

## IDEAL RIGHT-ANGLED PENTAGONS IN HYPERBOLIC 4-SPACE

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ABSTRACT. An ideal right-angled pentagon in hyperbolic 4-space  $\mathbb{H}^4$  is a sequence of oriented geodesics  $(L_1, \dots, L_5)$  such that  $L_i$  intersects  $L_{i+1}$ ,  $i = 1, \dots, 4$ , perpendicularly in  $\mathbb{H}^4$  and the initial point of  $L_1$  coincides with the endpoint of  $L_5$  in the boundary at infinity  $\partial\mathbb{H}^4$ . We study the geometry of such pentagons and the various possible augmentations and prove identities for the associated quaternion half side lengths as well as other geometrically defined invariants of the configurations. As applications we look at two-generator groups  $\langle A, B \rangle$  of isometries acting on hyperbolic 4-space such that  $A$  is parabolic, while  $B$  and  $AB$  are loxodromic.

### 1. Introduction

The main purpose of this paper is to investigate configurations of ideal right-angled pentagons and their applications to two-generator groups of isometries acting on hyperbolic 4-space. We define an *ideal right-angled pentagon* in hyperbolic 4-space  $\mathbb{H}^4$  to be a sequence of oriented geodesics  $(L_1, \dots, L_5)$  such that  $L_i$  intersects  $L_{i+1}$ ,  $i = 1, \dots, 4$ , perpendicularly in  $\mathbb{H}^4$  and the initial point of  $L_1$  coincides with the endpoint of  $L_5$  in the boundary at infinity  $\partial\mathbb{H}^4$ , see Figure 1. It can be viewed as a degeneration of a right-angled hexagon, namely, when one of the sides has (real) length approaching zero, while the adjacent sides have (real) lengths approaching infinity.

The geometry of right-angled hexagons is one of the fundamental elements in the study of hyperbolic geometry. For example, in dimension 2, it is used to study the discreteness of two-generator groups of isometries [4], the collar lemma [5, 12] and the Fenchel-Nielsen coordinates of Teichmüller space [10]. We can also see the applications and generalizations of the geometry of right-angled hexagons in hyperbolic 3-space [19, 20] and complex hyperbolic space

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[11, 15, 17]. An ideal right-angled pentagon naturally appears in the process of deforming a two-generator group or a geodesic on a surface.

As a generalization of such configurations in hyperbolic 2 and 3-space, right-angled hexagons in hyperbolic 4-space were studied by Tan, Wong and Zhang in [21]. In particular, they showed how to augment the alternate sides of the hexagon with geodesic planes which were perpendicular to the incident sides, and in this way, defined associated quaternion half lengths for these augmented hexagons. They then proved generalizations of the Delambre-Gauss formulas for these half lengths. In this paper, we augment an ideal right-angled pentagon  $(L_1, \dots, L_5)$  in two different ways and prove various formulas of the quaternion half lengths associated to the augmented ideal right-angled pentagons.

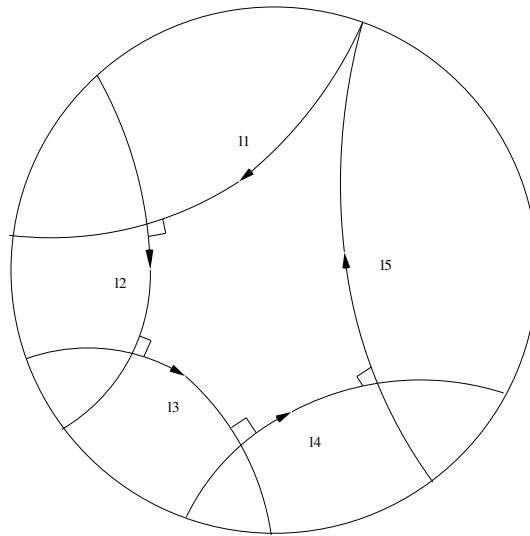


FIGURE 1. An ideal right-angled pentagon

Throughout the paper, we assume that all maps and isometries are orientation-preserving unless otherwise noted. We shall adopt the convention that all lines and all planes in hyperbolic 4-space are oriented geodesic lines and oriented geodesic 2-dimensional planes and they are denoted by  $L$  and  $\Pi$  respectively.

Firstly, we augment the pentagon  $(L_1, \dots, L_5)$  with two planes  $\Pi_2 \supset L_2$  and  $\Pi_4 \supset L_4$  such that  $\Pi_2$  is perpendicular to  $L_1$  and  $L_3$ , and  $\Pi_4$  is perpendicular to  $L_3$  and  $L_5$  (see Figure 3). In this way, we can define complex half lengths  $\delta_2$  and  $\delta_4$  for  $L_2$  and  $L_4$  respectively, and a quaternion half length  $\delta_3$  for  $L_3$ . Lemma 3.1 gives the relation between  $\delta_2$ ,  $\delta_3$  and  $\delta_4$ .

The lines  $L_1$  and  $L_5$  have infinite length, but we can still extract useful geometric invariants by introducing an oriented horocycle  $C_0$ , intersecting with and perpendicular to  $L_1$  and  $L_5$ , and with orientation from  $L_5$  to  $L_1$ . We can

make  $C_0$  unique by requiring it to have Euclidean length 1 between  $L_1$  and  $L_5$  (see Figure 2). We can also augment  $C_0$  with an oriented 2-dimensional horosphere  $S_0 \supset C_0$  which is perpendicular to both  $L_1$  and  $L_5$ . The choice for  $S_0$  is not canonical, but there is a one-dimensional family of choices.

Let  $p_{01}$  (resp.  $p_{50}$ ) be the point of intersection between  $C_0$  and  $L_1$  (resp.  $C_0$  and  $L_5$ ). For the chosen  $S_0$ , we take a geodesic  $L_0$  (resp.  $L_6$ ) which is tangent to  $C_0$  at  $p_{01}$  (resp.  $p_{50}$ ) and a plane  $\Pi_0$  (resp.  $\Pi_6$ ) which is tangent to  $S_0$  and contains  $L_0$  (resp.  $L_6$ ). In this way, we can choose canonically augmented pairs  $(L_0, \Pi_0)$  and  $(L_6, \Pi_6)$  although the choice of  $S_0$  is not canonical so that we may define quaternion half lengths  $\delta_1$  and  $\delta_5$  for the sides  $L_1$  and  $L_5$  (See Figure 4).

This is similar to the lambda length construction of Penner in the two dimensional case. Theorem 3.2 gives the formulas of the half lengths  $\delta_i, i = 1, \dots, 5$ . We will see in Remark 3.3 that Equation (3) gives us the same pentagon formula (43) in hyperbolic 2-space if the initial ideal right-angled pentagon  $(L_1, \dots, L_5)$  is embedded in a 2-dimensional geodesic plane of  $\mathbb{H}^4$ . Thus, this is a direct generalization into  $\mathbb{H}^4$  of the pentagon formula in hyperbolic 2-space.

**Theorem 3.2.** *For an augmented ideal right-angled pentagon  $(L_1, \Pi_2, L_3, \Pi_4, L_5)$ , the horocycle/horosphere pair  $C_0 \subset S_0$  assigned to the ideal vertex, and canonically chosen pairs  $(L_0, \Pi_0)$  and  $(L_6, \Pi_6)$ , let  $\delta_i, i = 1, \dots, 5$  be the associated quaternion or complex half lengths of the augmented pentagon. Then we have the following formulas:*

$$\begin{aligned}
 (1) \quad & \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm e_2; \\
 (2) \quad & \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \exp \delta_5^* \\
 & = \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4; \\
 (3) \quad & \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \\
 & \quad \times \left( \cosh \delta_4^* \exp \delta_3^* \sinh \delta_2^* + \sinh \delta_4^* \exp(-\delta_3) \cosh \delta_2^* \right) = -1; \\
 (4) \quad & \sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 = 0.
 \end{aligned}$$

On the other hand, we may also augment the sides  $L_1, L_3$  and  $L_5$  by three planes  $\Pi_1 \supset L_1, \Pi_3 \supset L_3$  and  $\Pi_5 \supset L_5$  (see Figure 7). The plane  $\Pi_3$  is chosen to be perpendicular to  $L_2$  and  $L_4$ , and is generically unique. A priori,  $\Pi_1$  and  $\Pi_5$  are only required to be perpendicular to  $L_2$  and  $L_4$  respectively, so generically, there is a one-dimensional family of choices for  $\Pi_1$  and  $\Pi_5$  respectively. This augmentation now gives quaternion half lengths  $\delta_2$  and  $\delta_4$  for  $L_2$  and  $L_4$  respectively and a complex half length  $\delta_3$  for  $L_3$ . Lemma 3.4 is about the relation between these lengths.

For the purpose of applications to two-generator groups of isometries acting on hyperbolic 4-space, we want to consider all possible choices for  $\Pi_1$  and  $\Pi_5$ . On the other hand, from a geometric point of view, there are more canonical choices which would pin down  $\Pi_1$  and  $\Pi_5$  precisely. We may use the horocycle  $C_0$  defined in the earlier augmentation to do this. Namely, we now require  $\Pi_1$  and  $\Pi_5$  to be perpendicular to  $C_0$  as well as  $L_2$  and  $L_4$  respectively (see Figure 8). Generically, this extra condition determines  $\Pi_1$  and  $\Pi_5$  and we now have complex half lengths  $\delta_1$  and  $\delta_5$  for  $L_1$  and  $L_5$  respectively. We also have an angle invariant  $\theta_0$  associated to  $C_0$  as generically, the isometry leaving  $C_0$  invariant and mapping the pair  $(L_5, \Pi_5)$  to  $(L_1, \Pi_1)$  is a composition of a parabolic isometry with the half rotation angle  $\theta_0$  and an involution. Here, the involution is needed only because of the orientation of  $L_1$  and  $L_5$ . Theorem 3.5 gives the formulas of the half lengths  $\delta_i, i = 1, \dots, 5$  and  $\theta_0$ .

**Theorem 3.5.** *Let  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  be an augmented ideal right-angled pentagon in  $\mathbb{H}^4$  by the horocycle  $C_0$  assigned to the ideal vertex and planes  $\Pi_1 \supset L_1, \Pi_3 \supset L_3$  and  $\Pi_5 \supset L_5$  as above. Let  $\delta_i, i = 1, \dots, 5$  be the associated quaternion half lengths of the augmented pentagon and  $\theta_0$  the half rotation angle associated to  $C_0$ . Then we have the following formulas:*

$$(5) \quad \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm \alpha e_2;$$

$$(6) \quad \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \exp \delta_5^* \\ = \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4;$$

$$(7) \quad \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \\ \times \left( \cosh \delta_4^* \exp \delta_3^* \sinh \delta_2^* + \sinh \delta_4^* \exp(-\delta_3) \cosh \delta_2^* \right) = -1;$$

$$(8) \quad \sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 = 0,$$

where  $\alpha = \cos \theta_0 + \sin \theta_0 e_1 e_2$ ,

If the half rotation angle  $\theta_0$  is 0, Equation (5) is the same as Equation (1). We also note that Equations (6), (7) and (8) are the same as (2), (3) and (4), respectively. This means that the formulas of side lengths of an ideal right-angled pentagon we induced do not depend on how we augment the given pentagon. Therefore, we call Equation (3) (or Equation (7)) a *generalized pentagon formula in hyperbolic 4-space*.

