n+2} \cdots d_{n+t}).$$

In particular, we have $\mathrm{Z}_n^1(X) = \ker d_n$, $\mathrm{Z}_n^N(X) = X_n$ and $\mathrm{B}_n^1(X) = \mathrm{Im} d_{n+1}$, $\mathrm{B}_n^N(X) = 0$. We also define $\mathrm{H}_n^t(X) = \mathrm{Z}_n^t(X)/\mathrm{B}_n^{N-t}(X)$ the amplitude homology objects of X for $t = 1, \ldots, N-1$. We say X is acyclic if $\mathrm{H}_n^t(X) = 0$ for all n and t.

Given an object A of $\mathscr{A}$, we define N-complexes $\mathrm{D}_n^t(A)$ for $t = 1, \ldots, N$ as follows. $\mathrm{D}_n^t(A)$ consists of A in degrees $n, n-1, \ldots, n-(t-1)$, all joined by identity morphisms, and 0 in every other degree.

Two chain maps $f, g : X \to Y$ of N-complexes are called chain homotopic, or simply homotopic if there exists a collection of morphisms $\{s_n : X_n \to Y_{n+N-1}\}$ such that

$$\begin{aligned} g_n - f_n &= d^{N-1}s_n + d^{N-2}s_{n-1}d + \cdots + s_{n-(N-1)}d^{N-1} \\ &= \sum_{i=0}^{N-1} d^{(N-1)-i}s_{n-i}d^i, \ \forall\ n. \end{aligned}$$

If f and g are homotopic, then we write $f \sim g$. We call f null homotopic if $f \sim 0$. There exists an additive category $\mathcal{K}_N(\mathscr{A})$, called the homotopy category of N-complexes, whose objects are the same as those of $\mathcal{C}_N(\mathscr{A})$ and whose Hom sets are the $\sim$ equivalence classes of Hom sets in $\mathcal{C}_N(\mathscr{A})$. An isomorphism in $\mathcal{K}_N(\mathscr{A})$ is called a homotopy equivalence.

The homotopy category $\mathcal{K}_N(\mathscr{A})$. Let (X, d^X), (Y, d^Y) be objects and $u : X \to Y$ a morphism in $\mathcal{C}_N(\mathscr{A})$. Then the mapping cone $C(u)$ of u is given as

$$C(u)_n = Y_n \oplus (\Sigma X)_n,\ d^{C(u)} = \begin{bmatrix} d & u & 0 & \cdots & 0 & 0 \\ 0 & -d & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -d^{N-2} & 0 & \cdots & 0 & 1 \\ 0 & -d^{N-1} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$\Sigma^{-1}C(u)_n = (\Sigma^{-1}Y)_n \oplus X_n,\ d^{\Sigma^{-1}C(u)} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -d^{N-1} & -d^{N-2} & -d^{N-3} & \cdots & -d & u \\ 0 & 0 & 0 & \cdots & 0 & d \end{bmatrix},$$

where $(\Sigma X)_n = X_{n-1} \oplus \cdots \oplus X_{n-(N-1)}$ and $(\Sigma^{-1}Y)_n = Y_{n+N-1} \oplus \cdots \oplus Y_{n+1}$ with

$$d^{\Sigma X} = \begin{bmatrix} -d & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -d^{N-2} & 0 & 0 & \cdots & 0 & 1 \\ -d^{N-1} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad d^{\Sigma^{-1}Y} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -d^{N-1} & -d^{N-2} & -d^{N-3} & \cdots & -d^2 & -d \end{bmatrix}.$$

We say a diagram in $\mathcal{K}_N(\mathscr{A})$ is a distinguished triangle if it is isomorphic to a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} C(u) \xrightarrow{w} \Sigma X$$

arising from a chain map $u : X \to Y$, where $v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

Lemma 2.1 ([15]). *Let $u \in \mathrm{Hom}_{\mathcal{C}_N(\mathscr{A})}(X, Z)$. Then the following are equivalent*:

(1) *u is null homotopic*;
(2) *The canonical exact sequence $0 \to Z \xrightarrow{v} C(u) \xrightarrow{w} \Sigma X \to 0$ splits.*

There is more structure on $\mathcal{C}_N(\mathscr{A})$ which we will need. Namely, instead of all short exact sequences $0 \to X \to Y \to Z \to 0$ of N-complexes, we sometimes consider only the sequences for which $0 \to X_n \to Y_n \to Z_n \to 0$ splits in $\mathscr{A}$ for each $n \in \mathbb{Z}$. These exact sequences make $\mathcal{C}_N(\mathscr{A})$ an exact category in the sense of [7, App. A], with the componentwise split exact structure, and allow us to define a corresponding variant of the Yoneda Ext which we denote by $\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A}),c.s.}$, to distinguish it from the usual Ext-functor on $\mathcal{C}_N(\mathscr{A})$ which we denote by $\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A})}$. We refer to [7, App. A] for details. Thus, for each pair $Z, X \in \mathcal{C}_N(\mathscr{A})$, the group $\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A}),c.s.}(Z, X)$ is naturally a subgroup of $\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A})}(Z, X)$.

Lemma 2.2. *For any N-complexes $X, Z \in \mathcal{C}_N(\mathscr{A})$, we have*

$$\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A}),c.s.}(\Sigma X, Z) \cong \mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(X, Z) = \mathrm{Hom}_{\mathcal{C}_N(\mathscr{A})}(X, Z)/\sim,$$

where $\sim$ is chain homotopy.

Proof. Let $\mu \in \mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(X, Z)$. It follows from Lemma 2.1 that the canonical sequence $0 \to Z \xrightarrow{v} C(u) \xrightarrow{w} \Sigma X \to 0$ is an element of $\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A}),c.s.}(\Sigma X, Z)$. On the other hand, given an element $0 \to Z \to Y \to \Sigma X \to 0$ of $\mathrm{Ext}^1_{\mathcal{C}_N(\mathscr{A}),c.s.}(\Sigma X, Z)$. [15, Theorem 2.22] yields a chain map $\tau : X \to Z$. If the above sequence is split, then $\tau \sim 0$, as claimed. □

3. A left adjoint to the inclusion $\mathcal{K}_N(\mathrm{Inj}\mathscr{A}) \to \mathcal{K}_N(\mathscr{A})$

Throughout this section, we will assume that $\mathscr{A}$ is a Grothendieck abelian category and $G \in \mathscr{A}$ is a fixed generator. Let $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ be the full subcategory of $\mathcal{K}_N(\mathscr{A})$ whose objects are the N-complexes of injectives. There is an obvious inclusion $\mathcal{K}_N(\mathrm{Inj}\mathscr{A}) \to \mathcal{K}_N(\mathscr{A})$. In this section, we will study its left adjoint.

We begin with some preliminaries.

Lemma 3.1. *Every bounded above and acyclic N-complex lies in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$.*

Proof. Let X be an acyclic N-complex and E an N-complex of injectives. If X vanishes in degrees $> n$, then the chain map $X \to E$ factors through the brutal truncation $E_{\leqslant n}$, the factorization is the obvious.

$$\begin{tikzcd}
X: \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X_n \arrow[r] \arrow[d] & X_{n-1} \arrow[r] \arrow[d] & \cdots \\
E_{\leqslant n} \ \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \cdots \\
E: \cdots \arrow[r] & E_{n+2} \arrow[r] & E_{n+1} \arrow[r] & E_n \arrow[r] & E_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $X \to E_{\leqslant n}$ is a chain map, it follows from [14, Lemma 3.3] that it is null-homotopic. □

The objects in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ do not have to be bounded above. However, the following result implies that they do have to be acyclic.

Lemma 3.2. *Every object in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ is acyclic.*

Proof. Let E be an injective cogenerator of the category $\mathscr{A}$. By [16, Lemma 4.4],

$$\mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(X, \mathrm{D}^t_{n+t-1}(E)) \cong \mathrm{H}^{N-t}_n(\mathrm{Hom}_{\mathscr{A}}(X, E)), \ \forall\ n, t.$$

Hence $\mathrm{H}^t_n(\mathrm{Hom}_{\mathscr{A}}(X, E)) = 0$ if and only if $\mathrm{H}^t_n(X) = 0$ for all n, t, as desired. □

Lemma 3.3. *Let $f : X \to Y$ be a quasi-isomorphism of N-complexes and E an N-complex of injectives. If f_i is an isomorphism for all $i \gg 0$, then the natural map*

$$\mathrm{Hom}(f, E) : \mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(Y, E) \longrightarrow \mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(X, E)$$

is an isomorphism.

Proof. Consider the triangle $W \to X \xrightarrow{f} Y \to \Sigma W$. Then W is acyclic by [15, Proposition 3.2]. Also W is homotopy equivalent to a bounded above N-complex since f_i is an isomorphism for all $i \gg 0$. Hence Lemma 3.1 implies that $W \in {}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. The result now follows by applying $\mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(-, E)$ to the triangle. □

Lemma 3.3 showed that any chain map $X \to E$ factors up to homotopy through $X \to Y$. Next we consider the factorizations not only up to homotopy but in the category $\mathcal{C}_N(\mathscr{A})$.

Lemma 3.4. *Let $f : X \to Y$ be a chain map in $\mathcal{C}_N(\mathscr{A})$ whose mapping cone lies in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. If f_i is a monomorphism in each degree i, then the natural map*

$$\mathrm{Hom}(f, E) : \mathrm{Hom}_{\mathcal{C}_N(\mathscr{A})}(Y, E) \to \mathrm{Hom}_{\mathcal{C}_N(\mathscr{A})}(X, E)$$

is surjective for any N-complex E of injectives.

