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CHARACTERIZATION OF FINITE COLORED SPACES WITH CERTAIN CONDITIONS

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Abstract. A *colored space* is a pair (X, r) of a set X and a function r whose domain is $\binom{X}{2}$. Let (X, r) be a finite colored space and $Y, Z \subseteq X$. We shall write $Y \simeq_r Z$ if there exists a bijection $f : Y \to Z$ such that $r(U) = r(f(U))$ for each $U \in \binom{Y}{2}$ where $f(U) = \{f(u) \mid u \in U\}$. We denote the numbers of equivalence classes with respect to $\simeq_r$ contained in $\binom{X}{i}$ by $a_i(r)$. In this paper we prove that $a_2(r) \leq a_3(r)$ when $5 \leq |X|$, and show what happens when equality holds.

1. Introduction

A *colored space* is a pair (X, r) of a set X and a function r whose domain is $\binom{X}{2}$, where we denote by $\binom{X}{k}$ the set of k-subsets of X for a positive integer k. Notice that r can be identified with an edge-coloring of the complete graph on X that leads us to call the elements of $\mathsf{Im}(r)$ *colors* where $\mathsf{Im}(r)$ is the image of r. From this point of view simple graphs induce colored spaces with at most two colors. On the other hand, metric spaces also induce colored space whose colors are defined by its metric function. Thus, we observe that colored spaces cover quite general objects of combinatorics and geometry.

Let (X, r) be a colored space. Then each subset Y of X induces a colored space (Y, r_Y), called a *subspace*, where r_Y is the restriction of r to $\binom{Y}{2}$. For $Y, Z \subseteq X$ we say that Y is *isometric* to Z if there exists a bijection $f : Y \to Z$ such that $r_Y(U) = r_Z(f(U))$ for each $U \in \binom{Y}{2}$ where $f(U) = \{f(u) \mid u \in U\}$ and we shall write

$$Y \simeq_r Z \text{ if } Y \text{ is isometric to } Z.$$

For a positive integer k we denote the class of isometric subspaces of size k by $A_k(r)$, in other words,

$$A_k(r) = \left\{[Y] \mid Y \in \binom{X}{k}\right\} \text{ where } [Y] = \{Z \mid Y \simeq_r Z\}.$$

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We denote the cardinality of $A_k(r)$ by $a_k(r)$. One may notice that $A_1(r)$ is a singleton, $A_2(r)$ can be identified with $\mathsf{Im}(r)$, and $A_k(r)$ is a finite set if $\mathsf{Im}(r)$ is a finite set. Thus, the following sequence of positive integers are defined when $|X| = n$:

$$(a_1(r), a_2(r), \ldots, a_n(r)),$$

which is called the *isometric sequence* of (X, r). As mentioned before, $a_1(r) = 1$, $a_2(r) = |\mathsf{Im}(r)|$, and it is easy to obtain the following:

Example 1.1. For each finite colored space (X, r) with $n = |X|$ we have

$$1 \leq a_2(r) \leq \binom{n}{2}.$$

One of the extremal cases has the isometric sequence $(1, 1, \ldots, 1)$ and the other has the isometric sequence

$$\left(1, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{k}, \ldots, \binom{n}{n-1}, 1\right).$$

Notice that, for $Y, Z \in \binom{X}{3}$, $Y \simeq_r Z$ if and only if $(r(U) \mid U \in \binom{Y}{2})$ coincides with $(r(V) \mid V \in \binom{Z}{2})$ as multi-sets, so that we obtain from the rule of combinations with repetitions that

$$1 \leq a_3(r) \leq \binom{a_2(r)+2}{3}.$$

One of the optimal cases appears in four points in Euclidean space such as the vertices of a rectangle, a tetrahedron with congruent faces or a regular simplex. On the other hand, the other can be attained when you have sufficiently many points, and one may notice that such colored spaces are conventional and not so curious. In this paper we focus on how isometric sequences change. We present the following conjecture.

Conjecture 1.1. *Any isometric sequences change unimodally, that is,*

$$a_1(r) \leq a_2(r) \leq \cdots \leq a_k(r) \geq \cdots \geq a_n(r)$$

holds for some $1 \leq k \leq n$.

The first step toward this conjecture must be to say what happens when $a_2(r) > a_3(r)$. That is exactly what is stated in our main theorem:

Theorem 1.2. *For each finite colored space* (X, r) *with* $5 \leq |X|$ *we have* $a_2(r) \leq a_3(r)$.

Namely, if $a_2(r) > a_3(r)$, then $|X| \leq 4$, and hence the isometric sequence is unimodal. In Theorem 1.2 it is natural to ask when equality holds. For example, the vertices of a regular octahedron and a regular hexagon in Euclidean space induce the isometric sequences

$$(1, 2, 2, 2, 1, 1) \text{ and } (1, 3, 3, 3, 1, 1),$$

respectively. In [3] the authors classify all colored spaces with $a_2(r) = a_3(r) \leq 3$ and $5 \leq |X|$ up to isomorphism (see Section 2 for the definition)[1]. Continuing these works we solved this problem completely in the following:

Theorem 1.3. *Every finite colored space (X, r) with $4 \leq a_2(r) = a_3(r)$ and $9 \leq |X|$ implies one of the following partitions of $\binom{X}{2}$:*

(i) *disjoint matchings and the remaining,*
(ii) *the disjoint union of two cliques, disjoint matchings between the two cliques and the remaining,*
(iii) *a singleton of one edge U, the edges from one end of the edge to $X \setminus U$, the edges from the other end to $X \setminus U$, and the remaining.*

Remark 1.2. We have $a_2(r) = a_3(r) = 4$ if (iii) happens in Theorem 1.3.

In relation to this topic we refer some articles on distance sets (see [1], [2], [4], and [5]). For a positive integer s we say that a finite subset X in Euclidean space $\mathbb{R}^N$ is an *s-distance set* if $a_2(r) = s$, where $r : \binom{X}{2} \to \mathbb{R}_{\geq 0}$ is defined by $r(\{x, y\}) = ||x - y||$. In [1], an upper bound for $|X|$ is given by a function on s and N. In [4] and [5] they show 2-distance sets in $\mathbb{R}^N$ that attain the upper bound. In [2] they show some criterion to embed finite colored spaces (X, r) with $a_2(r) = 2$ into Euclidean space of dimension less than $|X| - 1$. In such a way studies on distance sets focus on finite subsets in a Euclidean space with certain optimal conditions. From this point of view Theorem 1.3 gives a characterization of finite subsets X of a Euclidean space whose distances corresponds to the congruence classes of triangles derived from the elements in $\binom{X}{3}$.