As mentioned earlier, for the purposes of applications to linked two-generator groups of isometries acting on hyperbolic 4-space, we want generic planes  $\Pi_1 \supset L_1$  and  $\Pi_5 \supset L_5$ . We can see this as follows. Let  $\text{Isom}(\mathbb{H}^n)$  be the group of orientation-preserving isometries acting on hyperbolic  $n$ -space. Let  $\langle A, B, C \mid ABC = I \rangle$  be a linked two-generator subgroup of  $\text{Isom}(\mathbb{H}^4)$  such

that  $A$  is parabolic and  $B, C$  are loxodromic. By the definition of *linked*, this means that  $A = PQ, B = QR$  and  $C = RP$  where  $P, Q, R$  are half turns around planes, say,  $\Pi_5, \Pi_1$  and  $\Pi_3$  respectively. We note that for a plane  $\Pi \subset \mathbb{H}^4$ , a half turn around  $\Pi$  is a  $\pi$ -rotation around  $\Pi$  in this paper. Since  $A = PQ$  is parabolic,  $\overline{\Pi}_5 \cap \overline{\Pi}_1$  is a point  $p_\infty \in \partial\mathbb{H}^4$  (where  $\overline{\Pi}$  denotes  $\Pi \cup \partial\Pi \subset \mathbb{H}^4 \cup \partial\mathbb{H}^4$ ). Since  $B$  and  $C$  are loxodromic,  $\overline{\Pi}_1 \cap \overline{\Pi}_3 = \overline{\Pi}_3 \cap \overline{\Pi}_5 = \emptyset$ . Denote by  $L_2$  (respectively  $L_4$ ) the unique line perpendicular to  $\Pi_1$  and  $\Pi_3$  (respectively  $\Pi_3$  and  $\Pi_5$ ) and denote by  $p_{ij}$  the point of intersection of  $L_i$  with  $\Pi_j, |j - i| = 1$ . By construction, the line  $L_3$  passing through  $p_{23}$  and  $p_{34}$  lies in  $\Pi_3$  and is perpendicular to  $L_2$  and  $L_4$ . The line  $L_1$  from  $p_\infty$  through  $p_{12}$  lies in  $\Pi_1$  and is perpendicular to  $L_2$ , similarly, the line  $L_5$  through  $p_{45}$  to  $p_\infty$  lies in  $\Pi_5$  and is perpendicular to  $L_4$ . We hence recover the ideal right-angled pentagon  $(L_1, \dots, L_5)$  and the second augmentation by  $\Pi_1, \Pi_3, \Pi_5$ . Conversely, half turns around  $\Pi_1, \Pi_3$  and  $\Pi_5$  respectively coming from the augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  generates a linked two-generator subgroup of  $\text{Isom}(\mathbb{H}^4)$  with a parabolic isometry  $A$ . Note that while  $\Pi_3$  is generically well defined,  $\Pi_1$  and  $\Pi_5$  are not. Lemma 4.1 gives us the relations for this general configuration.

If the associated ideal right-angled pentagon  $(L_1, \dots, L_5)$  is embedded in a 2-dimensional plane of  $\mathbb{H}^4$ , the group generated by the three half turns around  $\Pi_1, \Pi_3$  and  $\Pi_5$  respectively is conjugate to a Fuchsian group by an isometry of  $\mathbb{H}^4$ . A deformation of the group in  $\text{Isom}(\mathbb{H}^4)$  is said to be *type-preserving* if  $A$  remains parabolic and  $B, C$  loxodromic. We then have the following result.

**Theorem 4.3.** *Let  $\Gamma = \langle A, B, C \mid ABC = 1 \rangle$  be a discrete two-generator subgroup of  $\text{Isom}(\mathbb{H}^2)$  where  $A$  is parabolic and  $B, C$  are loxodromic. Then there exists a 6-dimensional parameter space  $\mathcal{P}$  containing the identity representation in the deformation space  $\mathcal{D}(\Gamma)$  which is the set of discrete, faithful and type-preserving representations of  $\Gamma$  into  $\text{Isom}(\mathbb{H}^4)$  up to the conjugation action of  $\text{Isom}(\mathbb{H}^4)$ .*

## 2. Preliminaries and notations

In this section we will give definitions and basic facts of hyperbolic 4-space  $\mathbb{H}^4$ , Vahlen matrices and geometric configurations. For the basics on hyperbolic geometry, the reader is referred to [7, 8, 16, 18]; for Vahlen matrices, to [1–3, 9, 13, 14, 21–23]. We will follow the notation and terminology of [21] closely as many of the basic technical details for our results are worked out there.

### 2.1. Hyperbolic 4-space and Vahlen matrices

We first describe the setting for hyperbolic 4-space  $\mathbb{H}^4$  and its boundary at infinity  $\partial\mathbb{H}^4$ . An isometry of hyperbolic space can be represented as a  $2 \times 2$  matrix whose entries are the Clifford numbers satisfying some conditions. The action of the  $2 \times 2$  matrix is the usual action of Möbius transformations. This is a natural generalization of the classical settings,  $\text{PSL}(2, \mathbb{R})$  and  $\text{PSL}(2, \mathbb{C})$ , via

identifying the real numbers  $\mathbb{R}$  with the Clifford algebra  $\mathbb{A}_0$  and the complex numbers  $\mathbb{C}$  with the Clifford algebra  $\mathbb{A}_1$ .

The Clifford algebra  $\mathbb{A}_n$  is the associative algebra over the real numbers generated by the elements  $e_1, e_2, \dots, e_n$  subject to the relations  $e_i^2 = -1$  for all  $i = 1, \dots, n$  and  $e_i e_j = -e_j e_i$  for  $i \neq j$ . The Clifford algebra  $\mathbb{A}_2$  is isomorphic to the quaternions as an algebra, where we identify  $e_1, e_2$  and  $e_1 e_2$  with  $i, j$  and  $k$  respectively. This identification however does not take into account the grading of  $\mathbb{A}_2$ .

As a graded real algebra with  $e_1, \dots, e_n$  all having degree 1,  $\mathbb{A}_n$  decomposes as the direct sum of its degree  $p$  vector subspaces  $\mathbb{A}_n^{(p)}$ ,  $p = 0, 1, \dots, n$ , with  $\mathbb{A}_n^{(0)} = \mathbb{R}$ , and, every element  $a \in \mathbb{A}_n$  decomposes as  $a = a^{(0)} + a^{(1)} + \dots + a^{(n)}$ , where, for  $p = 0, 1, \dots, n$ ,  $a^{(p)} \in \mathbb{A}_n^{(p)}$  is the degree  $p$  part of  $a$ . We identify the Euclidean space  $\mathbb{R}^{n+1}$  with  $\mathbb{A}_n^{(0)} + \mathbb{A}_n^{(1)}$ , denoted by  $\mathbb{A}_n^{(0,1)}$ , which is a vector space with basis  $\{1, e_1, \dots, e_n\}$ .

We set the upper half-space model of the hyperbolic 4-space as

$$\mathbb{H}^4 = \{x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}, x_3 > 0\} \subset \mathbb{A}_3^{(0,1)}.$$

The boundary at infinity is

$$\partial \mathbb{H}^4 = \hat{\mathbb{R}}^3 = \mathbb{A}_2^{(0,1)} \cup \{\infty\} = \{x_0 + x_1 e_1 + x_2 e_2 \mid x_0, x_1, x_2 \in \mathbb{R}\} \cup \{\infty\}.$$

There are three involutions in the Clifford algebra  $\mathbb{A}_n$ :

- (1) The main involution  $a \mapsto a'$  is obtained by replacing each  $e_i$  with  $-e_i$ . Thus,  $(ab)' = a'b'$  and  $(a + b)' = a' + b'$ .
- (2) The reversion  $a \mapsto a^*$  is obtained by replacing each  $e_{v_1} e_{v_2} \cdots e_{v_p}$  with  $e_{v_p} e_{v_{p-1}} \cdots e_{v_1}$ . Therefore,  $(ab)^* = b^* a^*$  and  $(a + b)^* = a^* + b^*$ .
- (3) The conjugation  $a \mapsto \bar{a}$  is the composition of the main involution and the reversion, i.e.,  $\bar{a} = (a')^* = (a^*)'$ .

Every non-zero  $x \in \mathbb{A}_n^{(0,1)}$  is invertible with its multiplicative inverse  $x^{-1} = \frac{\bar{x}}{|x|^2}$  where  $|x|$  is the Euclidean norm of  $\mathbb{A}_n^{(0,1)} \simeq \mathbb{R}^{n+1}$ . A Clifford group  $\Gamma_n$  is a multiplicative group generated by all non-zero elements of  $\mathbb{A}_n^{(0,1)}$ . Note that  $\Gamma_n = \mathbb{A}_n \setminus \{0\}$  is true for only  $n = 0, 1, 2$ .

**Definition 2.1.** A matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is said to be a *Vahlen matrix* if the following conditions are satisfied:

- (1)  $a, b, c, d \in \Gamma_n \cup \{0\}$ .
- (2)  $ad^* - bc^* = 1$ .
- (3)  $ab^*, cd^*, c^* a, d^* b \in \mathbb{R}^{n+1}$ .

A Vahlen matrix  $A$  has a multiplicative inverse  $A^{-1} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$  which is also a Vahlen matrix. Hence, all Vahlen matrices form a group, denoted by  $SL(\Gamma_n)$ .

A Vahlen matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\Gamma_n)$  induces a Möbius transformation of  $\hat{\mathbb{R}}^{n+1}$  by  $Ax = (ax + b)(cx + d)^{-1}$  for any vector  $x = x_0 + x_1 e_1 + \dots +$

$x_n e_n \in \mathbb{R}^{n+1}$ , and  $\infty \mapsto \infty$  if  $c = 0$  and  $\infty \mapsto ac^{-1}$ ,  $-c^{-1}d \mapsto \infty$  if  $c \neq 0$ . Moreover, any orientation-preserving Möbius transformation of  $\hat{\mathbb{R}}^{n+1}$  can be presented as a Vahlen matrix. Replacing  $x$  with  $x + x_{n+1}e_{n+1} \in \mathbb{R}^{n+2}$ , we can automatically extend the action of  $A$  to a Möbius transformation of  $\hat{\mathbb{R}}^{n+2}$ :  $x + x_{n+1}e_{n+1} \mapsto (a(x + x_{n+1}e_{n+1}) + b)(c(x + x_{n+1}e_{n+1}) + d)^{-1}$ . The coefficient of the last generator  $e_{n+1}$  of the image is  $\frac{x_{n+1}}{|cx+d|^2}$ . This shows that the extension keeps hyperbolic space  $\mathbb{H}^{n+2}$  invariant. In fact, the group of Vahlen matrices modulo  $\pm I$  is isomorphic to the group of orientation-preserving isometries of  $\mathbb{H}^{n+2}$ , denoted by  $\text{Isom}(\mathbb{H}^{n+2})$ .

From now on, we will only consider the case where  $n = 2$  which corresponds to hyperbolic 4-space  $\mathbb{H}^4$ .

**Theorem 2.2** ([9]). (1) An isometry  $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda^{*-1} \end{pmatrix} \in \text{SL}(\Gamma_2)$  is loxodromic if and only if  $|\lambda| \neq 1$ .  
 (2) An isometry  $\begin{pmatrix} \lambda & \mu \\ 0 & \lambda' \end{pmatrix} \in \text{SL}(\Gamma_2)$  with  $|\lambda| = 1$  is  $\begin{cases} \text{strictly parabolic if } \lambda \in \mathbb{R}, \\ \text{screw parabolic if } \mu \notin \mathbb{R}^3, \\ \text{elliptic otherwise.} \end{cases}$

In particular, a parabolic isometry  $P_\theta$  which fixes  $\infty$ , rotates around  $\langle 1 \rangle$ -axis by  $2\theta$  and sends 0 to 1 is of the form  $P_\theta = \begin{pmatrix} \alpha & \alpha \\ 0 & \alpha \end{pmatrix}$  where  $\alpha = \cos \theta + \sin \theta e_1 e_2 \in \mathbb{A}_2$ ,  $0 \leq \theta < \pi$ . We call  $\theta$  the half rotation angle of  $P_\theta$ . Any parabolic isometry can be conjugate to  $P_\theta$  by an isometry.

For a loxodromic isometry  $f \in \text{Isom}(\mathbb{H}^4)$ , we denote the axis of  $f$  by  $\text{Axis}(f)$ , which is the geodesic line connecting the two fixed points.