Proof. Let $h : X \to E$ be a chain map. By Lemma 3.3, there exist a chain map $g : Y \to E$ and a collection of morphisms $\{s_i : X_i \to E_{i+N-1}\}$ such that

$$h_i - g_i f_i = d^{N-1} s_i + d^{N-2} s_{i-1} d + \cdots + s_{i-(N-1)} d^{N-1}, \ \forall\ i.$$

Since E_{i+N-1} is an injective object in $\mathscr{A}$ and f_i is a monomorphism, there is $r_i : Y_i \to E_{i+N-1}$ such that $r_i f_i = s_i$ for all i. Set

$$g'_i = g_i + d^{N-1} r_i + d^{N-2} r_{i-1} d + \cdots + r_{i-(N-1)} d^{N-1}, \ \forall\ i.$$

One can check $g' = \{g'_i\}_{i \in \mathbb{Z}}$ is a chain map and $h = g'f$. □

Next we think about uniqueness of the factorization $h = gf$ in Lemma 3.4.

Lemma 3.5. *Assume that $g, g' : Y \to E$ are two chain maps with gf and $g'f$ homotopic to h, and let $\{s_i : X_i \to E_{i+N-1}\}$ be a homotopy with*

$$g_i f_i - g'_i f_i = d^{N-1} s_i + d^{N-2} s_{i-1} d + \cdots + s_{i-(N-1)} d^{N-1}, \ \forall\ i,$$

where $f : X \to Y$ and $h : X \to E$ are as in Lemma 3.4. Then there exists a homotopy $\{r_i : Y_i \to E_{i+N-1}\}$ with $rf = s$ and so that

$$g_i - g'_i = d^{N-1} r_i + d^{N-2} r_{i-1} d + \cdots + r_{i-(N-1)} d^{N-1}, \ \forall\ i.$$

Proof. By Lemma 3.3, we have that $g \sim g'$. Choose a homotopy $\{r'_i : Y_i \to E_{i+N-1}\}$ connecting g with g'. Then $\{r'_i f_i : X_i \to E_{i+N-1}\}$ is a homotopy connecting gf with $g'f$ as is $\{s_i : X_i \to E_{i+N-1}\}$, which implies that

$$\alpha = \left(\begin{bmatrix} s_i - r'_i f_i \\ (s_{i-1} - r'_{i-1} f_{i-1})d \\ \vdots \\ (s_{i-N+2} - r'_{i-N+2} f_{i-N+2})d^{N-2} \end{bmatrix}\right)_{i\in\mathbb{Z}} : X \to \Sigma^{-1}E$$

is a chain map. By Lemma 3.4, $\alpha = \rho f$ for some chain map $\rho : Y \to \Sigma^{-1}E$ with $\rho_i = \begin{bmatrix} \rho_{i,i+N-1} \\ \rho_{i,i+N-2} \\ \vdots \\ \rho_{i,i+1} \end{bmatrix} : Y_i \to (\Sigma^{-1}E)_i$. Therefore $\{r_i = r'_i + \rho_{i,i+N-1} : Y_i \to E_{i+N-1}\}$ is a homotopy of g with g' and $s = rf$. □

Definition 3.6 ([12])**.** Let λ be an ordinal and $\mathscr{C}$ a category. A sequence of length λ in $\mathscr{C}$ is the following data:

(i) for every ordinal $i \leqslant \lambda$ an object $X_i \in \mathscr{C}$ and
(ii) for every pair of ordinals i and j with $i < j \leqslant \lambda$ a morphism $f^{ij} : X^i \to X^j$.
(iii) If $i < j < k \leqslant \lambda$, then the composite $X^i \xrightarrow{f^{ij}} X^j \xrightarrow{f^{jk}} X^k$ agrees with $f^{ik} : X^i \to X^k$.

Lemma 3.7. *Suppose that X is a sequence of length λ in $\mathcal{C}_N(\mathscr{A})$ and $X^j = \varinjlim_{i<j} X^i$ for every limit ordinal j. If each of the chain maps $X^i \to X^{i+1}$ is a degreewise monomorphism with the mapping cone in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, then the mapping cones of all $f^{ij} : X^i \to X^j$ belong to ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$.*

Proof. We prove that the statement is true for all f^{ij} with $i \leqslant j \leqslant k$ by induction on $k \leqslant \lambda$. If $k = 0$, there is nothing to prove. Suppose the statement is true for k. We wish to prove it for $k+1$. Choose any $i < j \leqslant k+1$. If $i < j \leqslant k$, then the mapping cone on f^{ij} lies in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ by the inductive hypothesis. If $j = k+1$, then $i \leqslant k$ and $f^{ij} = f^{k,k+1} f^{ik}$. Since the mapping cones on f^{ik} and on $f^{k,k+1}$ both lie in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, so does the mapping cone on $f^{ij} = f^{k,k+1} f^{ik}$. Next suppose k is a limit ordinal and the mapping cone on f^{ij} lies in ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ for all $i < j < k$. It suffices to prove that the induced map

$$\mathrm{Hom}(f, E) : \mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(X^k, E) \to \mathrm{Hom}_{\mathcal{K}_N(\mathscr{A})}(X^i, E)$$

is an isomorphism for every $E \in \mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. Let us first prove the surjectivity. Suppose we are given a chain map $h^i : X^i \to E$. By induction on j, h^i can be factored in $\mathcal{C}_N(\mathscr{A})$ as $X^i \xrightarrow{f^{ij}} X^j \xrightarrow{h^j} E$. Hence $h^j : X^j \to E$ can be factored as $X^j \xrightarrow{f^{j,j+1}} X^{j+1} \xrightarrow{h^{j+1}} E$ by Lemma 3.4. For limit ordinals ν, we can obtain the desired factorization since $X^\nu = \varinjlim_{i<\nu} X^i$. Thus we have a factorization of h^i as $X^i \xrightarrow{f^{ik}} X^k \xrightarrow{h^k} E$ in $\mathcal{C}_N(\mathscr{A})$, which induces a factorization in $\mathcal{K}_N(\mathscr{A})$.

Now we prove injectivity. Suppose we are given a chain map $h^i : X^i \to E$. By the preceding proof, we can choose an $h^k : X^k \to E$ such that $h^k f^{ik} = h^i$ in $\mathcal{C}_N(\mathscr{A})$. Now take any $h : X^k \to E$ with $hf^{ik} \sim h^i = h^k f^{ik}$. By choosing a homotopy $\{s_n^i : X_n^i \to E_{n+N-1}\}$ connecting hf^{ik} with $h^k f^{ik}$ and induction on j, we can get a homotopy connecting hf^{jk} with $h^k f^{jk}$ with $i \leqslant j \leqslant k$ by Lemma 3.5. This completes the proof. □

Next we will construct sequences to which we will apply Lemma 3.7.

Construction 3.8. Given an object $X \in \mathcal{C}_N(\mathscr{A})$, an integer n and a monomorphism $X_n \to A$ in $\mathscr{A}$, we form a chain map of N-complexes $f : X \to Y = B(X, n, X_n \to A)$ as follows:

(i) $f_i : X_i \to Y_i$ is the identity map $1_{X_i} : X_i \to X_i$ for all $i \neq n, \dots, n-N+1$.

(ii) In degrees $n, \dots, n-N+1$, each commutative square of the following diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\
X_{n-1} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
X_{n-N+1} \arrow[r] & Y_{n-N+1}
\end{tikzcd}$$

is just the pushout square

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & A \arrow[d] \\
X_{n-1} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
X_{n-N+1} \arrow[r] & Y_{n-N+1}.
\end{tikzcd}$$

So we have a short exact sequence of N-complexes

$$0 \to X \xrightarrow{f} Y \to \mathrm{D}_n^N(A/X_n) \to 0,$$

which implies that f is an injective quasi-isomorphism. But the mapping cone of f in $\mathcal{K}_N(\mathscr{A})$ is homotopic to a bounded N-complex and belongs to ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ by Lemma 3.1. Thus f is a suitable building block for constructing chains of N-complexes as in Lemma 3.7.

In fact, we could specify A, up to noncanonical isomorphism, by giving its class as an extension in $\mathrm{Ext}^1_{\mathscr{A}}(A/X_n, X_n)$. Let Q be the coproduct of all the quotients of the generator G. Set $A/X_n = Q^{(\beta)}$ be a large coproduct. It suffices to give a subset $\Lambda \subseteq \mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)$ of cardinality β. We denote the corresponding N-complex Y by $B(X, \Lambda)$. If $\Lambda = \mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)$, we denote the chain map as $X \to B(X, n)$. In this case, the functor $\mathrm{Ext}^1_{\mathscr{A}}(C, -)$ annihilates the map $X_n \to B(X, n)_n$ whenever C is a direct summand of Q.

Now we inductively define a sequence of length ω in $\mathcal{C}_N(\mathscr{A})$. At each step, let the chain map $X^i \to X^{i+1}$ be $X^i \to B(X^i, n)$ for some n depending on i. The precise recipe is:

(i) $X^0 = X$ and $X^0 \to X^1$ is the chain map $X \to B(X, 0)$.

(ii) For an integer $i > 0$, we define $X^{2i-1} \to X^{2i}$ to be $X^{2i-1} \to B(X^{2i-1}, i)$ while $X^{2i} \to X^{2i+1}$ is set to be $X^{2i} \to B(X^{2i}, -i)$.

(iii) $X^\omega = \varinjlim X^n$.

Let $f_X : X \to J(X)$ be the chain map $X \to X^\omega$ of above. Then f_X is a degreewise monomorphism since it is the colimit of degreewise monomorphisms. Let C be a quotient of G. Then either the chain map $X^{2|n|-1} \to X^{2|n|}$ or the chain map $X^{2|n|} \to X^{2|n|+1}$ induces zero in degree n under the functors $\mathrm{Ext}^1_{\mathscr{A}}(C, -)$. Thus f_X is annihilated by $\mathrm{Ext}^1_{\mathscr{A}}(C, -)$ in every degree n depending on whether n is positive or negative. Furthermore, the mapping cone lies in ${}^\perp\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ by Lemma 3.7.

The following result was proved by Neeman when $N = 2$ (see [12, Theorem 2.13]).