In Section 2 we prepare some definitions of graphs and some notation on colored spaces, and in Section 3 we give a proof of the first step of the induction to prove our main result in Section 4.

2. Preliminaries

We prepare some definitions for a graph $G = (V, E)$. Two edges e and e' are said to be *adjacent* if they have common vertices. An edge set $F \subset E$ is called a *matching* of G if e and e' are not adjacent for any distinct $e, e' \in F$. *Disjoint matchings* mean matchings whose union is also a matching. A matching F is called a *perfect matching* of G if every vertex of G is incident with an edge in F. A vertex set $W \subset V$ is called a *clique* of G if v and v' are adjacent for any distinct $v, v' \in W$.

Throughout this paper we assume that (X, r) is a finite colored space with $n = |X|$. For all distinct $x, y \in X$ we write $r(x, y)$ instead of $r(\{x, y\})$ for short. As mentioned in Section 1, $\mathsf{Im}(r)$ is identified with $A_2(r)$, so that an element $[\{x, y\}] \in A_2(r)$ is denoted by $r(x, y)$. For all distinct $x, y, z \in X$ we write

[1] In [3] only finite metric spaces are discussed. However, all results on the classifications of finite metric spaces can be translated to those on colored spaces, since the classification is done up to isomorphisms.

$[\{x,y,z\}] \in A_3(r)$ as $\alpha\beta\gamma$, where $\alpha = r(x,y)$, $\beta = r(y,z)$, and $\gamma = r(z,x)$, so that

$$\alpha\beta\gamma = \beta\gamma\alpha = \gamma\alpha\beta = \alpha\gamma\beta = \gamma\beta\alpha = \beta\alpha\gamma.$$

For each $\alpha \in A_2(r)$, we define

$$E_\alpha = r^{-1}(\alpha) \text{ and } R_\alpha = \{(x,y) \in X \times X \mid r(x,y) = \alpha\},$$

so that, (X, E_α) is a simple graph and R_α is a symmetric binary relation on X. For a binary relation R on X and $x \in X$ we set

$$R(x) = \{y \in X \mid (x,y) \in R\}.$$

For a finite colored space (X_1, r_1) we say that (X, r) is *isomorphic* to (X_1, r_1) if there exist bijections $f : X \to X_1$ and $g : \mathsf{Im}(r) \to \mathsf{Im}(r_1)$ with

$$g(r(U)) = r_1(f(U))$$

for each $U \in \binom{X}{2}$, where $f(U) = \{f(u) \mid u \in U\}$. Remark that any isometric subspaces are isomorphic, but the converse does not hold in general.

For a positive integer n, we denote the set $\{1, \dots, n\}$ by $[n]$. For $k \in [n]$ we set

$$M_k(r) = \{\alpha \in A_2(r) \mid (X, E_\alpha) \text{ has a vertex of degree at least } k\},$$

and we denote the size of $M_k(r)$ by $m_k(r)$.

Remark 2.1. For each $\alpha \in A_2(r)$,

$$\alpha \notin M_2(r) \text{ if and only if } (X, E_\alpha) \text{ is a matching on } X,$$

$$\text{and } \alpha\alpha\alpha \notin A_3(r) \text{ if and only if } (X, E_\alpha) \text{ is triangle-free.}$$

For $\emptyset \neq \Gamma \subseteq A_2(r)$, we say that Γ is *closed* if

$$\text{for all } \alpha, \beta \in \Gamma \text{ and } \gamma \in A_2(r),\ \alpha\beta\gamma \in A_3(r) \text{ implies } \gamma \in \Gamma.$$

Lemma 2.1. *For $\emptyset \neq \Gamma \subseteq A_2(r)$, Γ is closed if and only if $(X, \bigcup_{\gamma\in\Gamma} E_\gamma)$ is the disjoint union of complete graphs.*

Proof. Set $R_0 = \{(x,x) \mid x \in X\}$. Then Γ is closed if and only if $\bigcup_{\alpha\in\Gamma\cup\{0\}} R_\alpha$ is an equivalence relation on X. Since the equivalence classes correspond to the connected components of the graph $(X, \bigcup_{\gamma\in\Gamma} E_\gamma)$, the lemma holds. □

Lemma 2.2. *If $M_{k+1}(r) = \emptyset$, then $n \leq 1 + ka_2(r)$, and the equality does not hold when k is odd and $a_2(r)$ is even.*

Proof. Let $x \in X$. Since $M_{k+1}(r) = \emptyset$, we have

$$n = |X| = 1 + \left| \bigcup_{\alpha \in A_2(r)} R_\alpha(x) \right| \leq 1 + ka_2(r).$$

Suppose that the equality holds when k is odd and $a_2(r)$ is even. Then each vertex has exactly k neighbors in (X, E_α), which implies that

$$2|E_\alpha| = |R_\alpha| = nk = (1 + ka_2(r))k$$

is odd, a contradiction. □

Lemma 2.3. *If* $a_2(r) = a_3(r) = m_2(r)$*, then* $a_2(r) \leq 2$*.*

Proof. It is clear that, for all $\alpha, \beta, \gamma, \delta \in A_2(r)$, if $\alpha\alpha\beta = \gamma\gamma\delta$, then $\alpha = \gamma$ and $\beta = \delta$. This implies that $M_2(r)$ is embedded into $A_3(r)$. Since $A_2(r) = M_2(r)$ by $a_2(r) = m_2(r)$, it follows from $a_3(r) = m_2(r)$ that the mapping from $M_2(r)$ to $A_3(r)$ defined by

$$\alpha \mapsto \alpha\alpha\alpha'$$

for some $\alpha' \in A_2(r)$ is well-defined and bijective, so that there are no tricolored triangles where a tricolored triangle means a triangle T with $a_2(T) = 3$.