**2.2. The quaternion exponential and hyperbolic functions**

Following [21], we define the quaternion exponential function  $\exp : \mathbb{A}_2 \rightarrow \mathbb{A}_2$  by

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{A}_2.$$

In general,

(9)  $\exp x \exp(-x) = \exp(-x) \exp x = 1,$

(10)  $\exp(x^*) = (\exp x)^*,$

(11)  $\exp x \exp y \neq \exp y \exp x \neq \exp(x + y).$

The quaternion hyperbolic functions  $\cosh, \sinh : \mathbb{A}_2 \rightarrow \mathbb{A}_2$  are defined by

(12)  $\cosh x = \frac{\exp x + \exp(-x^*)}{2},$

(13)  $\sinh x = \frac{\exp x - \exp(-x^*)}{2}.$

We note that

- (14)  $\cosh(-x) = \cosh x^*$ ,
- (15)  $\sinh(-x) = -\sinh x^*$ ,
- (16)  $\cosh x + \sinh x = \exp x$ ,
- (17)  $\cosh x - \sinh x = \exp(-x^*)$ ,
- (18)  $\cosh x \cosh x^* - \sinh x \sinh x^* = 1$ .

Since the quaternions do not commute in general, we also note that  $\sinh x \neq \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$  and  $\cosh x \neq \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ .

### 2.3. Geometric configurations

Here we consider geometric configurations of lines and planes that we will investigate in what follows.

For distinct points  $u, v \in \partial\mathbb{H}^4$ , let  $L_{[u,v]}$  denote the oriented geodesic from  $u$  to  $v$ . If  $u, v, x, y$  are four distinct points in  $\partial\mathbb{H}^4$  such that  $L_1 := L_{[u,v]}$  and  $L_2 := L_{[x,y]}$  intersect in  $\mathbb{H}^4$ , denote the oriented plane spanned by  $L_1$  and  $L_2$  with orientation determined by the oriented frame  $(L_1, L_2)$  by  $L_1 \vee L_2$  or  $\Pi_{[u,v] \vee [x,y]}$ . In particular,  $L_1 \vee L_2$  and  $L_2 \vee L_1$  are the same planes but with opposite orientations. A point-line-plane flag configuration (a *PLP-configuration* for short) is a triple  $(p, L, \Pi)$  where  $p \in L \subset \Pi \subset \mathbb{H}^4$ .

**Definition 2.3.** An ideal right-angled pentagon in  $\mathbb{H}^4$  is a sequence of lines  $(L_1, \dots, L_5)$  such that  $L_i$  intersects  $L_{i+1}$ ,  $i = 1, \dots, 4$ , perpendicularly in  $\mathbb{H}^4$  and the initial point of  $L_1$  coincides with the endpoint of  $L_5$  in  $\partial\mathbb{H}^4$  (see Figure 1).

To the ideal vertex of an ideal right-angled pentagon  $(L_1, \dots, L_5)$  we assign the *horocycle/horosphere pair*  $(C_0, S_0)$  as follows. Let  $u_i$  and  $v_i$  ( $i = 1, \dots, 5$ ) be the starting and ending points of  $L_i$  in  $\partial\mathbb{H}^4$ , so  $L_i = L_{[u_i, v_i]}$  and  $u_1 = v_5$ . Denote by  $p_{ij}$  the point of intersection in  $\mathbb{H}^4$  between  $L_i$  and  $L_{i+1}$ ,  $1 \leq i \leq 4$ . Let  $C_0$  be the horocycle centered at  $u_1 = v_5$  of height 1 which intersects with  $L_1$  and  $L_5$  respectively. We define  $p_{01}$  (respectively  $p_{50}$ ) to be the point of intersection between  $C_0$  and  $L_1$  (respectively  $L_5$  and  $C_0$ ). Let  $S_0$  be any horosphere centred at  $u_1 = v_5$  containing  $C_0$ . If we normalize  $(L_1, \dots, L_5)$  so that  $L_1 = L_{[\infty, 0]}$  and  $L_5 = L_{[1, \infty]}$  (i.e.,  $u_1 = v_5 = \infty$ ,  $v_1 = 0$  and  $u_5 = 1$ ), then

$$\begin{aligned}
 (19) \quad & C_0 = \{-x + e_3 \mid x \in \mathbb{R}\}, \\
 & p_{01} = e_3, \quad p_{50} = 1 + e_3, \\
 & S_0 = \{x + y(ae_1 + be_2) + e_3 \mid x, y \in \mathbb{R}\},
 \end{aligned}$$

where  $a, b$  are fixed real constants. In particular, we often normalize them further and take  $a = 1$  and  $b = 0$  to define  $S_0$  (see Figure 2).



The horocycle/horosphere pair  $(C_0, S_0)$  assigned to the ideal vertex of  $(L_1, \dots, L_5)$  determines PLP-configurations at  $p_{01}$  and  $p_{50}$  as follows: The PLP-configuration at  $p_{01}$  is  $(p_{01}, L, \Pi)$  where  $L$  is tangent to  $C_0$  at  $p_{01}$  and  $\Pi$  is tangent to  $S_0$  at  $p_{01}$  and similarly for  $p_{50}$ . Again if we normalize as before, and choose  $a = 1$  and  $b = 0$  for  $S_0$ , then the PLP-configurations at  $p_{01}$  and  $p_{50}$  are

$$(20) \quad (e_3, L_{[1,-1]}, \Pi_{[1,-1] \vee [-e_1, e_1]}) \text{ and } (1 + e_3, L_{[2,0]}, \Pi_{[2,0] \vee [1-e_1, 1+e_1]})$$

respectively (see Figure 5). We will measure quaternion half lengths for the infinite lines  $L_1$  and  $L_5$  using these PLP-configurations later.

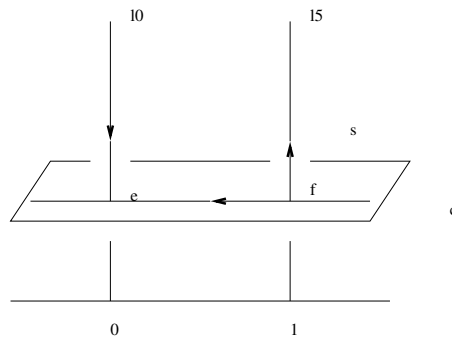


FIGURE 2. The horocycle/horosphere pair  $(C_0, S_0)$  assigned to the ideal vertex

### 2.4. Half side lengths of ideal right-angled pentagons

For a pair of lines  $L_1$  and  $L_2$  which share a common orthogonal line  $L$  in hyperbolic space, we may define a distance from  $L_1$  to  $L_2$  along  $L$  as a translation length of a loxodromic isometry which keeps  $L$  invariant and maps  $L_1$  to  $L_2$ . Since we consider oriented lines, such a loxodromic isometry might not exist in the isometries of hyperbolic 2-space. For example, let  $L_1$  and  $L_2$  be  $L_{[-1,1]}$  and  $L_{[4,-4]}$  respectively. Then the isometry which maps  $L_1$  to  $L_2$  and keeps the common orthogonal geodesic invariant cannot be a loxodromic isometry, but it is an elliptic isometry  $\begin{pmatrix} 0 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} \in \text{PSL}(2, \mathbb{R})$ . However, in contrast to hyperbolic 2-space, a loxodromic isometry  $A = \begin{pmatrix} 2e_1 & 0 \\ 0 & -\frac{1}{2}e_1 \end{pmatrix} \in \text{SL}(\Gamma_2)$  maps  $L_1$  to  $L_2$  and keeps the common orthogonal line  $L_{[0,\infty]}$  invariant in hyperbolic 4-space. Hence, we may define a quaternion distance from  $L_1$  to  $L_2$  along  $L$  as a translation length of  $A$ . Yet, such an isometry might not be unique. In fact, a loxodromic isometry  $B = \begin{pmatrix} 2e_2 & 0 \\ 0 & -\frac{1}{2}e_2 \end{pmatrix} \in \text{SL}(\Gamma_2)$  also maps  $L_1$  to  $L_2$  and keeps  $L_{[0,\infty]}$  invariant. The quaternion translation length of  $B$  is different from  $A$ . It is because a quaternion translation length reflects the rotational action of a loxodromic isometry. Note that  $A$  rotates around the plane  $\langle 1, e_2 \rangle$  and

$B$  around  $\langle 1, e_1 \rangle$ . Thus, we need to use flags to define a quaternion distance between lines.

We define a *flag*  $F = (L, \Pi)$  to be an ordered pair of a line  $L$  and a plane  $\Pi$  such that  $L$  is contained in  $\Pi$ . In particular, we define *the horizontal flag*  $F_h = (L_h, \Pi_h)$  and *the vertical flag*  $F_v = (L_v, \Pi_v)$  where

$$(21) \quad \begin{aligned} L_h &= L_{[-1,1]}, \\ L_v &= L_{[0,\infty]}, \\ \Pi_h &= \Pi_{[-1,1] \vee [-e_1, e_1]}, \\ \Pi_v &= \Pi_{[-e_2, e_2] \vee [0, \infty]}. \end{aligned}$$

The group of isometries  $\text{Isom}(\mathbb{H}^4)$  acts transitively on the set of all flags in  $\mathbb{H}^4$ . That is to say, for a given flag  $F = (L, \Pi)$  there exists an isometry which sends  $L$  to  $L_h$  and  $\Pi$  to  $\Pi_h$ . A line  $L'$  is said to be orthogonal to a flag  $F = (L, \Pi)$  if the line  $L'$  intersects with  $L$  orthogonally and is perpendicular to  $\Pi$ .

Let  $L_1$  and  $L_2$  be a pair of lines which share a common orthogonal line  $L$  in  $\mathbb{H}^4$ . For a pair of flags  $F_1 = (L_1, \Pi_1)$  and  $F_2 = (L_2, \Pi_2)$  which are orthogonal to  $L$ , we define a *quaternion distance from  $F_1$  to  $F_2$  along  $L$*  by a translation length of a loxodromic isometry which maps  $F_1$  to  $F_2$  and keeps  $L$  invariant. For a flag  $F = (L, \Pi)$  which is orthogonal to  $L_1$  and  $L_2$ , we define an  *$e_2$ -complex distance from  $L_1$  to  $L_2$  along  $F$*  by a translation length of a loxodromic isometry which maps  $L_1$  to  $L_2$  and keeps  $F$  invariant. We refer the precise definitions to [21] and provide here two propositions which we use in what follows.

**Proposition 5.12** ([21]). *Suppose  $F_1 = (L_1, \Pi_1)$  and  $F_2 = (L_2, \Pi_2)$  are two flags both orthogonal to a line  $L$  in  $\mathbb{H}^4$ . Let  $\tau \in \text{Isom}(\mathbb{H}^4)$  be such that  $\tau(L) = L$  and  $\tau(F_1) = F_2$ , and  $\iota \in \text{Isom}(\mathbb{H}^4)$  be such that  $\iota(L) = L_v$  and  $\iota(F_1) = F_h$ . Then the isometry  $\iota\tau\iota^{-1}$  has Vahlen matrices  $\pm \begin{pmatrix} \exp \delta & 0 \\ 0 & \exp(-\delta^*) \end{pmatrix}$  where  $\delta = d_L(F_1, F_2) \in \mathbb{A}_2$  is the quaternion half distance from  $F_1$  to  $F_2$  along  $L$ .*

**Proposition 5.17** ([21]). *Suppose  $L_1$  and  $L_2$  are two lines both orthogonal to a flag  $F = (L, \Pi)$  in  $\mathbb{H}^4$ . Let  $\tau \in \text{Isom}(\mathbb{H}^4)$  be such that  $\tau(F) = F$  and  $\tau(L_1) = L_2$ , and  $\iota \in \text{Isom}(\mathbb{H}^4)$  be such that  $\iota(F) = F_h$  and  $\iota(L_1) = L_v$ . Then the isometry  $\iota\tau\iota^{-1}$  has Vahlen matrices  $\pm \begin{pmatrix} \cosh \delta & \sinh \delta \\ \sinh \delta & \cosh \delta \end{pmatrix}$  where  $\delta = d_F(L_1, L_2) \in \mathbb{R} + \mathbb{R}e_2$  is the  $e_2$ -complex half distance from  $L_1$  to  $L_2$  along  $F$ .*

We will close this section with a couple of remarks on isometries we use in the rest of the paper.