Theorem 3.9. *The natural inclusion $i : \mathcal{K}_N(\mathrm{Inj}\mathscr{A}) \to \mathcal{K}_N(\mathscr{A})$ has a left adjoint i_ℓ. In particular, the category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ has arbitrary coproducts.*

Proof. Let X be an arbitrary object of $\mathcal{C}_N(\mathscr{A})$. By transfinite induction, we define a chain of N-complexes $J^\lambda(X)$ for every ordinal λ. The rule is:

(i) $J^0(X) = X$.

(ii) If $J^\lambda(X)$ has been defined, then the map $J^\lambda(X) \to J^{\lambda+1}(X)$ is just $J^\lambda(X) \to J(J^\lambda(X))$.

(iii) If λ is a limit ordinal, then $J^\lambda(X) = \varinjlim_{i<\lambda} J^i(X)$.

Let α be the regular cardinal of [12, Definition 1.4]. Consider the triangle

$$W \to X \to J^\alpha(X) \to \Sigma W.$$

Lemma 3.7 implies that W belongs to ${}^\perp\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. Since α is an α-filtered colimit of the ordinals $\lambda < \alpha$, it follows from [12, Lemma 1.10] that for each quotient C of G,

$$\mathrm{Ext}^1_{\mathscr{A}}(C, J^\alpha(X)_n) = \varinjlim_{\lambda<\alpha} \mathrm{Ext}^1_{\mathscr{A}}(C, J^\lambda(X)_n).$$

By construction, the map

$$\mathrm{Ext}^1_{\mathscr{A}}(C, J^\lambda(X)_n) = \mathrm{Ext}^1_{\mathscr{A}}(C, J^{\lambda+1}(X)_n)$$

is zero, and so the colimit vanishes. Thus $\mathrm{Ext}^1_{\mathscr{A}}(C, J^\alpha(X)_n) = 0$ and $J^\alpha(X)_n$ is injective for all n. Now define $i_\ell(X) = J^\alpha(X)$. It follows from the triangle and [10, Theorem 9.1.13] that i_ℓ is a left adjoint of i.

Finally, given a collection of objects $\{X^\lambda \mid \lambda \in \Lambda\}$ in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, we can form the coproduct in $\mathcal{K}_N(\mathscr{A})$. Applying i_ℓ to this coproduct gives the coproduct in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. □

4. The μ-compact generation of $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$

In this section, we show that there exists a regular cardinal μ for which the homotopy category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ is μ-compactly generated.

Construction 4.1. Choose a regular cardinal μ as in [12, Definition 1.11], define $\mathscr{B} \subseteq \mathscr{A}$ to be the full subcategory of $\mathscr{A}$ consisting of the objects A with $|\mathrm{Hom}_{\mathscr{A}}(G, A)| < \mu$. Then $\mathscr{B} = \mathscr{A}^\mu$ the full subcategory of μ-presentable objects in $\mathscr{A}$ by [12, Proposition 1.18]. Let X be an object in $\mathcal{C}_N(\mathscr{A})$ and $\mathscr{F}$ a full subcategory of subobjects $Y \subseteq X$ with $Y \in \mathcal{C}_N(\mathscr{B}) \subseteq \mathcal{C}_N(\mathscr{A})$. Assume $\mathscr{F}$ is μ-filtered and its colimit is X. Construct the category $\mathscr{L}(\mathscr{F}, n)$ whose objects are subobjects Y of $B(X, n)$ with the following properties:

(i) $Y \cap X$ belongs to $\mathscr{F}$.

(ii) The chain map $Y \cap X \to Y$ is an isomorphism in degrees $i \neq n, \ldots, n - N + 1$.

(iii) In degree n, we have a monomorphism $Y_n / Y_n \cap X_n \to B(X, n)_n / X_n$. Note that $B(X, n)_n / X_n = Q^{(\beta)} = \coprod_{\mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)} Q$ with Q as in Construction 3.8. We require that the monomorphism $Y_n / Y_n \cap X_n \to B(X, n)_n / X_n$ is the inclusion of a subcoproduct.

(iv) Each commutative square of the following diagram

$$\begin{tikzcd}
Y_n \cap X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\
Y_{n-1} \cap X_{n-1} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
Y_{n-N+1} \cap X_{n-N+1} \arrow[r] & Y_{n-N+1}
\end{tikzcd}$$

is a pushout square.

By Construction 3.8 and the construction of the category $\mathscr{L}(\mathscr{F}, n)$, we have a commutative diagram of short exact sequences in $\mathcal{C}_N(\mathscr{A})$:

$$\begin{tikzcd}
0 \arrow[r] & Y \cap X \arrow[r] \arrow[d, hook] & Y \arrow[r] \arrow[d, hook] & Y/Y \cap X \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X,n) \arrow[r] & \mathrm{D}_n^N(Q)^{(|\mathrm{Ext}^1_{\mathscr{A}}(Q,X_n)|)} \arrow[r] & 0,
\end{tikzcd}$$

where h must be the inclusion of a subcoproduct. In degree n, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & Y_n \cap X_n \arrow[r] \arrow[d] & Y_n \arrow[r] \arrow[d] & Q^{(|\Lambda'|)} \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X_n \arrow[r] & B(X,n)_n \arrow[r] & Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(Q,X_n)|)} \arrow[r] & 0.
\end{tikzcd}$$

The top row of this diagram defines a map $\varphi : \Lambda' \to \mathrm{Ext}^1_{\mathscr{A}}(Q, Y_n \cap X_n)$ giving the extension. Since h is an inclusion, the composite

$$\Lambda' \xrightarrow{\varphi} \mathrm{Ext}^1_{\mathscr{A}}(Q, Y_n \cap X_n) \to \mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)$$

is injective. Therefore φ is injective and Λ' is a subset of $\mathrm{Ext}^1_{\mathscr{A}}(Q, Y_n \cap X_n)$.

Lemma 4.2. *The objects of the category $\mathscr{L}(\mathscr{F}, n)$ all belong to $\mathcal{C}_N(\mathscr{B}) = \mathcal{C}_N(\mathscr{A}^\mu)$.*

Proof. Since $Y \cap X$ belongs to $\mathscr{F} \subseteq \mathcal{C}_N(\mathscr{B})$, all the objects $Y_i \cap X_i$ belong to $\mathscr{B}$. For $i \neq n, \ldots, n-N+1$, we have $Y_i = Y_i \cap X_i \in \mathscr{B}$. We need to show that $Y_n, \ldots, Y_{n-N+1} \in \mathscr{B}$. From the pushout squares

$$\begin{tikzcd}
Y_n \cap X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\
Y_{n-1} \cap X_{n-1} \arrow[r] \arrow[d] & Y_{n-1} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
Y_{n-N+1} \cap X_{n-N+1} \arrow[r] & Y_{n-N+1},
\end{tikzcd}$$

it follows that Y_{n-t} is a quotient of $Y_{n-t+1} \oplus (Y_{n-t} \cap X_{n-t})$ for $t = 1, \ldots, N-1$. In Construction 4.1, we saw that Y_n is an extension of $Q^{(|\Lambda'|)}$ by $Y_n \cap X_n \in \mathscr{B}$, where $\Lambda' \subseteq \mathrm{Ext}^1_{\mathscr{A}}(Q, Y_n \cap X_n)$. So $|\Lambda'| \leqslant |\mathrm{Ext}^1_{\mathscr{A}}(Q, Y_n \cap X_n)| < \mu$ by [12, Remark 1.17]. But Q is the coproduct of the $< \alpha$ quotients C of the generator G, it follows that $Q^{(|\Lambda'|)}$ is a coproduct of $< \mu$ objects in $\mathscr{B}$ and so $Q^{(|\Lambda'|)} \in \mathscr{B}$. Thus [12, Proposition 1.15(iii)] implies that $Y_n \in \mathscr{B}$, and hence $Y_{n-1}, \ldots, Y_{n-N+1}$ belong to $\mathscr{B}$ by [12, Proposition 1.15(ii)]. □

Lemma 4.3. *The category $\mathscr{L}(\mathscr{F}, n)$ is μ-filtered.*

Proof. Since $\mathscr{L}(\mathscr{F}, n)$ is equivalent to a partially ordered set, we need only show that every collection of fewer than μ objects in $\mathscr{L}(\mathscr{F}, n)$ is dominated by an object of $\mathscr{L}(\mathscr{F}, n)$. Suppose that we are given a set $\{Y^j \mid j \in J\}$ of $< \mu$ objects in $\mathscr{L}(\mathscr{F}, n)$. Since the objects $Y^j \cap X$ all belong to the μ-filtered category $\mathscr{F}$, we may choose a $Z \in \mathscr{F}$ dominating them. Note that $X_n = \varinjlim_{X^i \in \mathscr{F}} X_n^i$ and $\operatorname{Ext}^1_{\mathscr{A}}(Q, -)$ commutes with μ-filtered colimits. Hence

$$\operatorname{Ext}^1_{\mathscr{A}}(Q, X_n) = \varinjlim_{X^i \in \mathscr{F}} \operatorname{Ext}^1_{\mathscr{A}}(Q, X_n^i).$$

For any $k \in \operatorname{Ker}(\operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, X_n))$, we choose some morphism $Z \to Z^k$ in $\mathscr{F}$ so that k is annihilated by $\operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n^k)$. Note that $|\operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n)| < \mu$, there are $< \mu$ possible k. Since $\mathscr{F}$ is μ-filtered, Z^k are all dominated by some object $Z' \in \mathscr{F}$. Thus the chain map $Z \to Z'$ annihilates all the k, which implies that $\operatorname{Im}(\operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, Z'_n))$ maps injectively to $\operatorname{Ext}^1_{\mathscr{A}}(Q, X_n)$. For each Y^j, Y_n^j is an extension of $Q^{(|\Lambda_j|)}$ by $Y_n^j \cap X_n$, where $\Lambda_j \subseteq \operatorname{Ext}^1_{\mathscr{A}}(Q, Y_n^j \cap X_n)$ that maps injectively to $\operatorname{Ext}^1_{\mathscr{A}}(Q, X_n)$. We may take the image of Λ_j under the composite

$$\operatorname{Ext}^1_{\mathscr{A}}(Q, Y_n^j \cap X_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, Z'_n).$$