Suppose $a_2 > 1$. Then $\alpha\alpha\beta \in A_3(r)$ for some $\alpha, \beta \in A_2(r)$ with $\alpha \neq \beta$. Let $x_0, x_1, x_2, y \in X$ with

$$r(x_0, x_1) = r(x_0, x_2) = \alpha,\ r(x_1, x_2) = \beta, \text{ and } y \notin \{x_0, x_1, x_2\}.$$

Since $\{x_0, x_1, y\}$ is not tricolored, we have either

$$r(x_0, y) = \gamma \text{ or } r(x_0, y) = \alpha \text{ where } \gamma := r(x_1, y).$$

If $r(x_0, y) = \gamma$, then $r(y, x_2) \neq \gamma$ by the injectivity of the above mapping, and hence $r(y, x_2) = \alpha$, which implies that $\beta = \gamma$ or $\alpha = \gamma$ since $\{x_1, x_2, y\}$ is not tricolored. If $r(x_0, y) = \alpha$, then $r(y, x_2) = \gamma = \alpha$ by the injectivity. Since y is arbitrarily taken, we conclude that $r(x_1, y) \in \{\alpha, \beta\}$. Since there are no tricolored triangles, it follows that $A_2(r) = \{\alpha, \beta\}$, and hence $a_2(r) = 2$. □

Lemma 2.4. *For all* $\alpha, \beta, \gamma, \delta \in A_2(r)$*, if* $\beta\gamma\delta \in A_3(r)$ *is the unique element in which* δ *appears, and* $\alpha \notin \{\beta, \gamma, \delta\}$*, then either* $\alpha\beta\gamma \in A_3(r)$ *or* $\beta\beta\alpha, \gamma\gamma\alpha \in A_3(r)$*. Furthermore,* $\beta, \gamma \in M_2(r)$ *unless* $n \leq 4$*.*

Proof. Let $(x, y) \in R_\delta$ and $z \in X\backslash\{x, y\}$. Then, by the assumption, $[\{x, y, z\}] = \beta\gamma\delta$, and hence,

$$(r(x, z), r(y, z)) \in \{(\beta, \gamma), (\gamma, \beta)\}. \tag{2.1}$$

For all distinct $z, w \in X$ with $r(z, w) = \alpha$, we have $\{z, w\} \cap \{x, y\} = \emptyset$ by the assumption of $\alpha \notin \{\beta, \gamma, \delta\}$. Thus, $([\{x, z, w\}], [\{y, z, w\}])$ is either

$$(\alpha\beta\gamma, \alpha\beta\gamma) \text{ or } (\alpha\beta\beta, \alpha\gamma\gamma).$$

Moreover, if $n \geq 5$, then $\beta, \gamma \in M_2(r)$ by (2.1) and $|X \setminus \{x, y\}| \geq 3$. □

3. Proof of the first step of the induction

Throughout this section we assume that (X, r) is a finite colored space with $a_2(r) = a_3(r) = 4$.

Lemma 3.1. *For all distinct* $\alpha, \beta, \gamma \in A_2(r)$*, if* $\{\alpha, \beta, \gamma\}$ *is closed, then*

$$\alpha\delta\delta, \beta\delta\delta, \gamma\delta\delta \in A_3(r),$$

where δ *is the unique element in* $A_2(r) \setminus \{\alpha, \beta, \gamma\}$*.*

Proof. By Lemma 2.1, $(X, E_\alpha \cup E_\beta \cup E_\gamma)$ is the disjoint union of complete graphs. Since (X, E_δ) is its complement, each element in the conclusion can be realized as a triangle in this configuration. □

Lemma 3.2. *Suppose that $\alpha, \beta \in A_2(r)$ satisfy the following*:

(i) *$\{\alpha, \beta\}$ is closed and $\alpha \neq \beta$,*
(ii) *We have $\alpha\alpha\beta \in A_3(r)$,*
(iii) *There exists an element in $A_3(r) \setminus \{\alpha\alpha\beta\}$ that forms $\alpha_1\alpha_2\alpha_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \{\alpha, \beta\}$.*

Let Y be a connected component of $(X, E_\alpha \cup E_\beta)$ with $x_0, x_1, x_2 \in Y$,

$$r(x_0, x_1) = r(x_0, x_2) = \alpha,\ r(x_1, x_2) = \beta, \text{ and } x \in X \setminus Y.$$

Then $r(x_1, x) = r(x_2, x) \neq r(x_0, x) \notin M_2(r)$. In particular, $|X \setminus Y| = 1$ and

$$A_3(r) = \{\beta\beta\beta, \alpha\alpha\beta, \gamma\gamma\beta, \alpha\gamma\delta\},$$

where $\gamma = r(x_2, x)$ and $\delta = r(x_0, x)$.

Proof. We claim that $r(x_i, x) \neq r(x_0, x)$ for $i \in \{1, 2\}$. Suppose that $r(x_i, x) = r(x_0, x)$ for some $i \in \{1, 2\}$. Without loss of generality we may assume that $r(x_1, x) = r(x_0, x)$. Notice that

$$\delta\delta\alpha, \delta\gamma\alpha, \delta\gamma\beta \in A_3(r),$$

and they are distinct if $\gamma \neq \delta$, which contradicts $a_3(r) = 4$ by (ii) and (iii). On the other hand, if $\gamma = \delta$, then $\delta\delta\alpha, \delta\delta\beta \in A_3(r)$ are distinct, a contradiction to $a_2(r) = 4$ by (ii) and (iii).

Since $r(x_i, x) \in A_2(r) \setminus \{\alpha, \beta\}$ by (i), it follows from the above claim that

$$r(x_1, x) = r(x_2, x) \neq r(x_0, x), \text{ and } \gamma\gamma\beta, \gamma\delta\alpha \in A_3(r) \text{ are distinct.}$$

Therefore, we conclude from (ii) and (iii) that $\delta \notin M_2(r)$. Since x is arbitrarily taken in $X \setminus Y$, it follows from $\delta \notin M_2(r)$ that

$$|X \setminus Y| = 1.$$

This implies that

$$\alpha\alpha\alpha, \alpha\beta\beta \notin A_3(Y),$$

otherwise,

$$\alpha\gamma\gamma \in A_3(r) \text{ or } \beta\gamma\delta \in A_3(r),$$

a contradiction to $a_3(r) = 4$. Since $\gamma\gamma\beta, \gamma\delta\alpha \in A_3(r)$ are distinct, the last statement follows from (ii) and (iii) with $a_2(r) = a_3(r) = 4$. □

Lemma 3.3. *Suppose $m_3(r) > 0$ and there is no $\alpha \in A_2(r)$ such that $\alpha\alpha\alpha \in A_3(r)$. Then, for a suitable ordering of elements of $A_2(r)$, $A_3(r)$ equals one of the following*:

(i) *$\{\alpha\alpha\beta, \alpha\alpha\gamma, \alpha\alpha\delta, \beta\gamma\delta\}$ and $n \leq 8$,*
(ii) *$\{\alpha\alpha\beta, \alpha\alpha\gamma, \beta\beta\gamma, \alpha\alpha\delta\}$ and $n \leq 6$,*
(iii) *$\{\alpha\alpha\beta, \alpha\alpha\gamma, \beta\beta\gamma, \alpha\beta\delta\}$ and $n \leq 6$.*

Proof. Let $\alpha \in M_3(r)$ and $\{x, x_1, x_2, x_3\} \in \binom{X}{4}$ such that

$$r(x, x_i) = \alpha \text{ for } i \in [3].$$

Let $U = \{x_1, x_2, x_3\}$. Notice that, by the assumption,

$$\alpha \notin A_2(r_U) \text{ and } 1 < |A_2(r_U)| \leq 3.$$

If $|A_2(r_U)| = 3$, then we obtain the first case

$$A_3(r) = \{\alpha\alpha\beta, \alpha\alpha\gamma, \alpha\alpha\delta, \beta\gamma\delta\},$$

where $A_3(r) \setminus \{\alpha\} = \{\beta, \gamma, \delta\}$. Since $\{\beta, \gamma, \delta\}$ is closed and $\beta\gamma\delta$ is the unique element consisting of some of $\{\beta, \gamma, \delta\}$, it follows from [3, Theorem 3.1] and $\alpha\alpha\alpha \notin A_3(r)$ that $(X, E_\beta \cup E_\gamma \cup E_\delta)$ has exactly two connected components of size at most four. Therefore, $n \leq 8$.

Suppose $A_2(r_U) = \{\beta, \gamma\}$, where α, β, γ are distinct. Without loss of generality we may assume that

$$r(x_1, x_2) = r(x_2, x_3) = \beta, r(x_1, x_3) = \gamma.$$

Note that

$$\alpha\alpha\beta, \alpha\alpha\gamma, \beta\beta\gamma \in A_3(r) \text{ are distinct},$$

and the unique element $\delta \in A_2(r) \setminus \{\alpha, \beta, \gamma\}$ appears in the unique element in $A_3(r) \setminus \{\alpha\alpha\beta, \alpha\alpha\gamma, \beta\beta\gamma\}$. We shall write the unique element in $A_3(r)$ as $\mu\nu\delta$ where $\mu, \nu \in \{\alpha, \beta, \gamma, \delta\}$.

We claim that $\alpha \in \{\mu, \nu\}$. Otherwise, $\{\beta, \gamma, \delta\}$ is closed, by Lemma 3.1,

$$\alpha\alpha\delta \in A_3(r),$$

a contradiction.

Without loss of generality we may assume that $\mu = \alpha$ and $\nu \in \{\alpha, \beta, \gamma, \delta\}$. Then $\{\beta, \gamma\}$ is closed for each case. Since $\beta\beta\gamma$ is the unique element of $A_3(r)$ consisting of β and γ, it follows from [3, Theorem 3.1] that each connected component of $(X, E_\beta \cup E_\gamma)$ has size at most four.

If $\nu = \alpha$, then we obtain the second case. Since $\{\beta, \gamma, \delta\}$ is closed, it follows that $(X, E_\beta \cup E_\gamma \cup E_\delta)$ has exactly two connected components, one of which contains $\beta\beta\gamma$, and the other of which consists of two vertices with color δ. Therefore, $n \leq 6$.

From now on we assume that $\nu \neq \alpha$. We claim $(X, E_\beta \cup E_\gamma)$ has exactly two connected components. Clearly, it has more than one connected component. If y is an element of X belonging to neither of the connected components containing x nor x_1, then

$$r(y, x_i) = \alpha \text{ for some } i \in [3],$$

otherwise, $\delta\delta\beta \in A_3(r)$, a contradiction. Since $\alpha\alpha\alpha \notin A_3(r)$, it follows that $r(x, y) = \delta$, which implies that $\alpha\alpha\delta \in A_3(r)$, a contradiction to $\nu \neq \alpha$.

Since $\alpha\nu\delta \in A_3(r)$, it follows from the claim that one edge with color δ lies between the two connected components and $\nu \in \{\beta, \gamma\}$. Notice that $\alpha\beta\delta$ and $\alpha\gamma\delta$ are distinct so that only one of them belonged to $A_3(r)$. This implies that each connected component has size at most three and $|R_\delta(x_2)| = 1$. Therefore, $\nu = \beta$ and (iii) holds.

This completes the proof. □

Example 3.1. Let $\{Y, Z\}$ be a bipartition of X with $|Y| = |Z| = 4$. We set $E_\alpha = (Y, Z)$, where (Y, Z) is defined to be the set of edges between Y and Z, and E_β, E_γ, E_δ to be three disjoint perfect matching disjoint from E_α. Then each subset W of X with $|W| \geq 4$ and $|W \cap Y| \neq 2$ satisfies

$$A_3(W) = \{\alpha\alpha\beta, \alpha\alpha\gamma, \alpha\alpha\delta, \beta\gamma\delta\}.$$

Example 3.2. Let $\{Y, Z\}$ be a bipartition of X with $|Y| = 4$ and $|Z| = 2$. We set

$$E_\alpha = (Y, Z), \text{ and } E_\gamma \text{ to be a perfect matching on } Y,$$
$$E_\beta = \binom{Y}{2} \setminus E_\gamma,\ E_\delta = \binom{Z}{2}.$$

Then each subset W of X with $|W| \geq 5$ and $Z \subseteq W$ satisfies

$$A_3(W) = \{\alpha\alpha\beta, \alpha\alpha\gamma, \beta\beta\gamma, \alpha\alpha\delta\}.$$

Example 3.3. Let $\{Y, Z\}$ be a bipartition of X with $|Y| = |Z| = 3$ and $(y_0, z_0) \in Y \times Z$. We set

$$E_\alpha = (Y, Z) \setminus \{y_0z_0\},\ E_\delta = \{y_0z_0\},\ E_\gamma = \binom{Y\setminus\{y_0\}}{2} \cup \binom{Z\setminus\{z_0\}}{2},$$

and E_β to be the remaining. Then each subset W of X with $4 \leq |W|$, $\{y_0, z_0\} \subseteq W$, and $|Y \cap W| \neq 2$ satisfies

$$A_3(W) = \{\alpha\alpha\beta, \alpha\alpha\gamma, \beta\beta\gamma, \alpha\beta\delta\}.$$