Let  $i = \frac{e_1 + e_2}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \in \text{SL}(\Gamma_2)$  be the isometry which maps the PLP-configuration  $(e_3, L_h, F_h)$  to  $(e_3, L_v, F_v)$ . Note that  $i$  is an involution, i.e.,  $i^2 = id$ .

$$(22) \quad i : L_{[-1,1]} \leftrightarrow L_{[0,\infty]}, \quad L_{[-e_2, e_2]} \leftrightarrow L_{[-e_1, e_1]}.$$

For a plane  $\Pi \subset \mathbb{H}^4$ , we call a  $\pi$ -rotation around  $\Pi$  a *half turn* around  $\Pi$ . In particular, let  $I_v$  (respectively  $I_h$ ) be the half turn around  $\Pi_v$  (respectively  $\Pi_h$ ):

$$(23) \quad I_v = \begin{pmatrix} e_1 & 0 \\ 0 & -e_1 \end{pmatrix}, \quad I_h = iI_v i^{-1} = \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix}.$$

### 3. Augmented ideal right-angled pentagons

#### 3.1. Augmented pentagons with two planes

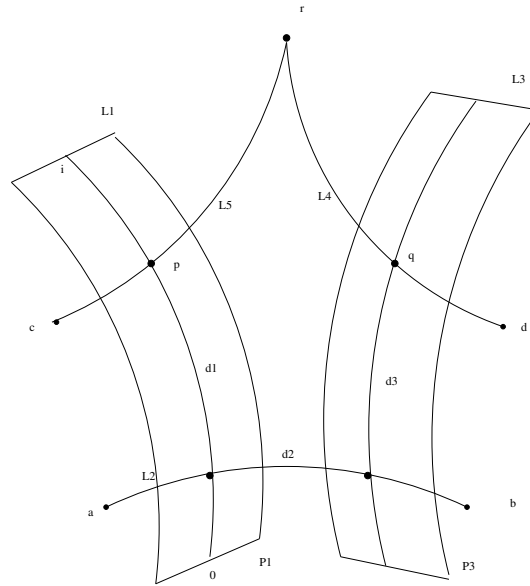


FIGURE 3. A normalized augmented ideal right-angled pentagon  $(L_1, \Pi_2, L_3, \Pi_4, L_5)$

Here, we augment an ideal right-angled pentagon  $(L_1, \dots, L_5)$  with two planes  $\Pi_2 \supset L_2$  and  $\Pi_4 \supset L_4$  such that  $\Pi_2$  is perpendicular to  $L_1$  and  $L_3$ , and  $\Pi_4$  is perpendicular to  $L_3$  and  $L_5$ . In this way, we can associate a quaternion half length  $\delta_3$  to  $L_3$  and  $e_2$ -complex half lengths  $\delta_2$  and  $\delta_4$  to  $L_2$  and  $L_4$  respectively.

**Lemma 3.1.** *For an augmented ideal right-angled pentagon  $(L_1, \Pi_2, L_3, \Pi_4, L_5)$  with a quaternion half length  $\delta_3 \in \mathbb{A}_2$  and  $e_2$ -complex half lengths  $\delta_2, \delta_4 \in \mathbb{R} + \mathbb{R}e_2$ , we have the following formula*

$$(24) \quad \exp \delta_3 (\sinh \delta_4^{-1} \cosh \delta_4) \exp \delta_3^* = \cosh \delta_2^{-1} \sinh \delta_2.$$

*Proof.* Without loss of generality, we may normalize the configuration so that  $(L_2, \Pi_2) = (L_h, \Pi_h)$  and  $L_3 = L_v$  (see Figure 3). Let  $\eta_i$  ( $i = 2, 3, 4$ ) be a loxodromic isometry such that  $\text{Axis}(\eta_i) = L_i$  and

$$(25) \quad \begin{aligned} \eta_2(\Pi_2) &= \Pi_2, \quad \eta_2(L_1) = L_3, \\ \eta_3(\Pi_2) &= \Pi_4, \quad \eta_3(L_2) = L_4, \\ \eta_4(\Pi_4) &= \Pi_4, \quad \eta_4(L_3) = L_5. \end{aligned}$$

Then  $\eta_2, \eta_3, \eta_4$  can be written as

$$(26) \quad \begin{aligned} \eta_2 &= \begin{pmatrix} \cosh \delta_2 & \sinh \delta_2 \\ \sinh \delta_2 & \cosh \delta_2 \end{pmatrix}, \\ \eta_3 &= \begin{pmatrix} \exp \delta_3 & 0 \\ 0 & \exp(-\delta_3^*) \end{pmatrix}, \\ \eta_4 &= \eta_3 \begin{pmatrix} \cosh \delta_4 & \sinh \delta_4 \\ \sinh \delta_4 & \cosh \delta_4 \end{pmatrix} \eta_3^{-1}. \end{aligned}$$

Then

$$(27) \quad \begin{aligned} \eta_2^{-1}(0) &= -\cosh \delta_2^{-1} \sinh \delta_2, \\ \eta_4(\infty) &= \exp \delta_3 (-\sinh \delta_4^{-1} \cosh \delta_4) \exp(-\delta_3^*)^{-1}. \end{aligned}$$

Equation (24) comes from the fact that  $\eta_2^{-1}(0) = \eta_4(\infty)$ . □

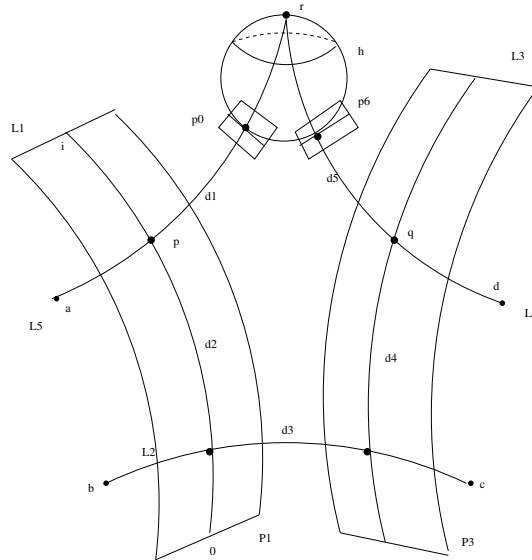


FIGURE 4. An augmented ideal right-angled pentagon  $(L_1, \Pi_2, L_3, \Pi_4, L_5)$  with  $(C_0, S_0)$

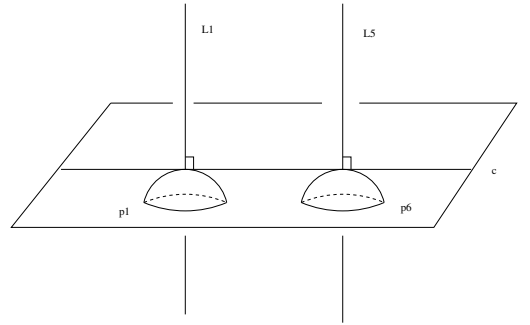


FIGURE 5. PLP-configurations at  $p_{01}$  and  $p_{50}$

For the augmented ideal right-angled pentagon  $(L_1, \Pi_2, L_3, \Pi_4, L_5)$ , consider the horocycle/horosphere pair  $C_0 \subset S_0$  assigned to the ideal vertex. Let  $p_{01}$  (resp.  $p_{50}$ ) be the point of intersection between  $C_0$  and  $L_1$  (resp.  $C_0$  and  $L_5$ ). For the chosen  $S_0$ , we take a geodesic  $L_0$  (resp.  $L_6$ ) which is tangent to  $C_0$  at  $p_{01}$  (resp.  $p_{50}$ ) and a plane  $\Pi_0$  (resp.  $\Pi_6$ ) which is tangent to  $S_0$  and contains  $L_0$  (resp.  $L_6$ ) (see Figure 4). In this way, we can choose canonically augmented pairs  $(L_0, \Pi_0)$  and  $(L_6, \Pi_6)$  although the choice of  $S_0$  is not canonical.

**Theorem 3.2.** *For an augmented ideal right-angled pentagon  $(L_1, \Pi_2, L_3, \Pi_4, L_5)$ , the horocycle/horosphere pair  $C_0 \subset S_0$  assigned to the ideal vertex, and canonically chosen pairs  $(L_0, \Pi_0)$  and  $(L_6, \Pi_6)$ , let  $\delta_i, i = 1, \dots, 5$  be the associated quaternion or complex half lengths of the augmented pentagon. Then we have the following formulas:*

$$(28) \quad \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm e_2;$$

$$(29) \quad \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \exp \delta_5^* \\ = \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4;$$

$$(30) \quad \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \\ \times \left( \cosh \delta_4^* \exp \delta_3^* \sinh \delta_2^* + \sinh \delta_4^* \exp(-\delta_3) \cosh \delta_2^* \right) = -1;$$

$$(31) \quad \sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 = 0.$$

*Proof.* Without loss of generality, we may assume that the common ideal vertex of  $L_1$  and  $L_5$  is the point  $\infty \in \partial \mathbb{H}^4$ ,  $L_0 = L_{[1, -1]}$ ,  $\Pi_0 = \Pi_{[1, -1] \vee [-e_1, e_1]}$ ,  $L_1 = L_{[\infty, 0]}$  and  $L_5 = L_{[1, \infty]}$  such that the intersection point of  $L_1$  and  $\Pi_0$  is  $e_3 \in \mathbb{H}^4$ . Let  $\eta_i$  ( $i = 1, \dots, 5$ ) be loxodromic isometries such that  $\text{Axis}(\eta_i) = L_i$

and

$$\begin{aligned}
 \eta_i(L_{i-1}) &= L_{i+1}, \quad i = 1, \dots, 5, \\
 \eta_i(\Pi_{i-1}) &= \Pi_{i+1}, \quad i = 1, 3, 5, \\
 \eta_i(\Pi_i) &= \Pi_i, \quad i = 2, 4.
 \end{aligned}
 \tag{32}$$

In particular, let  $\eta_6 = I_h \circ P$  where  $P$  is a strictly parabolic isometry which fixes  $\infty$  and translates by  $-1$  and  $I_h$  is the half turn around  $\Pi_h$ . So

$$\eta_6(L_5) = L_1, \quad \eta_6(L_6) = L_0, \quad \eta_6(\Pi_6) = \Pi_0.$$

Then  $\eta_6$  has a Vahlen matrix of the form  $\pm \begin{pmatrix} 0 & e_2 \\ e_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ . Since the isometry  $\eta_6 \cdots \eta_2 \eta_1$  maps not only the PLP-configuration  $(e_3, L_0, \Pi_0)$  onto itself, but also  $L_1$  onto itself, it is the identity map, *id*.