Since the image of each Λ_j is contained in $\operatorname{Im}(\operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, Z'_n))$, so is the union of the images $\Lambda' = \bigcup \operatorname{Im}(\Lambda_j)$. But $\operatorname{Im}(\operatorname{Ext}^1_{\mathscr{A}}(Q, Z_n) \to \operatorname{Ext}^1_{\mathscr{A}}(Q, Z'_n))$ maps injectively to $\operatorname{Ext}^1_{\mathscr{A}}(Q, X_n)$, so does its subset Λ'. Let $Y' = B(Z', \Lambda')$. For the objects Y^j, Y' and $B(X, n)$, we have three extension sequences:

$$0 \to Y^j \cap X \to Y^j \to \mathrm{D}_n^N(Q^{(|\Lambda_j|)}) \to 0,$$

$$0 \to Z' \to Y' \to \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \to 0,$$

$$0 \to X \to B(X, n) \to \mathrm{D}_n^N(Q^{(|\operatorname{Ext}^1_{\mathscr{A}}(C, X_n)|)}) \to 0.$$

Since the extension classes are all compatible, we may choose morphisms of extensions

$$\begin{tikzcd}
0 \arrow[r] & Y^j \cap X \arrow[r] \arrow[d, hook] & Y^j \arrow[r, "\delta^j"] \arrow[d, "f^j"] & \mathrm{D}_n^N(Q^{(|\Lambda_j|)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z' \arrow[r] \arrow[d, hook] & Y' \arrow[r] \arrow[d, "g"] & \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X, n) \arrow[r] & \mathrm{D}_n^N(Q^{(|\operatorname{Ext}^1_{\mathscr{A}}(C, X_n)|)}) \arrow[r] & 0.
\end{tikzcd}$$

For each subobject Y^j of $B(X, n)$, we have a monomorphism $h^j : Y^j \to B(X, n)$ and a commutative diagram in $\mathcal{C}_N(\mathscr{A})$:

$$\begin{tikzcd}
0 \arrow[r] & Y^j \cap X \arrow[r] \arrow[d] & Y^j \arrow[r, "\delta^j"] \arrow[d, "h^j"] & \mathrm{D}_n^N(Q^{(|\Lambda_j|)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X,n) \arrow[r] & \mathrm{D}_n^N(Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(C,X_n)|)}) \arrow[r] & 0.
\end{tikzcd}$$

Note that the chain maps f^j and g may be chosen as above, but they are not unique. Thus there is no reason to expect that h^j should equal gf^j. But $h^j - gf^j$ factors through a chain map $\mathrm{D}_n^N(Q^{(|\Lambda_j|)}) \to X$, so $h^j - gf^j$ is determined by a morphism $Q^{(|\Lambda_j|)} \to X_n$ in $\mathscr{A}$ by [14, Lemma 2.2]. Since $\mathrm{Hom}_{\mathscr{A}}(Q^{(|\Lambda_j|)}, -)$ commutes with μ-filtered colimits and X_n is the μ-filtered colimit of $\{X_n^i\}_{X^i \in \mathscr{F}}$, we choose a morphism $Z' \to Z^j$ in $\mathscr{F}$ for each j so that $h^j - gf^j$ factors through $\mathrm{D}_n^N(Q^{(|\Lambda_j|)}) \to Z^j \subseteq X$. Since there are fewer than μ objects $Z^j \in \mathscr{F}$, there is an object $Z'' \in \mathscr{F}$ dominating them. Let $W = B(Z'', \Lambda')$ and we have an extension $0 \to Z'' \to W \to \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \to 0$ corresponding to the image of $\Lambda' \subseteq \mathrm{Ext}^1_{\mathscr{A}}(Q, Z'_n)$ under the map $\mathrm{Ext}^1_{\mathscr{A}}(Q, Z'_n) \to \mathrm{Ext}^1_{\mathscr{A}}(Q, Z''_n)$. Because the extension classes are compatible, we construct the following morphisms of extensions:

$$\begin{tikzcd}
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Y' \arrow[r, "\eta"] \arrow[d, "\rho"] & \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z'' \arrow[r, "\tau"] \arrow[d] & W \arrow[r, "\eta'"] \arrow[d, "\sigma"] & \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X \arrow[r, "\theta"] & B(X,n) \arrow[r] & \mathrm{D}_n^N(Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(C,X_n)|)}) \arrow[r] & 0.
\end{tikzcd}$$

There is no reason to expect g to be equal to $\sigma\rho$. But $g - \sigma\rho$ factors through some $\varphi : \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \to X$. Set $\bar{\sigma} = \sigma + \theta\varphi\eta'$. Then $g = \bar{\sigma}\rho$. Now we have monomorphisms $Y^j \xrightarrow{f^j} Y' \xrightarrow{\rho} W \xrightarrow{\bar{\sigma}} B(X,n)$ and $h^j - gf^j = h^j - \bar{\sigma}\rho f^j$ factors through a chain map $\mathrm{D}_n^N(Q^{(|\Lambda_j|)}) \xrightarrow{\gamma^j} Z'' \to X$. Set $\bar{f}^j = \rho f^j + \tau\gamma^j\delta^j$, then $\bar{\sigma}\bar{f}^j = h^j$ for all $j \in J$. Thus the monomorphisms h^j all factor through $\bar{\sigma}$, and so the subobject $\bar{\sigma} : W \to B(X,n)$ belongs to $\mathscr{L}(\mathscr{F}, n)$. □

Lemma 4.4. *$B(X,n)$ is the colimit of its subobjects $Y \in \mathscr{L}(\mathscr{F}, n)$.*

Proof. Let Y be an object in $Y \in \mathscr{L}(\mathscr{F}, n)$. We have a monomorphism of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & Y \cap X \arrow[r] \arrow[d, "f_Y"] & Y \arrow[r] \arrow[d, "g_Y"] & \mathrm{D}_n^N(Q^{(|\Lambda'|)}) \arrow[r] \arrow[d, "h_Y"] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X,n) \arrow[r] & \mathrm{D}_n^N(Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(Q,X_n)|)}) \arrow[r] & 0,
\end{tikzcd}$$

where h_Y is the inclusion of a subcoproduct. Since the category $\mathscr{L}(\mathscr{F}, n)$ is filtered, the colimit in $\mathscr{L}(\mathscr{F}, n)$ of the top row is exact. We wish to show that the colimit of g_Y is an isomorphism, it suffices to prove that the colimits of f_Y and h_Y are isomorphisms. But f_Y and h_Y are monomorphisms, so are their colimits. It therefore suffices to prove that the colimits of f_Y and h_Y are epimorphisms. Since the category $\mathscr{F}$ embeds in the category $\mathscr{L}(\mathscr{F}, n)$, we can view a subobject $Y \subseteq X$ as a subobject of $B(X, n)$, where the corresponding $\Lambda' \subseteq \mathrm{Ext}^1_{\mathscr{A}}(Q, Y_n)$ is empty. But the colimit of $\mathscr{F}$ maps epimorphically to X and this epimorphism factors through the colimit of f_Y. Hence the colimit of f_Y is an epimorphism. Take any $\lambda \in \mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)$. Since $\mathrm{Ext}^1_{\mathscr{A}}(Q, -)$ commutes with μ-filtered colimits and X_n is the μ-filtered colimit of $\{X^i_n\}_{X^i \in \mathscr{F}}$, there exist a $Z \in \mathscr{F}$ and an element $e_\lambda \in \mathrm{Ext}^1_{\mathscr{A}}(Q, Z_n)$ mapping to λ. Form the extension $0 \to Z \to Y \to \mathrm{D}^N_n(Q) \to 0$ corresponding to e_λ. We have a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & Z \arrow[r] \arrow[d, "f_Y"] & Y \arrow[r] \arrow[d, "g_Y"] & \mathrm{D}^N_n(Q) \arrow[r] \arrow[d, "h_Y"] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X,n) \arrow[r] & \mathrm{D}^N_n(Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(Q,X_n)|)}) \arrow[r] & 0,
\end{tikzcd}$$

where h_Y is the inclusion of the subcoproduct over the singleton $\{\lambda\}$. Thus the image of the colimit of the h_Y contains the coproduct over every singleton in $\mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)$, and hence it must be epimorphism. $\square$

Lemma 4.5. *If filtered colimits of $< \mu$ objects in $\mathscr{F}$ belong to $\mathscr{F}$, then filtered colimits of $< \mu$ objects in $\mathscr{L}(\mathscr{F}, n)$ belong to $\mathscr{L}(\mathscr{F}, n)$.*

Proof. Let Y be an object in $\mathscr{L}(\mathscr{F}, n)$. We have a monomorphism of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & Y \cap X \arrow[r] \arrow[d, "f_Y"] & Y \arrow[r] \arrow[d, "g_Y"] & \mathrm{D}^N_n(Q^{(|\Lambda'|)}) \arrow[r] \arrow[d, "h_Y"] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X,n) \arrow[r] & \mathrm{D}^N_n(Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(Q,X_n)|)}) \arrow[r] & 0,
\end{tikzcd}$$

where h_Y is the inclusion of a subcoproduct and $Y \cap X \in \mathscr{F}$. If $|\Lambda| < \mu$ and each $Y_\lambda \cap X \in \mathscr{F}$, then a filtered colimit of objects $\{Y^\lambda \cap X\}_{\lambda \in \Lambda}$ belong to $\mathscr{F}$. Hence the filtered colimit of $< \mu$ monomorphisms of short exact sequences as above is such a monomorphism since filtered colimits are exact, as desired. $\square$

Construction 4.6. By Theorem 3.9, the object $i_\ell(X) = J^\alpha(X)$ can be constructed using a single sequence. Let us remember this sequence:

(i) $X^0 = X$.