Lemma 3.4. *Suppose that there exists $\alpha \in A_2(r)$ such that*

$$\alpha\alpha\alpha \in A_3(r) \textit{ and } \{\alpha\} \textit{ is closed.}$$

Then, for a suitable ordering of elements of $A_2(r)$, $A_3(r)$ is one of the following:

(i) $\{\alpha\alpha\alpha, \alpha\beta\gamma, \alpha\gamma\delta, \alpha\beta\delta\}$ *and $n \leq 6$,*
(ii) $\{\alpha\alpha\alpha, \alpha\beta\beta, \gamma\gamma\alpha, \beta\gamma\delta\}$,
(iii) $\{\alpha\alpha\alpha, \alpha\beta\beta, \alpha\beta\gamma, \alpha\beta\delta\}$.

Proof. Let Y be a connected component of (X, E_α) containing distinct elements x_1, x_2, x_3, and let $x \in X \setminus Y$. Then

$$\alpha \notin \{r(x, x_i) \mid i \in [3]\}.$$

If $|\{r(x, x_i) \mid i \in [3]\}| = 3$, then we obtain the first case

$$A_3(r) = \{\alpha\alpha\alpha, \alpha\beta\gamma, \alpha\gamma\delta, \alpha\delta\beta\},$$

where $\{r(x, x_i) \mid i \in [3]\} = \{\beta, \gamma, \delta\}$. Since $\beta, \gamma, \delta \notin M_2(r)$,

$$|Y| = |R_\beta(x)| + |R_\gamma(x)| + |R_\delta(x)| \leq 3.$$

This implies that $n \leq 6$.

If $|\{r(x, x_i) \mid i \in [3]\}| < 3$, then there exists $\beta \in A_2(r)$ such that $\alpha\beta\beta \in A_3(r)$.

Suppose that $\{\alpha, \beta\}$ is closed. Applying Lemma 3.2 for $\alpha\beta\beta \in A_3(r)$ we obtain the second case

$$A_3(r) = \{\alpha\alpha\alpha, \alpha\beta\beta, \gamma\gamma\alpha, \beta\gamma\delta\},$$

where $A_2(r) \setminus \{\alpha, \beta\} = \{\gamma, \delta\}$.

Suppose that $\{\alpha, \beta\}$ is not closed. Note that there is no $\gamma \in A_2(r)$ such that

$$\alpha\alpha\gamma \in A_3(r)$$

since $\{\alpha\}$ is closed. Thus there exists $\gamma \in A_2(r) \setminus \{\alpha, \beta\}$ such that one of the following holds:

(i) $\alpha\beta\gamma \in A_3(r)$,
(ii) $\beta\beta\gamma \in A_3(r)$.

Notice that, for each case, $\{\alpha, \beta, \gamma\}$ is not closed by Lemma 3.1. This implies $A_3(r)$ contains $\delta\mu\nu$ such that $\mu, \nu \in \{\alpha, \beta, \gamma\}$, where $\{\mu, \nu\}$ is one of the following:

$$\{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta\}, \{\beta, \gamma\}, \{\gamma\}.$$

Applying Lemma 2.4 for the unique element $\mu\nu\delta \in A_3(r)$ including δ we can eliminate the case of $\gamma\gamma\delta \in A_3(r)$, and hence, $\gamma \notin M_2(r)$. Since $n \geq 5$ unless the first case of this lemma holds, it follows from Lemma 2.4 that $\gamma \notin \{\mu, \nu\}$, and the following case can be eliminated:

(i) $\alpha\beta\gamma, \alpha\beta\delta \in A_3(r)$, which is the third case,
(ii) $\alpha\beta\gamma, \beta\beta\delta \in A_3(r)$ does not occur since $\beta\beta\gamma \notin A_3(r)$,
(iii) $\beta\beta\gamma, \alpha\beta\delta \in A_3(r)$ does not occur since $\alpha\beta\gamma, \alpha\alpha\gamma \notin A_3(r)$,
(iv) $\beta\beta\gamma, \beta\beta\delta \in A_3(r)$ does not occur since $\beta\beta\beta \notin A_3(r)$ and $\{\alpha, \delta, \gamma\}$ is closed.

This completes the proof. □

Example 3.4. Let $\{Y, Z\}$ be a bipartition of X with $|Y| = |Z| = 3$. We set

$$E_\alpha = \binom{Y}{2} \cup \binom{Z}{2},$$

E_β, E_γ, and E_δ to be disjoint perfect matchings on X connecting Y and Z. Then each subset W of X with $|W| \geq 4$ and $|Y \cap W| \neq 2$ satisfies $A_3(W) = \{\alpha\alpha\alpha, \alpha\beta\gamma, \alpha\gamma\delta, \alpha\delta\beta\}$.

Lemma 3.5. *Suppose that there exists $\alpha \in A_2(r)$ such that $\alpha\alpha\alpha \in A_3(r)$ and $\{\alpha\}$ is not closed. Then, for a suitable ordering of elements of $A_2(r)$,*

$$A_3(r) = \{\alpha\alpha\alpha, \alpha\alpha\beta, \alpha\alpha\gamma, \alpha\alpha\delta\}.$$

Proof. Since $\{\alpha\}$ is not closed, there exists $\beta \in A_2(r) \setminus \{\alpha\}$ such that

$$\alpha\alpha\beta \in A_3(r).$$

Applying Lemma 3.2 with the assumption of

$$\alpha\alpha\alpha, \alpha\alpha\beta \in A_3(r)$$

we obtain that $\{\alpha, \beta\}$ is not closed. Thus, there exists $\gamma \in A_2(r) \setminus \{\alpha, \beta\}$ such that one of the following holds:

(i) $\alpha\alpha\gamma \in A_3(r)$,
(ii) $\alpha\beta\gamma \in A_3(r)$,
(iii) $\beta\beta\gamma \in A_3(r)$.