Define  $\iota_0$  to be the identity map and for  $i = 1, \dots, 6$ ,

$$\begin{aligned}
 \iota_i &= (\eta_i \cdots \eta_2 \eta_1)^{-1}, \\
 \psi_i &= \iota_{i-1} \eta_i \iota_{i-1}^{-1}.
 \end{aligned}
 \tag{33}$$

Then

$$\begin{aligned}
 \iota_i(L_i) &= L_0, \quad i = 1, 3, 5, \\
 \iota_i : (L_i, \Pi_i) &\mapsto (L_0, \Pi_0), \quad i = 2, 4, 5.
 \end{aligned}
 \tag{34}$$

In particular, for  $i = 1, 3, 5$  (respectively  $i = 2, 4$ )  $\psi_i$  keeps  $0$  and  $\infty$  (respectively  $-1$  and  $1$ ). Thus,  $\psi_i$  has a Vahlen matrix of the following form:

$$\psi_i = \begin{cases} \begin{pmatrix} \exp \delta_i & 0 \\ 0 & \exp(-\delta_i^*) \end{pmatrix} & \text{if } i = 1, 3, 5, \\ \begin{pmatrix} \cosh \delta_i & \sinh \delta_i \\ \sinh \delta_i & \cosh \delta_i \end{pmatrix} & \text{if } i = 2, 4. \end{cases}
 \tag{35}$$

Therefore,

$$\begin{aligned}
 \psi_1 \psi_2 \cdots \psi_6 &= \eta_1 (\iota_1 \eta_2 \iota_1^{-1}) \cdots (\iota_5 \eta_6 \iota_5^{-1}) \\
 &= \eta_1 \eta_1^{-1} \eta_2 \eta_1 (\eta_2 \eta_1)^{-1} \eta_2 (\eta_1 \eta_1) \cdots (\eta_5 \cdots \eta_1)^{-1} \eta_6 (\eta_5 \cdots \eta_1) \\
 &= \eta_6 \eta_5 \cdots \eta_1 = id.
 \end{aligned}
 \tag{36}$$

From the fact that  $\psi_6 = \eta_6$  and  $\psi_1 \cdots \psi_5 = \psi_6^{-1}$ , we have

$$\begin{aligned}
 &\begin{pmatrix} \exp \delta_1 & 0 \\ 0 & \exp(-\delta_1^*) \end{pmatrix} \begin{pmatrix} \cosh \delta_2 & \sinh \delta_2 \\ \sinh \delta_2 & \cosh \delta_2 \end{pmatrix} \begin{pmatrix} \exp \delta_3 & 0 \\ 0 & \exp(-\delta_3^*) \end{pmatrix} \\
 &\begin{pmatrix} \cosh \delta_4 & \sinh \delta_4 \\ \sinh \delta_4 & \cosh \delta_4 \end{pmatrix} \begin{pmatrix} \exp \delta_5 & 0 \\ 0 & \exp(-\delta_5^*) \end{pmatrix} = \pm \begin{pmatrix} e_2 & e_2 \\ e_2 & 0 \end{pmatrix}.
 \end{aligned}$$

Therefore, we have

$$\exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm e_2,
 \tag{37}$$

$$(38) \quad \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \exp(-\delta_5^*) = \pm e_2,$$

$$(39) \quad \exp(-\delta_1^*) \left( \sinh \delta_2 \exp \delta_3 \cosh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm e_2,$$

$$(40) \quad \exp(-\delta_1^*) \left( \sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \exp(-\delta_5^*) = 0.$$

Equation (37) and (40) give us (28) and (31) respectively. Equation (29) comes from the fact that the left hand sides of (37) and (38) are the same. Using (38) and (39), we have Equation (30):

$$\begin{aligned} & \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \exp(-\delta_5^*) \\ & \times \left( \exp(-\delta_1^*) \left( \sinh \delta_2 \exp \delta_3 \cosh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \right)^* \\ & = e_2 e_2^* = -1. \end{aligned} \quad \square$$

*Remark 3.3.* In the above theorem, if the ideal right-angled pentagon  $(L_1, \dots, L_5)$  embeds in a plane in  $\mathbb{H}^4$ , we recover the pentagon formula (43) in hyperbolic 2-space (see Theorem 7.18.1 of [7]). We may assume that the ideal right-angled pentagon  $(L_1, \dots, L_5)$  embeds in the plane  $\langle 1, e_3 \rangle$  under our normalization during the proof. Then  $\delta_i$  ( $i = 2, 3, 4$ ) becomes an  $e_2$ -complex number  $d_i + \frac{\pi}{2}e_2$  for a real number  $d_i \in \mathbb{R}$  and they commute each other. We recall that for an  $e_2$ -complex number  $d + \frac{\pi}{2}e_2 \in \mathbb{R} + \mathbb{R}e_2$ ,

$$(41) \quad \begin{aligned} \exp(d + \frac{\pi}{2}e_2) &= e_2 \exp d, \\ \exp(-d - \frac{\pi}{2}e_2) &= -e_2 \exp(-d), \\ \cosh(d + \frac{\pi}{2}e_2) &= e_2 \sinh d, \\ \sinh(d + \frac{\pi}{2}e_2) &= e_2 \cosh d, \end{aligned}$$

(see Propositions 5.18 and 5.19 of [21]). Thus, Equation (30) becomes

$$(42) \quad \begin{aligned} & \left( -e_2 \sinh d_2 \exp d_3 \cosh d_4 + e_2 \cosh d_2 \exp(-d_3) \sinh d_4 \right) \\ & \times \left( -e_2 \sinh d_4 \exp d_3 \cosh d_2 + e_2 \cosh d_4 \exp(-d_3) \sinh d_2 \right) = -1. \end{aligned}$$

Therefore, we have

$$2 \cosh(2d_3) \cosh d_2 \sinh d_2 \cosh d_4 \sinh d_4 = \sinh^2 d_2 \sinh^2 d_4 + \cosh^2 d_2 \cosh^2 d_4$$

which induces the pentagon formula associated to the ideal right-angled pentagon in hyperbolic 2-space (see Figure 6).

$$(43) \quad \sinh 2d_2 \cosh 2d_3 \sinh 2d_4 = \cosh 2d_2 \cosh 2d_4 + 1.$$

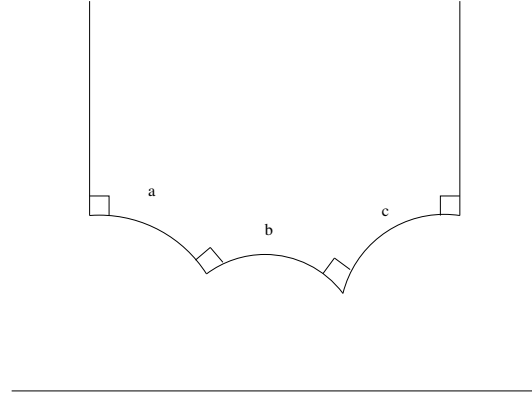


FIGURE 6. An ideal right-angled pentagon in hyperbolic 2-space

### 3.2. Augmented pentagons with three planes

Now, we augment an ideal right-angled pentagon  $(L_1, \dots, L_5)$  with three planes  $\Pi_1 \supset L_1$ ,  $\Pi_3 \supset L_3$  and  $\Pi_5 \supset L_5$  (See Figure 7). The plane  $\Pi_3$  is chosen to be perpendicular to  $L_2$  and  $L_4$ , and is generically unique. A priori,  $\Pi_1$  and  $\Pi_5$  are only required to be perpendicular to  $L_2$  and  $L_4$  respectively, so generically, there is a one dimensional family of choices for  $\Pi_1$  and  $\Pi_5$  respectively. This augmentation gives us quaternion half lengths  $\delta_2$  and  $\delta_4$  for  $L_2$  and  $L_4$  respectively, and an  $e_2$ -complex half length  $\delta_3$  for  $L_3$ . Lemma 3.4 is about the relation between these three lengths.

**Lemma 3.4.** *For an augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  with quaternion half lengths  $\delta_2, \delta_4 \in \mathbb{A}_2$  and an  $e_2$ -complex half length  $\delta_3 \in \mathbb{R} + \mathbb{R}e_2$ , we have the following formula*

$$(44) \quad \begin{aligned} & \left( \sinh \delta_3 \exp \delta_4 + \cosh \delta_3 \exp(-\delta_4^*) \right) \\ & + \exp \delta_2^* \exp \delta_2 \left( \cosh \delta_3 \exp \delta_4 + \sinh \delta_3 \exp(-\delta_4^*) \right) = 0. \end{aligned}$$

*Proof.* Without loss of generality, we may assume that  $L_2 = L_{[\infty,0]}$ ,  $L_3 = L_h$  and  $\Pi_3 = \Pi_h$  (see Figure 7). Let  $\eta_i$  ( $i = 2, 3, 4$ ) be a loxodromic isometry such



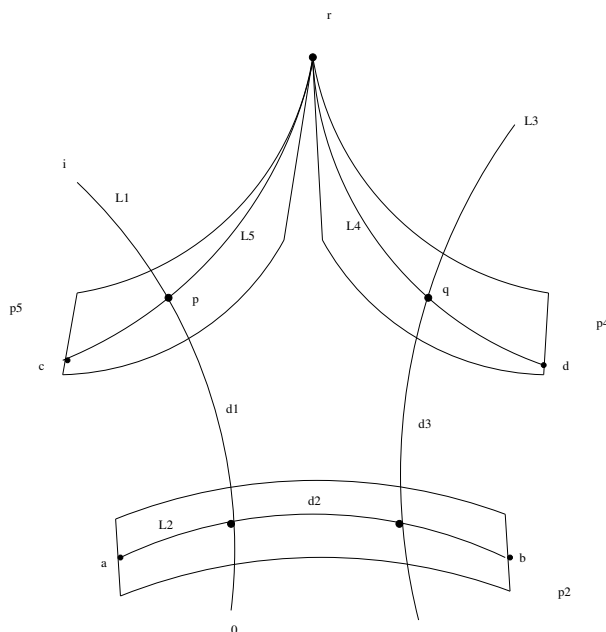


FIGURE 7. A normalized augmented pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$

that  $\text{Axis}(\eta_i) = L_i$  and

$$(45) \quad \begin{aligned} \eta_2 &: (L_1, \Pi_1) \mapsto (L_3, \Pi_3), \\ \eta_3(L_2) &= L_4, \quad \eta_3(\Pi_3) = \Pi_3, \\ \eta_4 &: (L_3, \Pi_3) \mapsto (L_5, \Pi_5). \end{aligned}$$

Then  $\eta_2, \eta_3, \eta_4$  can be written as

$$(46) \quad \begin{aligned} \eta_2 &= \begin{pmatrix} \exp \delta_2 & 0 \\ 0 & \exp(-\delta_2^*) \end{pmatrix}, \\ \eta_3 &= \begin{pmatrix} \cosh \delta_3 & \sinh \delta_3 \\ \sinh \delta_3 & \cosh \delta_3 \end{pmatrix}, \\ \eta_4 &= \eta_3 \begin{pmatrix} \exp \delta_4 & 0 \\ 0 & \exp(-\delta_4^*) \end{pmatrix} \eta_3^{-1} \\ &= \begin{pmatrix} \cosh \delta_3 \exp \delta_4 & \sinh \delta_3 \exp(-\delta_4^*) \\ \sinh \delta_3 \exp \delta_4 & \cosh \delta_3 \exp(-\delta_4^*) \end{pmatrix} \eta_3^{-1}. \end{aligned}$$

Since  $\eta_2^{-1}(-1) = \eta_4(1)$ ,

$$(47) \quad -\exp(-\delta_2) \exp(-\delta_2^*) = \begin{pmatrix} \cosh \delta_3 \exp \delta_4 + \sinh \delta_3 \exp(-\delta_4^*) \end{pmatrix}$$

$$\left( \sinh \delta_3 \exp \delta_4 + \cosh \delta_3 \exp(-\delta_4^*) \right)^{-1}$$

which implies Equation (44).