(ii) $X^{i+1} = B(X^i, n)$ for some n depending on i. The precise relation is that if $i = \nu + m$, where ν is a limit ordinal and m is an integer, then $n = -m/2$ if m is even and $n = (m+1)/2$ if m is odd.

(iii) For limit ordinals j we have $X^j = \varinjlim_{i<j} X^i$.

Suppose we are given an α-filtered category $\mathscr{F}$ of subobjects of X, whose colimit is X. For every ordinal i, we establish a subcategory $\mathscr{F}_i$ of subobjects of X^i as follows:

(i) $\mathscr{F}_0 = \mathscr{F}$.

(ii) If n is the integer for which $X^{i+1} = B(X^i, n)$, then $\mathscr{F}_{i+1} = \mathscr{L}(\mathscr{F}_i, n)$.

(iii) Let j be a limit ordinal. A subobject $Y \subseteq X^j$ belongs to $\mathscr{F}^j$ if and only if $Y \cap X^i$ belongs to $\mathscr{F}_i$ for all $i < j$.

Lemma 4.7. *Suppose $Y \subseteq X^j$ lies in $\mathscr{F}_i$ as the notation in Construction 4.6. Consider the triangle $\Sigma^{-1}W \to Y \cap X \to Y \to W$, we have W belongs to ${}^{\perp}\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$.*

Proof. Set $Y^i = Y \cap X^i$. By assumption, $Y^{i+1} \in \mathscr{L}(\mathscr{F}_i, n)$ and $Y^{i+1} = B(Y \cap X^i, \Lambda') = B(Y^i, \Lambda')$ for some subset $\Lambda' \subseteq \mathrm{Ext}^1_{\mathscr{A}}(Q, Y^i_n)$ mapping injectively to $\mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)$ by Construction 4.1. For limit ordinals ν, we have

$$Y^\nu = Y \cap X^\nu = \varinjlim_{i<\nu}(Y \cap X^i) = \varinjlim_{i<\nu} Y^i.$$

The lemma now follows from Lemma 3.7 and Construction 3.8. $\square$

Lemma 4.8. *Let X be an object of $\mathcal{C}_N(\mathscr{A})$ and $\mathscr{F} \subseteq \mathcal{C}_N(\mathscr{B}) = \mathcal{K}_N(\mathscr{A}^\mu)$ a full subcategory of the subobjects of X. Assume that $\mathscr{F}$ is μ-filtered with colimit X and that filtered colimits of $< \mu$ objects in $\mathscr{F}$ belong to $\mathscr{F}$. Then the full subcategory of $\mathscr{F}_\alpha$ whose objects are in $\mathcal{C}_N(\mathrm{Inj}\mathscr{A})$ is cofinal.*

Proof. Let Y be an object in $\mathscr{F}_\alpha$. We need to produce a morphism $Y \to Z$ in $\mathscr{F}_\alpha$ with $Z \in \mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. We inductively define a sequence $\{Z^i\}$ of objects in $\mathscr{F}_i$.

(i) Put $Z^0 = Y \cap X$.

(ii) Assume n is the integer for which $\mathscr{F}_{i+1} = \mathscr{L}(\mathscr{F}_i, n)$, and suppose we have defined $Z^i \in \mathscr{F}_i$. Choose an object $W^i \in \mathscr{F}_i$ containing Z^i and $Y \cap X^i$. There exists a morphism $W^i \to V^i$ in $\mathscr{F}_i$ annihilating $\mathrm{Ker}(\mathrm{Ext}^1_{\mathscr{A}}(Q, W^i_n) \to \mathrm{Ext}^1_{\mathscr{A}}(Q, X^i_n))$ by the proof of Lemma 4.3. Let $\Lambda' = \mathrm{Im}(\mathrm{Ext}^1_{\mathscr{A}}(Q, W^i_n) \to \mathrm{Ext}^1_{\mathscr{A}}(Q, V^i_n))$. Then Λ' maps injectively to $\mathrm{Ext}^1_{\mathscr{A}}(Q, X^i_n)$. Define $Z^{i+1} = B(V^i, \Lambda') \in \mathscr{F}_{i+1}$. Then Z^{i+1} is given by a morphism of extensions

$$\begin{tikzcd}
0 \arrow[r] & V^i \arrow[r] \arrow[d, "f"] & Z^{i+1} \arrow[r] \arrow[d, "g"] & \mathrm{D}^N_n(Q^{(|\Lambda'|)}) \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X \arrow[r] & B(X, n) \arrow[r] & \mathrm{D}^N_n(Q^{(|\mathrm{Ext}^1_{\mathscr{A}}(Q, X_n)|)}) \arrow[r] & 0.
\end{tikzcd}$$

Note that the monomorphisms f and h are given, and we make a choice of a compatible g.

(iii) For limit ordinals λ, define $Z^\lambda = \varinjlim_{i<\lambda} Z^i$. We have a morphism $Z^i \to Z^{i+1}$ that factors as $Z^i \to W^i \to V^i \to Z^{i+1}$. By construction, the morphism $W^i \to V^i$ kills the kernel of the map $\varphi : \mathrm{Ext}^1_{\mathscr{A}}(Q, W^i_n) \to \mathrm{Ext}^1_{\mathscr{A}}(Q, X^i_n)$

while the morphism $V^i \to Z^{i+1}$ kills the image of φ. It follows that the composite $Z^i_n \to Z^{i+1}_n$ is annihilated by $\mathrm{Ext}^1_{\mathscr{A}}(Q,-)$ for some suitable n for which $X^{i+1} = B(X^i, n)$ and n occurs between i and $i+\omega$ for any limit ordinal i. If we restrict to limit ordinals, then $\mathrm{Ext}^1_{\mathscr{A}}(Q,-)$ annihilates $Z^i_n \to Z^j_n$ for any integer n and any pair $i < j$ of limit ordinals. But Z^α is the α-filtered colimit of the limit ordinals $< \alpha$ and $\mathrm{Ext}^1_{\mathscr{A}}(C,-)$ commutes with α-filtered colimits for each quotient C of the generator G. It follows that $\mathrm{Ext}^1_{\mathscr{A}}(C, Z^\alpha_n) = 0$ for all C and all n, and hence Z^α is an N-complex of injectives. Therefore by construction, $Y = \underrightarrow{\mathrm{colim}}(Y \cap X^i)$ maps in $\mathscr{F}_\alpha$ to $Z = \underrightarrow{\mathrm{colim}} Z^i$. □

Lemma 4.9. *Let $i_\ell : \mathcal{K}_N(\mathscr{A}) \to \mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ be the functor of Theorem 3.9. Then the objects $\{i_\ell(S) | S \in \mathcal{K}_N(\mathscr{B})\}$ generate the category $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$.*

Proof. For every nonzero object $X \in \mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, we need to produce a nonzero morphism $i_\ell(S) \to X$ in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ or equivalently a nonzero morphism $S \to X$ in $\mathcal{K}_N(\mathscr{A})$. If X is not acyclic, there is a nontrivial amplitude homology group. Without loss of generality, we may assume $\mathrm{H}^t_0(X) \neq 0$ for some t. Choose a map $G \to \mathrm{Z}^t_0(X)$ that does not factor through $\mathrm{B}^{N-t}_0(X)$. Then $G \to \mathrm{Z}^t_0(X) \to X_0$ extends to a chain map $\mathrm{D}^t_0(G) \to X$ that is nonzero in amplitude homology, and also $\mathrm{D}^t_0(G) \in \mathcal{C}_N(\mathscr{B})$. Next suppose X is acyclic. If $X \neq 0$ in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, then X is not contractible. So there exists an n such that $\mathrm{Z}^1_n(X)$ is not injective in $\mathscr{A}$. Thus there is a quotient C of G and a nonzero element of $\mathrm{Ext}^1_{\mathscr{A}}(C, \mathrm{Z}^1_n(X))$. But elements of $\mathrm{Ext}^1_{\mathscr{A}}(C, \mathrm{Z}^1_n(X))$ are in a bijection with morphisms $\mathrm{D}^1_n(C) \to X$ in $\mathcal{K}_N(\mathscr{A})$. Therefore we have a nonzero morphism $\mathrm{D}^1_n(C) \to X$ where $\mathrm{D}^1_n(C) \in \mathcal{C}_N(\mathscr{B})$, as claimed. □

Let $\mathscr{T}$ be a triangulated category with arbitrary coproducts and suspension functor Σ, α an infinite regular cardinal. The category $\mathscr{T}$ is α-compactly generated if there is a set of objects $\mathscr{S}$ such that $\Sigma\mathscr{S} = \mathscr{S}$, satisfying the conditions:

(G1) an object $X \in \mathscr{T}$ is zero if $\mathscr{T}(S, X) = 0$ for all $S \in \mathscr{S}$;

(G2) for each family of morphisms $\{f_i : X_i \to Y_i\}_{i \in I}$, the induced morphism

$$\mathscr{T}(S, \coprod_{i \in I} X_i) \to \mathscr{T}(S, \coprod_{i \in I} Y_i)$$

is surjective for all $S \in \mathscr{S}$ if the morphisms

$$\mathscr{T}(S, X_i) \to \mathscr{T}(S, Y_i)$$

are surjective for all $i \in I$ and all $S \in \mathscr{S}$;

(G3) for each family of objects $\{X_i\}_{i \in I}$ of $\mathscr{T}$, each morphism $S \to \coprod_{i \in I} X_i$ factors through $S \to \coprod_{i \in J} X_i$ for some subset $J \subseteq I$ with $|J| < \alpha$.