Notice that, for each case, $\{\alpha, \beta, \gamma\}$ is not closed, otherwise, we have $a_3(r) > 4$, a contradiction. This implies that $A_3(r)$ contains $\delta\mu\nu$ such that $\mu, \nu \in \{\alpha, \beta, \gamma\}$ where $\{\mu, \nu\}$ is one of the following:

$$\{\alpha\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta\}, \{\beta, \gamma\}, \{\gamma\}.$$

Applying Lemma 2.4 we can show the following cases never occur:

$$\gamma\gamma\delta \in A_3(r),\ \beta\beta\delta \in A_3(r) \text{ or } \beta\beta\gamma \in A_3(r),$$

and hence, $\gamma \notin M_2(r)$ and $\beta \notin M_2(r)$. Since $n \geq 5$ by $a_2(r) = a_3(r) = 4$ and the assumption on α, it follows from Lemma 2.4 that $\gamma \notin \{\mu, \nu\}$, and the following cases can be eliminated:

(i) $\alpha\alpha\gamma, \alpha\alpha\delta \in A_3(r)$, which is the same as in the statement,
(ii) $\alpha\alpha\gamma, \alpha\beta\delta \in A_3(r)$ does not occur since $\alpha\beta\gamma, \beta\beta\gamma \notin A_3(r)$,
(iii) $\alpha\beta\gamma, \alpha\alpha\delta \in A_3(r)$ does not occur since $\alpha\alpha\gamma \notin A_3(r)$,
(iv) $\alpha\beta\gamma, \alpha\beta\delta \in A_3(r)$ does not occur since $\beta \notin M_2(r)$.

This completes the proof. □

Theorem 3.6. *The statement of Theorem 1.3 holds when $a_2(r) = a_3(r) = 4$.*

Proof. Suppose $m_3(r) = 0$. Then $n = 9$ since $a_2(r) = 4$ and $9 \leq n$. This implies $m_2(r) = 4$, which contradicts Lemma 2.3.

Suppose $m_3(r) > 0$. Applying Lemma 3.3 with $9 \leq n$ we conclude that there exists $\alpha \in A_2(r)$ such that $\alpha\alpha\alpha \in A_3(r)$. By Lemmas 3.4 and 3.5, $A_3(r)$ is one of the following:

(i) $\{\alpha\alpha\alpha, \alpha\beta\beta, \alpha\gamma\gamma, \beta\gamma\delta\}$,
(ii) $\{\alpha\alpha\alpha, \alpha\beta\beta, \alpha\beta\gamma, \alpha\beta\delta\}$,
(iii) $\{\alpha\alpha\alpha, \alpha\alpha\beta, \alpha\alpha\gamma, \alpha\alpha\delta\}$.

(i) Let $z_1, z_2 \in X$ with $r(z_1, z_2) = \delta$ and Y be a connected component of (X, E_α) with size at least three. Since no element of $A_3(r)$ contains both α and δ, we have $\{z_1, z_2\} \cap Y = \emptyset$. Since $\delta \notin M_2(r)$,

$$r(y, z_i) \in \{\beta, \gamma\} \text{ for each } y \in Y.$$

Since $\alpha\beta\gamma \notin A_3(r)$,

$$r(y, z_i) \text{ is constant whenever } y \in Y.$$

Since β and γ are symmetric, we may assume that $r(y, z_1) = \beta$ for each $y \in Y$. Since $\beta\beta\delta \notin A_3(r)$, it follows that $r(y, z_2) = \gamma$ for each $y \in Y$. Thus, the partition induces the one given in (iii) of Theorem 1.3.

(ii) Since each element of $A_3(r)$ contains α, it follows from Lemma 2.1 that (X, E_α) has exactly two connected components, say Y and Z. Note that E_γ and E_δ are matchings on X and $E_\beta \cup E_\gamma$ is also a matching on X since no element of $A_3(r)$ contains both β and γ. Therefore, (X, r) is the one given in (ii) of Theorem 1.3.

(iii) Note that E_β, E_γ, and E_δ are matchings on X and $E_\beta \cup E_\gamma \cup E_\delta$ is also a matching on X since no element of $A_3(r)$ contains two of β, γ, and δ. Therefore, (X, r) coincides with the one given in (i) of Theorem 1.3. $\square$

4. Proof of our main results

For functions r, r_1 whose domain is $\binom{X}{2}$ we say that r_1 is a *fusion* of r if $\{r^{-1}(\alpha) \mid \alpha \in \mathsf{Im}(r)\}$ is a refinement of $\{r_1^{-1}(\alpha) \mid \alpha \in \mathsf{Im}(r_1)\}$, in other words, for each $\alpha \in \mathsf{Im}(r)$, there exists $\Gamma \subset \mathsf{Im}(r_1)$ such that $r^{-1}(\alpha) = \cup_{\beta\in\Gamma} r_1^{-1}(\beta)$.

Lemma 4.1. *Let (X, r) be a finite colored space with*

$$2 \le a_2(r),\ 0 < m_2(r),\ \text{and } 5 \le n.$$

Then there exists a fusion r_1 of r such that

$$a_2(r) - a_2(r_1) = 1 \text{ and } 1 \le a_3(r) - a_3(r_1).$$

Proof. Let $\{x_1, x_2, x_3, x_4\} \in \binom{X}{4}$ with $r(x_1, x_2) = r(x_2, x_3)$.

If $r(x_4, x_1) \ne r(x_4, x_3)$, then the identification of $r(x_4, x_1)$ and $r(x_4, x_3)$ is required since

$$[\{x_1, x_2, x_4\}] = [\{x_3, x_2, x_4\}] \text{ in } (X, r_1) \text{ but not in } (X, r).$$

If $r(x_4, x_1) = r(x_4, x_3) \ne r(x_1, x_2)$, then the identification of $r(x_1, x_2)$ and $r(x_1, x_4)$ is required since

$$[\{x_1, x_2, x_3\}] = [\{x_1, x_4, x_3\}] \text{ in } (X, r_1) \text{ but not in } (X, r).$$

If $r(x_4, x_1) = r(x_4, x_3) = r(x_1, x_2)$ and $r(x_1, x_3) \ne r(x_2, x_4)$, then the identification of $r(x_1, x_3)$ and $r(x_2, x_4)$ is required since

$$[\{x_1, x_2, x_3\}] = [\{x_2, x_1, x_4\}] \text{ in } (X, r_1) \text{ but not in } (X, r).$$