□

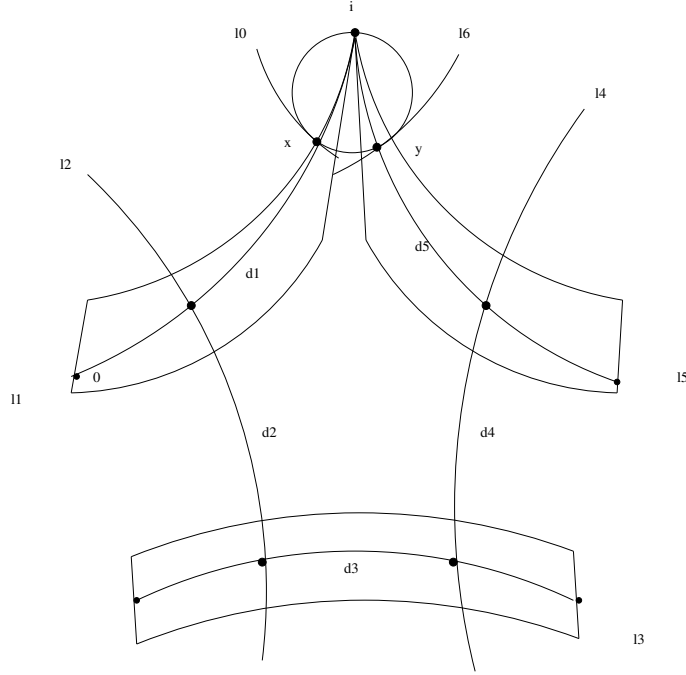


FIGURE 8. An augmented ideal right-angled pentagon with three planes and a horocycle

Now we may assume that the common ideal vertex of  $L_1$  and  $L_5$  is the point  $\infty \in \hat{\mathbb{R}}^3$  and will consider a horocycle  $C_0$  based at the ideal vertex  $\infty$  so that we choose planes  $\Pi_1$  and  $\Pi_5$  uniquely as follows (See Figure 8). Let  $C_0$  be the horocycle based at the ideal vertex  $\infty$  with height 1 after normalization and intersect with  $L_1$  and  $L_5$ . We call the intersection points  $C_0 \cap L_1$  and  $C_0 \cap L_5$  by  $p_{01}$  and  $p_{50}$  respectively. Let  $L_0$  (respectively  $L_6$ ) be the geodesic passing through  $p_{01}$  (respectively  $p_{50}$ ) and be tangent to the horocycle  $C_0$  at  $p_{01}$  (respectively  $p_{50}$ ). Then, we choose uniquely a plane  $\Pi_1$  (respectively  $\Pi_5$ ) which is perpendicular  $L_2$  and  $L_0$  (respectively  $L_4$  and  $L_6$ ) and contains  $L_1$  (respectively  $L_5$ ). We consider an isometry which maps a PLP-configuration  $(p_{50}, L_5, \Pi_5)$  to a PLP-configuration  $(p_{01}, L_1, \Pi_1)$  and keeps  $C_0$  invariant. In particular, this isometry can be seen as a composition of a parabolic isometry which fixes  $\infty$  and a half-turn around  $\Pi_1$ . Using this isometry, we assign an

angle invariant  $\theta_0$  associated to  $C_0$  as the rotation angle of the parabolic isometry. Theorem 3.5 gives the relations between the half lengths  $\delta_i$ ,  $i = 1, \dots, 5$  and  $\theta_0$ .

**Theorem 3.5.** *Let  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  be an augmented ideal right-angled pentagon in  $\mathbb{H}^4$  by the horocycle  $C_0$  assigned to the ideal vertex and planes  $\Pi_1 \supset L_1$ ,  $\Pi_3 \supset L_3$  and  $\Pi_5 \supset L_5$  as above. Let  $\delta_i$ ,  $i = 1, \dots, 5$  be the associated quaternion half lengths of the augmented pentagon and  $\theta_0$  the half rotation angle associated to  $C_0$ . Then we have the following formulas:*

$$(48) \quad \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 = \pm \alpha e_2;$$

$$(49) \quad \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \exp \delta_5^* = \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4;$$

$$(50) \quad \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \times \left( \cosh \delta_4^* \exp \delta_3^* \sinh \delta_2^* + \sinh \delta_4^* \exp(-\delta_3) \cosh \delta_2^* \right) = -1;$$

$$(51) \quad \sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 = 0,$$

where  $\alpha = \cos \theta_0 + \sin \theta_0 e_1 e_2$ .

*Remark 3.6.* If the half rotation angle  $\theta_0$  is 0, Equation (48) is the same as Equation (28). We also note that Equations (49), (50) and (51) are the same as (29), (30) and (31), respectively.

*Proof.* Without loss of generality, we may assume that the common ideal vertex of  $L_1$  and  $L_5$  is  $\infty$ ,  $L_1 = L_{[\infty,0]}$ ,  $L_5 = L_{[1,\infty]}$ ,  $C_0 = \{-x_0 + e_3 : x_0 \in \mathbb{R}\}$ ,  $L_0 = L_{[1,-1]}$ ,  $p_{01} = e_3$ ,  $L_6 = L_{[2,0]}$ ,  $p_{50} = 1 + e_3$  and  $\Pi_1 = \Pi_{[\infty,0] \vee [-e_2, e_2]}$  (see Figure 8).

Let  $\eta_i$  ( $i = 1, \dots, 5$ ) be an isometry such that  $\text{Axis}(\eta_i) = L_i$  and

$$(52) \quad \begin{aligned} \eta_i(L_{i-1}) &= L_{i+1}, \quad i = 1, \dots, 5, \\ \eta_i(\Pi_i) &= \Pi_i, \quad i = 1, 3, 5, \\ \eta_i(\Pi_{i-1}) &= \Pi_{i+1}, \quad i = 2, 4. \end{aligned}$$

Let  $\eta_6 = I_h \circ P$  where  $P$  is a screw parabolic isometry which maps a PLP-configuration  $(1 + e_3, [1, \infty], \Pi_5)$  to a PLP-configuration  $(e_3, [0, \infty], \Pi_1)$ . Note that  $P$  has a Vahlen matrix of the form  $\pm \begin{pmatrix} \alpha^* & -\alpha^* \\ 0 & \alpha^* \end{pmatrix}$  where  $\alpha = \cos \theta_0 + \sin \theta_0 e_1 e_2$  for some real number  $\theta_0 \in \mathbb{R}$ . Then, the isometry  $\eta_6 \cdots \eta_2 \eta_1$  maps the PLP-configuration  $(e_3, L_1, \Pi_1)$  onto itself. In addition, since  $\eta_6 \cdots \eta_2 \eta_1$  maps  $L_0$  onto itself, it is the identity map  $id$ .

Define  $\iota_0 = \text{id}$  and for  $i = 1, \dots, 6$ ,

$$(53) \quad \begin{aligned} \iota_i &= (\eta_i \cdots \eta_2 \eta_1)^{-1}, \\ \psi_i &= \iota_{i-1} \eta_i \iota_{i-1}^{-1}. \end{aligned}$$

Then

$$(54) \quad \begin{aligned} \iota_i(L_i) &= L_0, \quad i = 1, 3, 5, \\ \iota_i : (L_{i+1}, \Pi_{i+1}) &\mapsto (L_1, \Pi_1), \quad i = 2, 4. \end{aligned}$$

In particular,  $\psi_i (i = 1, 3, 5)$  fixes 0 and  $\infty$ , and  $\psi_i (i = 2, 4)$  fixes 1 and  $-1$ . So, their Vahlen matrices are of the form

$$(55) \quad \psi_i = \begin{cases} \begin{pmatrix} \exp \delta_i & 0 \\ 0 & \exp(-\delta_i^*) \end{pmatrix} & \text{if } i = 1, 3, 5, \\ \begin{pmatrix} \cosh \delta_i & \sinh \delta_i \\ \sinh \delta_i & \cosh \delta_i \end{pmatrix} & \text{if } i = 2, 4. \end{cases}$$

Then

$$(56) \quad \begin{aligned} \psi_1 \psi_2 \cdots \psi_6 &= \eta_1 (\iota_1 \eta_2 \iota_1^{-1}) \cdots (\iota_5 \eta_6 \iota_5^{-1}) \\ &= \eta_6 \cdots \eta_2 \eta_1 = \text{id} \end{aligned}$$

and  $\psi_6 = \eta_6$ . Since  $\psi_1 \cdots \psi_5 = \psi_6^{-1}$ ,

$$\begin{aligned} &\begin{pmatrix} \exp \delta_1 & 0 \\ 0 & \exp(-\delta_1^*) \end{pmatrix} \begin{pmatrix} \cosh \delta_2 & \sinh \delta_2 \\ \sinh \delta_2 & \cosh \delta_2 \end{pmatrix} \begin{pmatrix} \exp \delta_3 & 0 \\ 0 & \exp(-\delta_3^*) \end{pmatrix} \\ &\begin{pmatrix} \cosh \delta_4 & \sinh \delta_4 \\ \sinh \delta_4 & \cosh \delta_4 \end{pmatrix} \begin{pmatrix} \exp \delta_5 & 0 \\ 0 & \exp(-\delta_5^*) \end{pmatrix} \\ &= \pm P^{-1} I_h^{-1} = \pm \begin{pmatrix} \alpha e_2 & \alpha e_2 \\ \alpha e_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \exp \delta_1 \cosh \delta_2 \exp \delta_3 & \exp \delta_1 \sinh \delta_2 \exp(-\delta_3^*) \\ \exp(-\delta_1^*) \sinh \delta_2 \exp \delta_3 & \exp(-\delta_1^*) \cosh \delta_2 \exp(-\delta_3^*) \end{pmatrix} \\ &\quad \begin{pmatrix} \cosh \delta_4 \exp \delta_5 & \sinh \delta_4 \exp(-\delta_5^*) \\ \sinh \delta_4 \exp \delta_5 & \cosh \delta_4 \exp(-\delta_5^*) \end{pmatrix}. \end{aligned}$$

Thus, we have

$$(57) \quad \begin{aligned} &\exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \cosh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \\ &= \pm \alpha e_2, \end{aligned}$$

$$(58) \quad \begin{aligned} &\exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \exp(-\delta_5^*) \\ &= \pm \alpha e_2, \end{aligned}$$

$$(59) \quad \begin{aligned} &\exp(-\delta_1^*) \left( \sinh \delta_2 \exp \delta_3 \cosh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \\ &= \pm \alpha e_2, \end{aligned}$$

$$(60) \quad \exp(-\delta_1^*) \left( \sinh \delta_2 \exp \delta_3 \sinh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \exp(-\delta_5^*) = 0.$$

Equation (57) and (60) give us Equation (48) and (51) respectively. Equation (49) comes from that fact that the left hand sides of (57) and (58) are the same. Using (58) and (59), we have (50):

$$\begin{aligned} & \exp \delta_1 \left( \cosh \delta_2 \exp \delta_3 \sinh \delta_4 + \sinh \delta_2 \exp(-\delta_3^*) \cosh \delta_4 \right) \exp(-\delta_5^*) \\ & \times \left( \exp(-\delta_1^*) \left( \sinh \delta_2 \exp \delta_3 \cosh \delta_4 + \cosh \delta_2 \exp(-\delta_3^*) \sinh \delta_4 \right) \exp \delta_5 \right)^* \\ & = (\alpha e_2)(e_2 \alpha^*) = -1. \end{aligned} \quad \square$$

#### 4. Two-generator subgroups of $\text{Isom}(\mathbb{H}^4)$

A group  $\langle A, B, C \mid ABC = 1 \rangle$  is said to be a *linked two-generator group* if there exist three involutions  $P, Q, R$  so that  $A = PQ$ ,  $B = QR$  and  $C = RP$ . Depending on the dimension of hyperbolic space where the isometries act, involutions may be orientation-reversing. A reflection in a geodesic in hyperbolic 2-space is an orientation-reversing involution. A  $\pi$ -rotation around a geodesic in hyperbolic 3-space is an orientation-preserving involution. Finally, a  $\pi$ -rotation around a plane in hyperbolic 4-space is also an orientation-preserving involution which is called as a half turn in this paper.

Throughout this section, we consider an augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ . That is, we augment an ideal right-angled pentagon  $(L_1, \dots, L_5)$  with three planes  $\Pi_1 \supset L_1$ ,  $\Pi_3 \supset L_3$  and  $\Pi_5 \supset L_5$  in  $\mathbb{H}^4$  such that  $\Pi_1$  and  $\Pi_5$  are only required to be perpendicular to  $L_2$  and  $L_4$  respectively, and  $\Pi_3$  is generically uniquely chosen to be perpendicular to  $L_2$  and  $L_4$  (see Figure 7). This gives us a linked two-generator subgroup of  $\text{Isom}(\mathbb{H}^4)$  as follows.