Theorem 4.10. *Let $\mathscr{B} = \mathscr{A}^\mu$ the category of μ-presentable objects in $\mathscr{A}$. Then the objects $\{i_\ell(S) | S \in \mathcal{C}_N(\mathscr{B})\}$ form a μ-compact generating set in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$.*

Proof. Suppose we are given a set $\{X^\lambda | \lambda \in \Lambda\}$ of objects in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. Then the coproduct of these objects in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ is formed by applying the functor

i_ℓ of Theorem 3.9 to the ordinary coproduct in $\mathcal{K}_N(\mathscr{A})$ or $\mathcal{C}_N(\mathscr{A})$. Set $X = \coprod_{\lambda\in\Lambda} X^\lambda$ and the category $\mathscr{F}$ as

$$\mathscr{F} = \{\coprod_{\lambda\in\Lambda'} S^\lambda \,|\, \Lambda' \subseteq \Lambda \text{ with } |\Lambda'| < \mu \text{ and each } S^\lambda \in \mathcal{C}_N(\mathscr{B}) \text{ is a subobject of } X^\lambda\}.$$

Since each object X^λ in $\mathcal{C}_N(\mathscr{A})$ is the μ-filtered colimit of its subobjects $\{S \to X^\lambda \,|\, S \in \mathcal{C}_N(\mathscr{B})\}$, and the coproduct of the X^λ satisfies

$$\coprod_{\lambda\in\Lambda} X^\lambda = \varinjlim_{\substack{\Lambda'\subseteq\Lambda \\ |\Lambda'|<\mu}} \coprod_{\lambda\in\Lambda'} \varinjlim_{\substack{S^\lambda\to X^\lambda \\ S^\lambda\in\mathcal{C}_N(\mathscr{B})}} S^\lambda,$$

it follows from Lemmas 4.2-4.5 that $i_\ell(X) = X^\alpha$ is the colimit in $\mathcal{C}_N(\mathscr{A})$ of the μ-filtered category $\mathscr{F}_\alpha$, and hence any chain map $S \to i_\ell(X)$ factors through some object $Y \in \mathscr{F}_\alpha$ where $S \in \mathcal{C}_N(\mathscr{B})$. By Lemma 4.7, we have a triangle $\Sigma^{-1}W \to Y \cap X \xrightarrow{\iota} Y \to W$ with $W \in {}^\perp\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, and so $i_\ell(\iota)$ is an isomorphism. Thus the chain map $S \to i_\ell(X)$ factors as $S \to i_\ell(Y\cap X) \to i_\ell(X)$. But $Y \cap X \in \mathscr{F}$, it follows that $\coprod_{\lambda\in\Lambda'} S^\lambda \in \mathscr{F}$. We have factored the chain map as

$$i_\ell(S) \to i_\ell(\coprod_{\lambda\in\Lambda'} S^\lambda) \to i_\ell(\coprod_{\lambda\in\Lambda} X^\lambda)$$

in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. Now suppose that we are given in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ a vanishing composite

$$i_\ell(S) \xrightarrow{\theta} i_\ell(\coprod_{\lambda\in\Lambda'} S^\lambda) \xrightarrow{\tau} i_\ell(\coprod_{\lambda\in\Lambda} X^\lambda),$$

where $U = \coprod_{\lambda\in\Lambda'} S^\lambda \in \mathscr{F} = \mathscr{F}_0$. By Lemma 4.8, in $\mathscr{F}_\alpha$ the objects that belong to $\mathcal{C}_N(\mathrm{Inj}\mathscr{A})$ are cofinal. There exists a chain map $U \to Y$ in $\mathscr{F}_\alpha$ with Y in the subcategory. We have a morphism $S \to i_\ell(S) \to i_\ell(Y) = Y$ in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$. Choose a representative $S \to Y$ in $\mathcal{C}_N(\mathscr{A})$. Then the composite $S \to Y \to i_\ell(X)$ is null-homotopic. But $i_\ell(X)$ is the μ-filtered colimit of $\mathscr{F}_\alpha$ and S is μ-presentable. There is a chain map $Y \to Z$ in $\mathscr{F}_\alpha$ so that the composite $S \to Y \to Z$ is already null-homotopic. Since $i_\ell(Y\cap X) \to i_\ell(Y) = Y$ and $i_\ell(Z\cap X) \to i_\ell(Z)$ are isomorphisms in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$, the chain map $S \to i_\ell(Y\cap X) \to i_\ell(Z\cap X)$ is zero in $\mathcal{K}_N(\mathrm{Inj}\mathscr{A})$ for some $Z \in \mathscr{F}_\alpha$. The proof is complete by Lemma 4.9 and [8, Theorem A]. □

5. The compact generation of $\mathcal{K}_N(\mathrm{Prj}R)$

In this section, we show that the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$ of N-complexes of projectives is compactly generated whenever R is right coherent.

Let $\mathscr{X} = \mathrm{Prj}(R)$ (resp. $\mathrm{Inj}(R)$, $\mathrm{Flat}(R)$) be the subcategory of projective (resp. injective, flat) R-modules. We define the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$, $\mathcal{K}_N(\mathrm{Inj}R)$, $\mathcal{K}_N(\mathrm{Flat}R)$ respectively. We next investigate compactness of the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$. The following lemmas will be handy to prove a couple of little lemmas about chain maps from N-complexes of finitely generated projective R-modules.

Lemma 5.1. *Let X be an N-complex of finitely generated projective R-modules and Z an N-complex of flat R-modules. Given a chain map $f : X \to Z$ and some integer j. If f_i vanishes for $i > j$, then the map f factors as $X \xrightarrow{g} Y \xrightarrow{h} Z$ with Y an N-complex of finitely generated projective R-modules, and Y may be chosen so that $Y_i = 0$ for $i > j$.*

Proof. Without loss of generality we may assume $j = 0$. If $i > 0$, then the map $f_i : X_i \to Z_i$ vanishes by assumption, and hence factors through $Y_i = 0$. It therefore suffices to show, by induction, that if we can define a factorization up to some integer i, then we can extend to $i - 1$. Consider the following commutative diagram:

$$\begin{tikzcd}
X_{i+N-2} \arrow[r, "d^X_{i+N-2}"] \arrow[d, "g_{i+N-2}"] & X_{i+N-3} \arrow[r] \arrow[d, "g_{i+N-3}"] & \cdots \arrow[r] & X_{i+1} \arrow[r, "d^X_{i+1}"] \arrow[d, "g_{i+1}"] & X_i \arrow[r, "d^X_i"] \arrow[d, "g_i"] & X_{i-1} \arrow[dd, "f_{i-1}"] \\
Y_{i+N-2} \arrow[r, "d^Y_{i+N-2}"] \arrow[d, "h_{i+N-2}"] & Y_{i+N-3} \arrow[r] \arrow[d, "h_{i+N-3}"] & \cdots \arrow[r] & Y_{i+1} \arrow[r, "d^Y_{i+1}"] \arrow[d, "h_{i+1}"] & Y_i \arrow[d, "h_i"] & \\
Z_{i+N-2} \arrow[r, "d^Z_{i+N-2}"] & Z_{i+N-3} \arrow[r] & \cdots \arrow[r] & Z_{i+1} \arrow[r, "d^Z_{i+1}"] & Z_i \arrow[r, "d^Z_i"] & Z_{i-1}.
\end{tikzcd}$$

An easy diagram chase tells us that the maps

$$Y_{i+N-1} \oplus \cdots \oplus Y_{i+2} \oplus Y_{i+1} \oplus X_i \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -d^{N-1} & -d^{N-2} & -d^{N-3} & \cdots & -d & -g_i \\ 0 & 0 & 0 & \cdots & 0 & d \end{pmatrix}} Y_{i+N-2} \oplus \cdots \oplus Y_{i+1} \oplus Y_i \oplus X_{i-1} \xrightarrow{(d^{N-1}h_{i+N-2}, \ldots, d^2 h_{i+1}, dh_i, f_{i-1})} Z_{i-1}$$

must compose to zero. Note that $Y_{i+N-1} \oplus \cdots \oplus Y_{i+1} \oplus X_i$ and $Y_{i+N-2} \oplus \cdots \oplus Y_i \oplus X_{i-1}$ are finitely generated and projective and Z_{i-1} is flat, it follows from [11, Corollary 3.3] that there is a factorization of $(d^{N-1}h_{i+N-2}, \ldots, d^2 h_{i+1}, dh_i, f_{i-1})$ as

$$Y_{i+N-2} \oplus \cdots \oplus Y_{i+1} \oplus Y_i \oplus X_{i-1} \xrightarrow{(d^{N-1}, \ldots, d^2, d, g_{i-1})} Y_{i-1} \xrightarrow{h_{i-1}} Z_{i-1},$$

where Y_{i-1} is finitely generated and projective, and the composite

$$Y_{i+N-1} \oplus \cdots \oplus Y_{i+2} \oplus Y_{i+1} \oplus X_i \xrightarrow{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ -d^{N-1} & -d^{N-2} & -d^{N-3} & \cdots & -d & -g_i \\ 0 & 0 & 0 & \cdots & 0 & d \end{pmatrix}} Y_{i+N-2} \oplus \cdots \oplus Y_{i+1} \oplus Y_i \oplus X_{i-1} \xrightarrow{(d^{N-1}, d^{N-2}, \ldots, d, g_{i-1})} Y_{i-1}$$

vanishes. This defines for us the maps g_{i-1}, h_{i-1} and d^Y_i in the diagram

$$\begin{tikzcd}
X_{i+1} \arrow[r] \arrow[d] & X_i \arrow[r, "d_i^X"] \arrow[d, "g_i"] & X_{i-1} \arrow[d, "g_{i-1}"] \\
Y_{i+1} \arrow[r] \arrow[d] & Y_i \arrow[r, "d_i^Y"] \arrow[d, "h_i"] & Y_{i-1} \arrow[d, "h_{i-1}"] \\
Z_{i+1} \arrow[r] & Z_i \arrow[r, "d_i^Z"] & Z_{i-1}.
\end{tikzcd}$$

Then the above diagram is commutative, $h_{i-1}g_{i-1} = f_{i-1}$ and Y is an N-complex. □

Lemma 5.2. *Let $\mathscr{S}$ be the full subcategory of $\mathrm{Prj}(R)$ consisting of free modules of finite rank. Then each $X \in \mathcal{K}_N(\mathrm{Prj}R)$ is filtered by bounded above N-complexes in $\mathscr{S}$.*

Proof. Let $\mathscr{U}$ be the full subcategory of $\mathcal{K}_N(\mathrm{Prj}R)$ consisting of bounded above N-complexes in $\mathscr{S}$. Given $X \in \mathcal{K}_N(\mathrm{Prj}R)$, we fix for each component X_n an $\mathscr{S}$-filtration. We will construct by induction a $\mathscr{U}$-filtration $(X^\alpha \,|\, \alpha \leqslant \sigma)$ of X such that for each $\alpha < \sigma$

(1) $X^{\alpha+1}/X^\alpha$ is bounded above,

(2) $X_n^\alpha \in \mathrm{Prj}(R)$ and $X_n^{\alpha+1}/X_n^\alpha \in \mathscr{S}$ for all n.