Since x_4 is taken arbitrarily, we may assume that, for every $x_5 \in X \setminus \{x_i \mid i \in [4]\}$, we have $r(x_5, x_1) = r(x_5, x_3) = r(x_1, x_2)$ and

$$r(x_2, x_4) = r(x_1, x_3) = r(x_2, x_5) = r(x_4, x_5).$$

If $r(x_1, x_2) \ne r(x_2, x_4)$, then the identification of $r(x_1, x_2)$ and $r(x_2, x_4)$ is required since

$$[\{x_1, x_2, x_3\}] = [\{x_2, x_4, x_5\}] \text{ in } (X, r_1) \text{ but not in } (X, r).$$

If $r(x_1, x_2) = r(x_2, x_4)$, then $a_2(r) = 1$, a contradiction. $\square$

Lemma 4.2. *Let (X, r) be a finite colored space with*

$$a_2(r) < \binom{n}{2},\ m_2(r) = 0,\ \text{and } 5 \le n.$$

Then there exists a fusion r_1 of r such that

$$a_2(r) - a_2(r_1) \le 2, \text{ and } a_2(r) - a_2(r_1) \le a_3(r) - a_3(r_1).$$

Proof. The condition $m_2(r) = 0$ implies that, for each $\alpha \in \mathsf{Im}(r)$, E_α is a matching on X. The condition $a_2(r) < \binom{n}{2}$ implies that there exists $\alpha \in \mathsf{Im}(r)$ such that $|r^{-1}(\alpha)| \geq 2$. Let

$$x_1x_2, y_1y_2 \in r^{-1}(\alpha) \text{ with } \{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset,$$

so that, for $i \in [2]$,

$$r(x_i, y_1) \neq r(x_i, y_2) \text{ and } r(x_1, y_i) \neq r(x_2, y_i).$$

If $|\{r(x_i, y_j) \mid i, j \in [2]\}| = 4$, then the identification of $r(x_1, y_1)$ and $r(x_2, y_2)$ is required since

$$[\{x_1, x_2, y_1\}] = [\{y_1, y_2, x_2\}] \text{ in } (X, r_1) \text{ but not in } (X, r).$$

If $|\{r(x_i, y_j) \mid i, j \in [2]\}| = 3$ so that we may assume $r(x_1, y_1) = r(x_2, y_2)$, then the identification of $r(x_1, y_2)$ and $r(x_2, y_1)$ is required since

$$[\{x_1, x_2, y_1\}] = [\{x_1, x_2, y_2\}] \text{ in } (X, r_1) \text{ but not in } (X, r).$$

If $|\{r(x_i, y_j) \mid i, j \in [2]\}| = 2$ so that we may assume

$$r(x_1, y_1) = r(x_2, y_2) \text{ and } r(x_1, y_2) = r(x_2, y_1),$$

then we take $z \in X \setminus \{x_1, x_2, y_1, y_2\}$, so that

$$|\{r(z, x_i), r(z, y_i) \mid i \in [2]\}| = 4 \text{ and}$$

$$\{r(z, x_i), r(z, y_i) \mid i \in [2]\} \cap \{r(x_1, x_2), r(x_1, y_1), r(x_1, y_2)\} = \emptyset.$$

Furthermore, the fusion r_1 of r obtained by identifying $r(z, x_1)$ with $r(z, y_2)$, and $r(z, x_2)$ with $r(z, y_1)$, is required since

$$[\{x_1, x_2, z\}] = [\{y_1, y_2, z\}] \text{ and } [\{x_1, y_1, z\}] = [\{x_2, y_2, z\}] \text{ in } (X, r_1)$$

but not in (X, r).

Since $|\{r(x_i, y_j) \mid i, j \in [2]\}| > 1$, this completes the proof. □

Theorem 4.3. *For each finite colored space* (X, r) *with* $5 \leq n$ *we have* $a_2(r) \leq a_3(r)$.

Proof. Suppose that $a_2(r) > a_3(r)$. If $a_2(r) = \binom{n}{2}$, then

$$a_3(r) = \binom{n}{3}, \text{ so that } a_2(r) \leq a_3(r) \text{ unless } n \leq 4.$$

Applying Lemmas 4.1 and 4.2 for (X, r) we obtain a fusion r_1 of r such that

$$a_2(r_1) = a_2(r) - 1 > a_3(r) - 1 \geq a_3(r_1) \text{ if } a_2(r) - a_2(r_1) = 1,$$

and

$$a_2(r_1) = a_2(r) - 2 > a_3(r) - 2 \geq a_3(r_1) \text{ if } a_2(r) - a_2(r_1) = 2.$$

Repeating this argument we have, for some positive integer k,

$$2 = a_2(r_k) > a_3(r_k) \text{ or } 3 = 2_3(r_k) > a_3(r_k),$$

which contradicts [3, Theorem 3.5]. □

Now we present a proof of Theorem 1.3.

Proof of Theorem 1.3. Use induction on $a_2(r)$. Theorem 3.6 proves the first step of the induction. Suppose $5 \le a_2(r)$.

We claim that $m_2(r) > 0$. Suppose that E_α is a matching on X for each $\alpha \in A_2(r)$. Then

$$\binom{n}{2} = \sum_{\alpha \in \mathsf{Im}(r)} |r^{-1}(\alpha)| \le a_2(r)\frac{n}{2},$$

and hence, $8 \le n-1 \le a_2(r)$. Therefore, by Lemmas 4.1 and 4.2 and Theorem 1.2, there exists a fusion r_1 of r such that

$$6 \le a_2(r_1) = a_3(r_1) < a_2(r).$$

By the inductive hypothesis, (X, r_1) induces a partition given in Theorem 1.3. However, it is impossible to obtain (X, r) with $m_2(r) = 0$ by separating one or two elements of $A_2(r_1)$ into two parts, a contradiction.

By the claim, we conclude from Lemma 4.1 that (X, r) is obtained by separating only one of $r_1^{-1}(\alpha)$ with $\alpha \in \mathsf{Im}(r_1)$ into two parts.