**Lemma 4.1.** *For an augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ , let  $I_i$  ( $i = 1, 3, 5$ ) be a half turn around  $\Pi_i$  in  $\mathbb{H}^4$  and  $A = I_5 I_1$ ,  $B = I_1 I_3$ ,  $C = I_3 I_5$ . Then  $\langle A, B, C \rangle$  is a two-generator subgroup of  $\text{Isom}(\mathbb{H}^4)$  such that  $A$  is parabolic and  $B, C$  are loxodromic.*

The proof is a straightforward application of the following Propositions.

**Proposition 4.1** ([6]). *Let  $I_1$  and  $I_2$  be involutions with fixed point sets  $P_1$  and  $P_2$  in  $\mathbb{H}^4$  respectively and let  $f = I_2 I_1$ . Then  $f$  is elliptic if and only if  $P_1 \cap P_2 \neq \emptyset$ ;  $f$  is parabolic if and only if  $P_1 \cap P_2 = \emptyset$  and the hyperbolic distance  $d(P_1, P_2) = 0$ ;  $f$  is loxodromic if and only if the hyperbolic distance  $d(P_1, P_2) > 0$ .*

Applying the above proposition to the boundary at infinity  $\partial\mathbb{H}^4$ , we have:

**Proposition 4.2.** *Let  $P_i$  ( $i = 1, 2$ ) be a circle or a line in  $\hat{\mathbb{R}}^3$  and  $I_i$  be a half turn around  $P_i$ . Then the isometry  $f = I_2I_1$  is*

- (1) *elliptic if  $P_1$  and  $P_2$  are disjoint and linked (i.e., a unique fixed point in  $\mathbb{H}^4$ ),*
- (2) *loxodromic if  $P_1$  and  $P_2$  are disjoint and non-linked,*
- (3) *parabolic if  $P_1$  and  $P_2$  intersect at a single point of  $\hat{\mathbb{R}}^3$ ,*
- (4) *elliptic if  $P_1$  and  $P_2$  intersect at two points of  $\hat{\mathbb{R}}^3$  (i.e., an element in  $SO(2)$ ).*

*Proof.* To show (1), suppose that  $P_1$  and  $P_2$  are disjoint and linked. Without loss of generality, we may assume  $\infty \in P_2$ . Applying a dilation, a rotation or a translation to  $P_1$  and  $P_2$  in  $\hat{\mathbb{R}}^3$  if necessary, we may normalize so that  $P_1$  is a unit circle belonging to  $\langle 1, e_1 \rangle$  and centered at the origin, and  $P_2$  is a line. Since  $P_1$  and  $P_2$  are disjoint and linked,  $P_2$  must intersect the open unit 2-dimensional disk whose boundary is  $P_1$  in  $\langle 1, e_1 \rangle$ . Let  $Q$  be the intersection point in  $\partial\mathbb{H}^4$ . Then there exist  $x_0, y_0 \in \mathbb{R}$  such that  $x_0^2 + y_0^2 < 1$  and  $Q = x_0 + y_0e_1$ . Thus, we can write  $P_1 = \{x + ye_1 \in \mathbb{R}^3 : x, y \in \mathbb{R}, x^2 + y^2 = 1\}$  and  $P_2 = \{t(v_0 + v_1e_1 + v_2e_2) + x_0 + y_0e_1 \in \mathbb{R}^3 : t \in \mathbb{R}\} \cup \{\infty\}$  for a unit vector  $v_0 + v_1e_1 + v_2e_2$ . In  $\mathbb{H}^4$ , let  $HP_i$  ( $i = 1, 2$ ) be the plane whose boundary at infinity is  $P_i$ :

$$HP_1 = \{x + ye_1 + ue_3 \in \mathbb{H}^4 \mid x, y, u \in \mathbb{R}, x^2 + y^2 + u^2 = 1, u > 0\},$$

$$HP_2 = \{t(v_0 + v_1e_1 + v_2e_2) + x_0 + y_0e_1 + se_3 \in \mathbb{H}^4 \mid t, s \in \mathbb{R}, s > 0\}.$$

We see that  $HP_1$  and  $HP_2$  intersect at a point  $E = x_0 + y_0e_1 + s_0e_3 \in \mathbb{H}^4$  where  $s_0 = \sqrt{1 - x_0^2 - y_0^2} > 0$ . Then  $E \in \mathbb{H}^4$  is the fixed point of the isometry  $f$ , so  $f$  is elliptic.

To show (2), now suppose that  $P_1$  and  $P_2$  are disjoint and non-linked. As we did before, we may normalize  $P_1$  and  $P_2$  by isometries so that  $P_1$  is the same unit circle belonging to  $\langle 1, e_1 \rangle$  and centered at the origin, and  $P_2$  is a line. Only this time, because of the condition that  $P_1$  and  $P_2$  are not linked,  $P_2$  does not intersect with the open unit disk in  $\langle 1, e_1 \rangle$ . This implies that the planes  $HP_1$  and  $HP_2$  are disjoint in  $\mathbb{H}^4$ , and hence they have a non-zero hyperbolic distance. Applying Proposition 4.1 of [6], the isometry  $f$  is loxodromic.

Items (3) and (4) come directly from Proposition 4.1 of [6]. □

Therefore, we obtain a linked two-generator subgroup of  $\text{Isom}(\mathbb{H}^4)$  from an augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$ . However, Lemma 4.1 does not say anything about the discreteness of the group which is our goal in what follows.

Suppose that  $\Gamma = \langle A, B, C \mid ABC = 1 \rangle$  is a discrete two-generator subgroup of  $\text{Isom}(\mathbb{H}^2)$  such that  $A$  is parabolic and  $B, C$  are loxodromic. All non-elementary discrete two-generator groups are linked in  $\mathbb{H}^2$ . That is, we can write  $A = I_5I_1, B = I_1I_3$  and  $C = I_3I_5$  where  $I_i$  ( $i = 1, 3, 5$ ) is a reflection in a geodesic  $L_i$  in  $\mathbb{H}^2$ . Here,  $L_1$  and  $L_5$  have a common boundary point at infinity

which is the fixed point of the parabolic isometry  $A$ .  $L_1$  and  $L_3$  (respectively,  $L_3$  and  $L_5$ ) are disjoint and hence they have a unique common orthogonal geodesic, called  $L_2$  (respectively,  $L_4$ ) in  $\mathbb{H}^2$ . In this way, we associate an ideal right-angled pentagon  $(L_1, L_2, L_3, L_4, L_5)$  to the group  $\Gamma$  in  $\mathbb{H}^2$ .

The domain  $D$  bounded by  $L_1, L_3$  and  $L_5$  is the fundamental domain for the action of the group generated by three reflections,  $\langle I_1, I_3, I_5 \rangle$  which is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ . Since the group  $\Gamma = \langle A, B, C : ABC = 1 \rangle$  is an index 2 subgroup of  $\langle I_1, I_3, I_5 \rangle$ , we can obtain a fundamental domain of  $\Gamma$  by doubling  $D$ .

Let  $\mathcal{T}(\Gamma)$  be the Teichmüller space of  $\Gamma$  :

$$\{\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^2) : \text{discrete, faithful and type-preserving}\} / \text{Isom}(\mathbb{H}^2)$$

which is the set of discrete, faithful and type-preserving representations into  $\text{Isom}(\mathbb{H}^2)$  up to the conjugation action of  $\text{Isom}(\mathbb{H}^2)$ . It has dimension 2. So, we can parameterize it by a pair of two positive numbers  $(d_2, d_4)$  where  $d_2$  (respectively,  $d_4$ ) is the hyperbolic distance between  $L_1$  and  $L_3$  (respectively, between  $L_3$  and  $L_5$ ). Applying isometries if necessary, we may normalize the configuration so that  $L_1 = L_{[\infty, 0]}$ ,  $L_3 = L_{[1, s]}$  and  $L_5 = L_{[t, \infty]}$  for some  $1 < s < t$ . Hence, we may also parameterize  $\mathcal{T}(\Gamma)$  by the pair of two boundary points  $(s, t)$  ( $1 < s < t$ ) (see Figure 9).

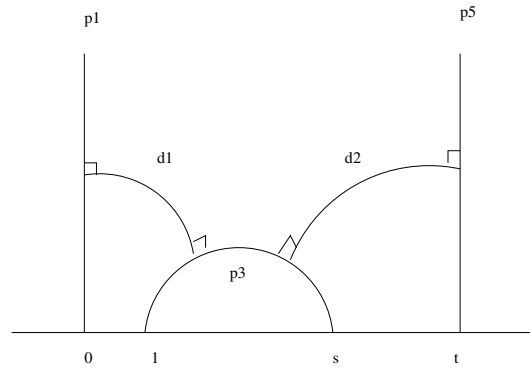


FIGURE 9. An ideal right-angled pentagon associated to a linked two-generator group in  $\mathbb{H}^2$

In hyperbolic 4-space, we define the deformation space  $\mathcal{D}(\Gamma)$  of  $\Gamma$  by the set of all discrete, faithful and type-preserving representation of  $\Gamma$  into  $\text{Isom}(\mathbb{H}^4)$  up to the conjugation action of  $\text{Isom}(\mathbb{H}^4)$ . That is,

$$\mathcal{D}(\Gamma) = \{\rho : \Gamma \rightarrow \text{Isom}(\mathbb{H}^4) : \text{discrete, faithful and type-preserving}\} / \text{Isom}(\mathbb{H}^4).$$

By assigning an augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  to  $\Gamma$ , we may consider  $\Gamma$  itself as an image of the identity representation into  $\text{Isom}(\mathbb{H}^4)$  as follows. Without loss of generality, we may assume that the ideal

right-angled pentagon  $(L_1, L_2, L_3, L_4, L_5)$  associated to  $\Gamma$  is embedded in the plane  $\langle 1, e_3 \rangle \subset \mathbb{H}^4 = \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}, x_3 > 0\}$ . As we see in Figure 9, we may normalize it so that  $L_1 = L_{[\infty, 0]}$ ,  $L_3 = L_{[1, s]}$  and  $L_5 = L_{[t, \infty]}$  for some  $1 < s < t$ . We augment the ideal right-angled pentagon with three planes  $\Pi_i$  ( $i = 1, 3, 5$ ). In particular, we choose  $\Pi_1$  and  $\Pi_5$ , on the boundary at infinity, to be a line parallel to  $\langle e_1 \rangle$ . Note that  $\Pi_1$  (respectively  $\Pi_5$ ) passes through 0 and  $\infty$  (respectively  $t$  and  $\infty$ ). Let  $\Pi_3$  be a 1-dimensional circle embedded in  $\langle 1, e_1 \rangle$  and having the segment between 1 and  $s$  a diameter on  $\hat{\mathbb{R}}^3$ . Now, we extend  $I_i$  ( $i = 1, 3, 5$ ) as a half turn around  $\Pi_i$  in  $\mathbb{H}^4$ . By construction, the group  $\langle I_1, I_3, I_5 \rangle$  keeps the plane  $\langle 1, e_3 \rangle$  invariant, so does  $\Gamma$ . On the plane  $\langle 1, e_3 \rangle$ , the action of group  $\langle I_1, I_3, I_5 \rangle$  is the same as the original Fuchsian group. This shows how we extend the group  $\Gamma$  into a discrete subgroup of  $\text{Isom}(\mathbb{H}^4)$ .