To construct the filtration, we put $X^0 = 0$ and $X^\alpha = \bigcup_{\gamma<\alpha} X^\gamma$ for limit ordinals $\alpha \leqslant \sigma$. For non-limit steps, assume we have constructed $X^\alpha \subsetneqq X$ and we take an integer n and a submodule $W \subseteq X_n$ with $|W| < \aleph_0$ such that $W \nsubseteq X_n^\alpha$. Then we put $X_m^{\alpha+1} = X_m^\alpha$ for $m > n$. Note that $(W + X_n^\alpha)/X_n^\alpha \subseteq X_n/X_n^\alpha$ and $|(W + X_n^\alpha)/X_n^\alpha| < \aleph_0$, it follows from [13, Corollary 2.7] that there is $X_n^{\alpha+1} \in \mathrm{Prj}(R)$ such that $W + X_n^\alpha \subseteq X_n^{\alpha+1}$ and $|X_n^{\alpha+1}/X_n^\alpha| < \aleph_0$. Further note that, up to isomorphism, $X_n^{\alpha+1}/X_n^\alpha \in \mathscr{S}$. For $m < n$ we proceed by induction. Suppose we have already constructed $X_{m+1}^{\alpha+1}$ such that $X_{m+1}^{\alpha+1}/X_{m+1}^\alpha \in \mathscr{S}$ up to isomorphism. Then there exists a submodule $W' \subseteq X_{m+1}^{\alpha+1}$ with $|W'| < \aleph_0$ such that $X_{m+1}^{\alpha+1} = X_{m+1}^\alpha + W'$. But $|d_{m+1}^X(W')| < \aleph_0$, we can again use [13, Corollary 2.7] to find $X_m^{\alpha+1} \in \mathrm{Prj}(R)$ such that $X_m^\alpha + d_{m+1}^X(W') \subseteq X_m^{\alpha+1}$ and $X_m^{\alpha+1}/X_m^\alpha$ is isomorphic to a module from $\mathscr{S}$. This finishes the induction. It is easy to check that $X^{\alpha+1} \subseteq X$ is a subobject and $X^{\alpha+1}/X^\alpha$ is isomorphic to an object of $\mathscr{U}$. This shows our claim. □

Lemma 5.3. *Suppose that Z is an N-complex of flat R-modules. Then Z is pure acyclic if and only if $Z \in \mathcal{K}_N(\mathrm{Prj}R)^\perp \subseteq \mathcal{K}_N(\mathrm{Flat}R)$.*

Proof. "If" part. Let X be a pure projective R-module. Then by [16, Lemma 4.4], $0 = \mathrm{Hom}_{\mathcal{K}_N(R)}(\mathrm{D}_n^t(X), Z) \cong \mathrm{H}_n^t(\mathrm{Hom}_R(X, Z))$ for all n and t. Therefore $\mathrm{Hom}_R(X, Z)$ is acyclic and hence Z is pure acyclic.

"Only if" part. Let Z be a pure acyclic N-complex in $\mathcal{C}_N(\mathrm{Flat}R)$. Consider the full subcategory $\mathscr{R} \subseteq \mathcal{K}_N(\mathrm{Flat}R)$ defined by

$$\mathrm{ob}(\mathscr{R}) = \{X \in \mathcal{K}_N(\mathrm{Prj}R) \,|\, \mathrm{Hom}_{\mathcal{K}_N(R)}(\Sigma^n X, Z) = 0, \ \forall\, n \geqslant 0\}.$$

If $X \in \mathrm{ob}(\mathscr{R})$, then clearly $\Sigma X \in \mathrm{ob}(\mathscr{R})$. Next we show $\Omega X \in \mathscr{R}$. Given a chain map u in $\mathrm{Hom}_{\mathcal{K}_N(R)}(\Sigma^n(\Omega X), Z) \cong \mathrm{Hom}_{\mathcal{K}_N(R)}(\Omega X, \Omega^n Z)$. If $n = 0$, then $\mathrm{Hom}_{\mathcal{K}_N(R)}(\Omega X, Z) \cong \mathrm{Hom}_{\mathcal{K}_N(R)}(\Sigma X, Z) = 0$ since $\phi : \Omega X \to \Sigma X$ with $\phi_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ d & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d^{N-2} & d^{N-3} & d^{N-4} & \cdots & d & 1 \end{pmatrix} : (\Omega X)_n \to (\Sigma X)_{n+N}$ is an isomorphism. Assume $n \geqslant 1$. We have a canonical exact sequence $0 \to \Omega^{n+1} Z \to F_u \to \Omega X \to 0$. Then by Lemma 2.2, $\mathrm{Ext}^1_{\mathcal{C}_N(R)}(X, \Omega^n Z) \cong \mathrm{Ext}^1_{\mathcal{C}_N(R),c.s.}(X, \Omega^n Z) \cong \mathrm{Hom}_{\mathcal{K}_N(R)}(\Sigma^{n-1} X, Z) = 0$, it follows that $\mathrm{Ext}^1_{\mathcal{C}_N(R)}(\Omega X, \Omega^{n+1} Z) = 0$, and hence $u \sim 0$ by [15, Proposition 2.14]. Thus the category $\mathscr{R}$ is a pretriangulated subcategory, and is closed in $\mathcal{K}_N(\mathrm{Prj}R)$ under coproducts. We wish to show that $\mathcal{K}_N(\mathrm{Prj}R) \subseteq \mathscr{R}$. By Lemma 5.2 it suffices to prove that $\mathscr{U} \subseteq \mathscr{R}$, where $\mathscr{U}$ is the set of bounded above N-complexes with components finitely generated projective.

For any object $X \in \mathscr{U}$ and any morphism $u : X \to Z$, we want to produce a null homotopy $\{s_n : X_n \to Z_{n+N-1}\}$ by induction on n. Since $X \in \mathscr{U}$, we know that $X_i = 0$ for $i \gg 0$. We may choose an integer j so that $X_i = 0$ and define $s_i = 0$ for $i \geqslant j$. Suppose s_i has been defined for $i \geqslant k$. Suppose further that $u_{i+N-2} = d^{N-1} s_{i+N-2} + d^{N-2} s_{i+N-3} d + \cdots + s_{i-1} d^{N-1}$ for $i > k$. If $X_{i+N-2} = 0$ for $i \geqslant k$, then this identity is automatic. We will show that we can extend the homotopy. Precisely, we can define $r_i : X_i \to Z_{i+N-1}$ for all $i \geqslant k-1$, so that

(i) $r_i = s_i$ for all $i \geqslant k + N - 1$.

(ii) $u_{i+N-2} = d^{N-1} r_{i+N-2} + d^{N-2} r_{i+N-3} d + \cdots + r_{i-1} d^{N-1}$ for all $i > k-1$.

Note that we cannot guarantee that $s_{k+t} = r_{k+t}$ for $t = 0, 1, \ldots, N-2$. We need to modify s_k to extend the homotopy. But for $i > k + N - 2$ the s_i are stable. It remains to prove the induction step. Suppose that for some k, we have defined s_i for all $i \geqslant k$, in such a way that

$$u_{i+N-2} = d^{N-1} s_{i+N-2} + d^{N-2} s_{i+N-3} d + \cdots + s_{i-1} d^{N-1}, \ \forall\ i > k.$$

Extend s to all i by defining $s_i = 0$ for all $i < k$. Let $\bar{u} : X \to Z$ be the chain map given by $\bar{u}_i = u_i - d^{N-1} s_i - d^{N-2} s_{i-1} - \cdots - s_{i-N+1} d^{N-1}$. We have a chain map $\bar{u} : X \to Z$. In the induction step we replace u_i by $\bar{u}_i$, and r_i by $\bar{r}_i = r_i - s_i$. The induction assertion becomes that there exist maps $\bar{r}_i : X_i \to Z_{i+N-1}$ for all $i \geqslant k-1$, so that

(i′) $\bar{r}_i = 0$ for all $i \geqslant k + N - 1$.

(ii′) $\bar{u}_{i+N-2} = d^{N-1} \bar{r}_{i+N-2} + d^{N-2} \bar{r}_{i+N-3} d + \cdots + \bar{r}_{i-1} d^{N-1}$ for all $i > k-1$.

Now observe that $\bar{u} : X \to Z$ is a chain map from an object X in $\mathscr{U}$. Furthermore $\bar{u}_i = 0$ for all $i > k + N - 2$. By Lemma 5.1, we can factor $\bar{u}$ as $X \xrightarrow{v} Y \xrightarrow{w} Z$, where $Y \in \mathscr{U}$ and $Y_i = 0$ for $i > k + N - 2$. The assertions (i′) and (ii′) above would immediately follow if we could produce maps $\tilde{r}_i : Y_i \to Z_{i+N-1}$ for all $i \geqslant k-1$, so that

(i″) $\tilde{r}_i = 0$ for all $i \geqslant k + N - 1$.

(ii$''$) $w_{i+N-2} = d^{N-1}\tilde{r}_{i+N-2} + d^{N-2}\tilde{r}_{i+N-3}d + \cdots + \tilde{r}_{i-1}d^{N-1}$ for all $i > k-1$. Next note that $Y_i = 0$ for $i > k+N-2$, which means that (i$''$) has no choice but to be true. And furthermore the equality in (ii$''$) is also immediate for $i > k$. There is only one map from $Y_{i+N-2} = 0$ to Z_{i+N-2}. Therefore (i$''$) and (ii$''$) come down to showing that there exist maps $\tilde{r}_{k+N-2}, \ldots, \tilde{r}_k, \tilde{r}_{k-1}$, satisfying the single identity $w_{k+N-2} = d^{N-1}\tilde{r}_{k+N-2} + d^{N-2}\tilde{r}_{k+N-3}d + \cdots + \tilde{r}_{k-1}d^{N-1}$. Next we prove the existence of the maps $\tilde{r}_{k+N-2}, \ldots, \tilde{r}_k, \tilde{r}_{k-1}$. Consider the chain map $w : Y \to Z$:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & Y_{k+N-2} \arrow[r, "d^Y_{k+N-2}"] \arrow[d, "w_{k+N-2}"] & Y_{k+N-3} \arrow[r] \arrow[d, "w_{k+N-3}"] & \cdots \arrow[r] & Y_k \arrow[r, "d^Y_k"] \arrow[d, "w_k"] & Y_{k-1} \arrow[r] \arrow[d, "w_{k-1}"] & \cdots \\
\cdots \arrow[r] & Z_{k+N-1} \arrow[r, "d^Z_{k+N-1}"] & Z_{k+N-2} \arrow[r, "d^Z_{k+N-2}"] & Z_{k+N-3} \arrow[r] & \cdots \arrow[r] & Z_k \arrow[r, "d^Z_k"] & Z_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

We have the following commutative diagram

$$\begin{tikzcd}
Y_{k+N-2} \arrow[r, "d^Y"] \arrow[d, "w_{k+N-2}"] & Y_{k+N-3} \arrow[r] \arrow[d, "w_{k+N-3}"] & \cdots \arrow[r] & Y_k \arrow[r, "d^Y"] \arrow[d, "w_k"] & Y_{k-1} \arrow[d, "w_{k-1}"] & & & & \\
Z_{k+N-2} \arrow[r, "d^Z"] & Z_{k+N-3} \arrow[r] & \cdots \arrow[r] & Z_k \arrow[r, "d^Z"] & Z_{k-1} \arrow[r, "d^Z"] & \mathrm{B}^1_{k-2}(Z) \arrow[r, "d^Z"] & \mathrm{B}^2_{k-3}(Z) \arrow[r] & \cdots \arrow[r] & \mathrm{B}^{N-1}_{k-N}(Z) \subset Z_{k-N}.
\end{tikzcd}$$

This yields a vanishing composite $Y_{k+N-2} \to Y_{k-1} \to \mathrm{B}^1_{k-2}(Z)$ with both Y_{k+N-2}, Y_{k-1} finitely generated and projective and $\mathrm{B}^1_{k-2}(Z)$ flat. It allows us to factor the map $Y_{k-1} \to \mathrm{B}^1_{k-2}(Z)$ as $Y_{k-1} \to Y_{k-2} \to \mathrm{B}^1_{k-2}(Z)$ with Y_{k-2} finitely generated and projective by [11, Corollary 3.3], and in such a way that the composite $Y_{k+N-2} \to Y_{k-1} \to Y_{k-2}$ vanishes. We also have a vanishing composite $Y_{k+N-3} \to Y_{k-2} \to \mathrm{B}^2_{k-3}(Z)$ with both Y_{k+N-3}, Y_{k-2} finitely generated and projective and $\mathrm{B}^2_{k-3}(Z)$ flat. It allows us to factor the map $Y_{k-2} \to \mathrm{B}^2_{k-3}(Z)$ as $Y_{k-2} \to Y_{k-3} \to \mathrm{B}^2_{k-3}(Z)$ with Y_{k-3} finitely generated and projective by [11, Corollary 3.3], and in such a way that the composite $Y_{k+N-3} \to Y_{k-2} \to Y_{k-3}$ vanishes. Continuing this process, we have a vanishing composite $Y_k \to Y_{k-N+1} \to \mathrm{B}^{N-1}_{k-N}(Z)$ with both Y_k, Y_{k-N+1} finitely generated and projective and $\mathrm{B}^{N-1}_{k-N}(Z)$ flat. It allows us to factor the map $Y_{k-N+1} \to \mathrm{B}^{N-1}_{k-N}(Z)$ as $Y_{k-N+1} \to Y_{k-N} \to \mathrm{B}^{N-1}_{k-N}(Z)$ with Y_{k-N} finitely generated and projective by [11, Corollary 3.3], and in such a way that the composite $Y_k \to Y_{k-N+1} \to Y_{k-N}$ vanishes. In other words we can form a commutative diagram

$$\begin{tikzcd}
Y_{k+N-2} \arrow[r, "d^Y"] \arrow[d, "w_{k+N-2}"] & \cdots \arrow[r] & Y_k \arrow[r, "d^Y"] \arrow[d, "w_k"] & Y_{k-1} \arrow[r] \arrow[d, "w_{k-1}"] & Y_{k-2} \arrow[r] \arrow[d, "\bar{w}_{k-2}"] & Y_{k-3} \arrow[r] \arrow[d, "\bar{w}_{k-3}"] & \cdots \arrow[r] & Y_{k-N} \arrow[r] \arrow[d, "\bar{w}_{k-N}"] & 0 \arrow[d] \\
Z_{k+N-2} \arrow[r, "d^Z"] & \cdots \arrow[r] & Z_k \arrow[r, "d^Z"] & Z_{k-1} \arrow[r] & \mathrm{B}^1_{k-2}(Z) \arrow[r, "d^Z"] & \mathrm{B}^2_{k-3}(Z) \arrow[r] & \cdots \arrow[r] & \mathrm{B}^{N-1}_{k-N}(Z) \arrow[r] & Z_{k-N-1},
\end{tikzcd}$$

where the horizontal any N-consecutive composites vanish. We deduce a chain map of N-complexes

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & Y_{k+N-2} \arrow[r, "d^Y"] \arrow[d, "w_{k+N-2}"] & \cdots \arrow[r] & Y_k \arrow[r, "d^Y"] \arrow[d, "w_k"] & Y_{k-1} \arrow[r] \arrow[d, "w_{k-1}"] & Y_{k-2} \arrow[r] \arrow[d, "w_{k-2}"] & Y_{k-3} \arrow[r] \arrow[d, "w_{k-3}"] & \cdots \arrow[r] & Y_{k-N} \arrow[r] \arrow[d, "w_{k-N}"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Z_{k+N-1} \arrow[r, "d^Z"] & Z_{k+N-2} \arrow[r, "d^Z"] & \cdots \arrow[r] & Z_k \arrow[r, "d^Z"] & Z_{k-1} \arrow[r] & Z_{k-2} \arrow[r, "d^Z"] & Z_{k-3} \arrow[r] & \cdots \arrow[r] & Z_{k-N} \arrow[r] & Z_{k-N-1} \arrow[r] & \cdots .
\end{tikzcd}$$

Then [16, Lemma 5.2] implies that this chain map is null homotopic. Therefore we have defined $\tilde{r}_{k+N-2}, \dots, \tilde{r}_k, \tilde{r}_{k-1}$, such that

$$w_{k+N-2} = d^{N-1}\tilde{r}_{k+N-2} + d^{N-2}\tilde{r}_{k+N-3}d + \cdots + \tilde{r}_{k-1}d^{N-1}.$$

This completes the induction. □

The following result was proved by Neeman when $N = 2$ (see [11, Proposition 7.14]).

Theorem 5.4. *Let R be right coherent. Then the homotopy category $\mathcal{K}_N(\mathrm{Prj}R)$ is compactly generated.*

Proof. Let A be a finitely presented R-module. Then [16, Proposition 3.4] yields a quasi-isomorphism $\mathrm{D}_n^t(A) \to p_n^t A$ with $p_n^t A$ a bounded below N-complex in $\mathrm{Prj}(R)$ for all n, t, and each N-complex $p_n^t A$ is a compact object in $\mathcal{K}_N(\mathrm{Prj}R)$ by [16, Lemma 5.5]. It remains to show that $\mathscr{G} = \{p_n^t A \mid A \in R\text{-Mod}$ is finitely presented$\}$ is a set of generators. Suppose that X in $\mathcal{K}_N(\mathrm{Prj}R)$ has $\mathrm{Hom}_{\mathcal{K}_N(\mathrm{Prj}R)}(G, X) = 0$ for every $G \in \mathscr{G}$. We show $X \cong 0$ in $\mathcal{K}_N(\mathrm{Prj}R)$.

First $0 = \mathrm{Hom}_{\mathcal{K}_N(\mathrm{Prj}R)}(p_n^t R, X) \cong \mathrm{H}_n^t(\mathrm{Hom}_R(R, X)) \cong \mathrm{H}_n^t(X)$ for all n and t. So X is acyclic. Again by [16, Lemma 5.5], for any finitely presented left R-module A,

$$0 = \mathrm{Hom}_{\mathcal{K}_N(\mathrm{Prj}R)}(p_n^t A, X) \cong \mathrm{H}_n^t(\mathrm{Hom}_R(A, X)),\ \forall\ n, t.$$

Then [4, Theorem 6.4] implies that $0 \to \mathrm{Z}_n^t(X) \to X_n \to \mathrm{Z}_{n-t}^{N-t}(X) \to 0$ is pure and $\mathrm{Z}_n^t(X)$ is flat for all n, t. Thus $\mathrm{Z}_n^t(X)$ is projective for all n, t by Lemma 5.3. This implies that X is a projective N-complex by the dual of [1, Theorem 4.5], and hence $X \cong 0$ in $\mathcal{K}_N(\mathrm{Prj}R)$, as desired. □

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