For the augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  in  $\mathbb{H}^4$ , we have associated half turns to each plane and the group generated by these half turns. Hence, deforming the augmented ideal right-angled pentagon may give us a new point in the deformation space  $\mathcal{D}(\Gamma)$ . To obtain a new point in  $\mathcal{D}(\Gamma)$ , we will consider a configuration which is deformed continuously from the initial augmented pentagon and impose a condition that  $\Pi_1$  passes through 0,  $\infty$ ,  $\Pi_3$  through 1,  $s$  and  $\Pi_5$  through  $t, \infty$ . In other words, we fix the initial three geodesics  $L_1 = L_{[\infty, 0]}$ ,  $L_3 = L_{[1, s]}$  and  $L_5 = L_{[t, \infty]}$ . Since there are 2-dimensional family of choices for each  $\Pi_i$  ( $i = 1, 3, 5$ ), we have 6-dimensional family of choices for  $(\Pi_1, \Pi_3, \Pi_5)$  in total. However, we have to consider the action of a rotation around  $\langle 1 \rangle$  in  $\partial\mathbb{H}^4$  since it fixes 0, 1,  $s, t$  and  $\infty$ . Suppose that one configuration is an image of another configuration under a rotation around  $\langle 1 \rangle$  in  $\partial\mathbb{H}^4$ . Then the two configurations might be different, but the two groups associated to each configurations are conjugate to each other by the rotation. Therefore, we have a 5-dimensional parameter space for the configurations of  $(\Pi_1, \Pi_3, \Pi_5)$  up to the conjugation action of  $\text{Isom}(\mathbb{H}^4)$ . Now, considering the parameter  $s$  and  $t$  we have a 7-dimensional parameter space for the deformation space  $\mathcal{D}(\Gamma)$ . An arbitrary group constructed in this way might not be discrete. So, we deform the pentagon in a special way to obtain discrete groups in the next theorem.

**Theorem 4.3.** *Let  $\Gamma = \langle A, B, C \mid ABC = 1 \rangle$  be a discrete two-generator subgroup of  $\text{Isom}(\mathbb{H}^2)$  where  $A$  is parabolic and  $B, C$  are loxodromic. Then there exists a 6-dimensional parameter space  $\mathcal{P}$  containing the identity representation in the deformation space.*

*Proof.* We identify the group  $\Gamma$  with the image of the identity representation and consider its associated augmented ideal right-angled pentagon  $(\Pi_1, L_2, \Pi_3, L_4, \Pi_5)$  as above: Let  $L_1 = L_{[\infty, 0]}$ ,  $L_3 = L_{[1, s_0]}$ ,  $L_5 = L_{[t_0, \infty]}$  for some  $1 < s_0 < t_0$  and  $L_2$  (respectively  $L_5$ ) be a line perpendicular to  $L_1$  and  $L_3$  (respectively,  $L_3$  and  $L_5$ ). In the boundary at infinity  $\partial\mathbb{H}^4 = \hat{\mathbb{R}}^3$ ,  $\Pi_1$  (respectively  $\Pi_5$ ) is a line parallel to  $\langle e_1 \rangle$  and passes through 0 and  $\infty$  (respectively  $t$  and  $\infty$ ),  $\Pi_3$  is a



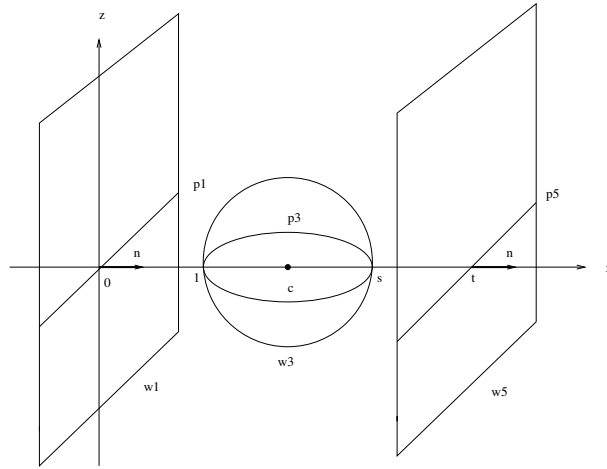


FIGURE 10.  $D$ : the boundary of the fundamental domain of  $\langle I_1, I_3, I_5 \rangle$  in  $\partial\mathbb{H}^4$

1-dimensional circle embedded in  $\langle 1, e_1 \rangle$  and having the segment between 1 and  $s$  a diameter,  $I_i$  ( $i = 1, 3, 5$ ) is a half-turn around  $\Pi_i$ , and  $A = I_5 I_1, B = I_1 I_3$  and  $C = I_3 I_5$ . Let  $L_2$  (respectively,  $L_4$ ) be the unique geodesic orthogonal to  $L_1$  and  $L_3$  (respectively  $L_3$  and  $L_5$ ). Here, the pentagon  $(L_1, \dots, L_5)$  is embedded into  $\langle 1, e_3 \rangle \subset \mathbb{H}^4$  and the group action of  $\Gamma$  keeps  $\langle 1, e_3 \rangle$  invariant.

First, we will construct the fundamental domain for the action of  $\langle I_1, I_3, I_5 \rangle$ . In the boundary  $\mathbb{R}^3$ , let  $W_1$  be the 2-dimensional subspace  $\langle e_1, e_2 \rangle$ ,  $W_5$  the subspace  $t_0 + W_1$  and  $W_3$  the 2-dimensional sphere having  $\Pi_3$  a great circle. Let  $D$  be the domain bounded by  $W_1, W_3$  and  $W_5$  in  $\hat{\mathbb{R}}^3$ . Since the half turn  $I_i$  ( $i = 1, 3, 5$ ) maps  $W_i$  to itself and  $I_i(D) \cap D = \emptyset$ ,  $D$  is the fundamental domain of the group  $\langle I_1, I_3, I_5 \rangle$  (See Figure 10).

Now we will deform the domain  $D$  together with the pentagon to obtain a new point in  $\mathcal{D}(\Gamma)$  (See Figure 11). Recall that we identify  $\mathbb{R}^3$  with the vector space  $\{x_0 + x_1 e_1 + x_2 e_2 \mid x_0, x_1, x_2 \in \mathbb{R}\}$ . We denote a unit vector in  $\mathbb{R}^3$  by  $n(\phi, \psi) = (\cos \phi \cos \psi, \cos \phi \sin \psi, \sin \phi)$  for some real numbers  $\phi, \psi$ . For a unit vector  $n(\phi, \psi) \in \mathbb{R}^3$ , there is an element  $\xi_{\phi, \psi} \in \text{SO}(3)$  such that  $\xi_{\phi, \psi}(1) = n(\phi, \psi)$ . Note that  $\xi_{0,0}(1) = 1$ . Let

$$\begin{aligned} W'_1 &= \langle \xi_{\phi, \psi}(e_1), \xi_{\phi, \psi}(e_2) \rangle = \langle n(\phi, \psi) \rangle^\perp, \\ \Pi'_1 &= \langle \xi_{\phi, \psi}(e_1) \rangle, \\ W'_5 &= t_0 + W'_1, \\ \Pi'_5 &= t_0 + \langle \cos \theta \xi_{\phi, \psi}(e_1) + \sin \theta \xi_{\phi, \psi}(e_2) \rangle, \end{aligned}$$

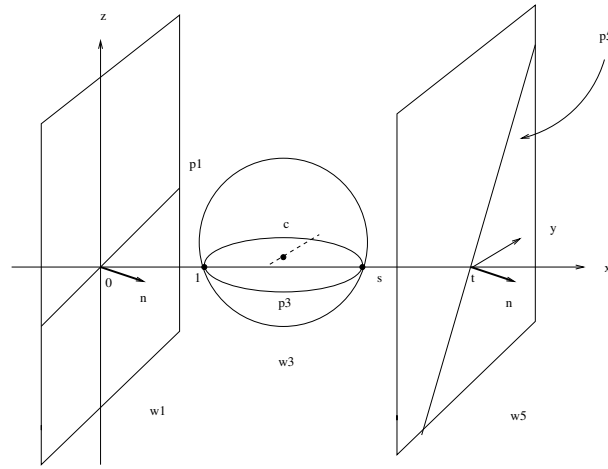


FIGURE 11. The fundamental domain  $D'$  in  $\mathbb{R}^3$

where  $0 \leq \theta < \pi$ . Then  $\Pi'_i \subseteq W'_i$  ( $i = 1, 5$ ) and  $W'_1$  and  $W'_5$  are parallel in  $\mathbb{R}^3$ . Let  $W'_3$  be the sphere centered at  $\frac{1}{2}(s_0 + 1) + ce_1$  for  $c \in \mathbb{R}$  and passing through 1,  $s_0$ , and  $\Pi'_3$  be the great circle of  $W'_3$  passing through 1 and  $s_0$ . Note that  $\Pi'_i \subset W'_i$  for  $i = 1, 3, 5$ . Since  $1 < s_0 < t_0$ , there are small positive numbers  $\epsilon_0, \delta_0$  such that  $W'_1, W'_3$  and  $W'_5$  are mutually disjoint in  $\mathbb{R}^3$  for any  $-\epsilon_0 < \phi, \psi < \epsilon_0$  and  $-\delta_0 < c < \delta_0$ .

Let  $D'$  be the domain bounded by  $W'_1, W'_3$  and  $W'_5$  in  $\hat{\mathbb{R}}^3$  for  $-\epsilon_0 < \phi, \psi < \epsilon_0$  and  $-\delta_0 < c < \delta_0$  and associate the half turns  $I'_i$  around  $\Pi'_i$  to each side  $W'_i$ . Then, the domain  $D'$  is the fundamental domain for the action of the group  $\langle I'_1, I'_3, I'_5 \rangle$ . Since  $W'_i$  and  $\Pi'_i$  ( $i = 1, 3, 5$ ) depend on  $\phi, \psi, c$  and  $\theta$ , let  $P$  be the 4-dimensional parameter space,

$$P = \{(\phi, \psi, c, \theta) \in \mathbb{R}^4 \mid -\epsilon_0 < \phi, \psi < \epsilon_0, -\delta_0 < c < \delta_0, 0 \leq \theta < \pi\}.$$

Note that the group associated  $(0, 0, 0, 0) \in P$  is the initial group  $\Gamma$  in  $\mathbb{H}^4$ .

We will show that two points in  $P$  give us two distinct groups in  $\mathcal{D}(\Gamma)$ . Suppose  $p_k = (\phi_k, \psi_k, c_k, \theta_k)$ ,  $k = 1, 2$  are two points in  $P$ . For  $k = 1, 2$ , let  $D_k$  be the fundamental domain associated to  $p_k$  with  $W_i^k$  ( $i = 1, 3, 5$ ), and  $G_k$  be the group generated by three half turns  $I_i^k$  ( $i = 1, 3, 5$ ) around  $\Pi_i^k$  associated to each sides of  $D_k$ . If  $G_1$  is conjugate to  $G_2$  by an isometry  $f$ , then  $D_1$  is the image of  $D_2$  under the isometry  $f$ . In particular,  $f(\Pi_i^1) = \Pi_i^2$  for  $i = 1, 3, 5$ . Since we impose the condition that  $\Pi_1^k$  passes through  $0, \infty$ ,  $\Pi_3^k$  through  $1, s_0$  and  $\Pi_5^k$  through  $t_0, \infty$  during the deformation,  $f$  fixes  $0, 1, s_0, t_0$  and  $\infty$ . So,  $f$  is a rotation around  $\langle 1 \rangle$  by some angle  $\alpha$ . However, since  $\Pi_3^k \subset \langle 1, e_1 \rangle$  during the deformation,  $f$  keeps the 2-dimensional subspace  $\langle 1, e_1 \rangle$  invariant. Thus,

$\alpha$  should be 0 which implies that  $f$  is the identity map. This proves that each point in  $P$  gives us a distinct point in the deformation space  $\mathcal{D}(\Gamma)$ .

Since  $1 < s_0 < t_0$  were arbitrary, this locally gives us a 6-dimensional parameter space  $\mathcal{P} = \{(s, t) : 1 < s < t\} \times P$  for the deformation space  $\mathcal{D}(\Gamma)$ .  $\square$

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