First, we claim that (X, r_1) does not induce a partition given in Theorem 1.3(iii). Suppose that (X, r_1) induces a partition given in Theorem 1.3(iii). For convenience we assume that

$$\mathsf{Im}(r_1) = \{\alpha, \beta, \gamma, \delta\},\ r_1^{-1}(\delta) = \{y, z\},$$
$$r_1^{-1}(\gamma) \text{ is the edges from } y \text{ to } U \text{ where } U := X \setminus \{y, z\},$$
$$r_1^{-1}(\beta) \text{ is the edges from } z \text{ to } U \text{ and } r_1^{-1}(\alpha) = \tbinom{U}{2}.$$

Notice that $r_1^{-1}(\delta)$ is not separated since it is only one edge. If $r_1^{-1}(\gamma)$ is separated into two parts, say $r^{-1}(\gamma_1)$ and $r^{-1}(\gamma_2)$, then

$$a_3(r) - a_3(r_1) \ge 2$$

since

$$\gamma_1\gamma_2\alpha,\ \gamma_1\beta\delta, \text{ and } \gamma_2\beta\delta$$

are non-isometric in (X, r). This implies

$$a_3(r) \ge a_3(r_1) + 2 = 6 > 5 = 1 + a_2(r_1) = a_2(r), \tag{4.1}$$

a contradiction to $a_2(r) = a_3(r)$. By symmetry of β and γ, we can show that $r_1^{-1}(\beta)$ is not separated. If $r_1^{-1}(\alpha)$ is separated into two parts, say $r^{-1}(\alpha_1)$ and $r^{-1}(\alpha_2)$, then $a_3(r) - a_3(r_1) \ge 2$ since $\alpha_1\beta\gamma$ and $\alpha_2\beta\gamma$ are not isometric in (X, r), and $a_3(r_U) \ge 2$ since $|U| \ge 5$ and $a_2(r_U) = 2$. Thus, we have the same contradiction as (4.1).

Second, we claim that, if (X, r_1) induces a partition given in Theorem 1.3(ii), then (X, r_1) induces a partition given in Theorem 1.3(ii). For convenience we assume that

$$\mathsf{Im}(r_1) = \{\alpha, \beta\} \cup \{\gamma_i \mid i \in [a_2(r) - 3]\},$$
$$r_1^{-1}(\alpha) = \tbinom{Y}{2} \cup \tbinom{Z}{2},\ \textstyle\bigcup_{i=1}^{a_2(r)-3} r^{-1}(\gamma_i) \text{ is a matching},$$
$$\text{and } r_1^{-1}(\beta) \text{ is the remaining edges.}$$

If $r_1^{-1}(\gamma_i)$ is separated, then (X, r) induces a partition given in Theorem 1.3(ii). If $r_1^{-1}(\beta)$ is separated into two parts, say $r^{-1}(\beta_1)$ and $r^{-1}(\beta_2)$, then one of them should be a matching disjoint from

$$\bigcup_{i=1}^{a_2(r)-3} r^{-1}(\gamma_i),$$

otherwise, the following non-isometric elements in (X, r) would induce a contradiction:

$$\alpha\beta_1\beta_1,\ \alpha\beta_2\beta_2, \text{ and } \alpha\beta_1\beta_2 \text{ or}$$
$$\alpha\beta_i\beta_i,\ \alpha\beta_i\beta_j,\ \alpha\beta_i\gamma_k, \text{ and } \alpha\beta_j\gamma_l \text{ for some } i, j, k, l \text{ with } i \neq j.$$

If $r_1^{-1}(\alpha)$ is separated into two parts, say $r^{-1}(\alpha_1)$ and $r^{-1}(\alpha_2)$, then we have a contradiction because of the following non-isometric elements:

$$\alpha_1\beta_i\beta_j,\ \alpha_2\beta_i\beta_j \text{ for some } i, j$$
$$\text{and } |A_3(r_Y) \cup A_3(r_Z)| \geq 2 \text{ since } \max\{|Y|, |Z|\} \geq 5.$$

It remains to eliminate the case where (X, r_1) induces a partition given in Theorem 1.3(i), and we shall prove that (X, r) induces a partition given in Theorem 1.3(i),(ii). For convenience we assume that

$$\mathsf{Im}(r_1) = \{\alpha\} \cup \{\gamma_i \mid i \in [a_2(r) - 2]\},$$
$$\textstyle\bigcup_{i=1}^{a_2(r)-2} r^{-1}(\gamma_i) \text{ is a matching, and}$$
$$r_1^{-1}(\alpha) \text{ is the remaining edges.}$$

If $r_1^{-1}(\gamma_i)$ is separated, then (X, r) induces a partition given in Theorem 1.3(i). Suppose $r_1^{-1}(\alpha)$ is separated into two parts, say $r^{-1}(\alpha_1)$ and $r^{-1}(\alpha_2)$. If $\alpha_i \notin M_2(r)$, then

$$\left(\bigcup_{i=1}^{a_2(r)-2} r^{-1}(\gamma_i)\right) \cup r^{-1}(\alpha_i)$$

is a matching, otherwise, the following non-isometric elements would induce a contradiction:

$$\alpha_i\alpha_j\alpha_j,\ \alpha_j\alpha_j\alpha_j,\ \alpha_j\alpha_j\gamma_k, \text{ and } \alpha_i\alpha_j\gamma_k \text{ for some } j, k \text{ with } i \neq j.$$

Thus, we may assume that $\alpha_i \in M_2(r)$ for $i \in [2]$.

Since $a_2(r) - 2 \geq 3$ and $a_2(r) = a_3(r)$, there exists k such that γ_k appears only once in $A_3(r)$.

We claim that $\gamma_k\alpha_i\alpha_i \notin A_3(r)$ for $i \in [2]$. Let $\{x, y\} \in E_{\gamma_k}$. If $\gamma_k\alpha_1\alpha_1 \in A_3(r)$, then all edges from $\{x, y\}$ to other belong to E_{α_1}, which implies

$$\gamma_i\alpha_1\alpha_1, \alpha_1\alpha_1\alpha_2 \in A_3(r) \text{ for each } i \in [a_2(r) - 2].$$

Since $9 \leq n$, we can take $V \in \binom{X}{4}$ with $V \cap \{x, y\} = \emptyset$ such that $A_2(r_V) \subseteq \{\alpha_1, \alpha_2\}$. This implies that $\alpha_i\alpha_i\alpha_i \in A_3(r)$ for some $i \in [2]$. Since $\alpha_2 \in M_2(r)$ and

$$|\{\alpha_1\alpha_1\alpha_2, \alpha_i\alpha_i\alpha_i\} \cup \{\gamma_j\alpha_1\alpha_1 \mid j \in [a_2(r) - 2]\}| = a_2(r),$$

it follows that $i = 2$. This implies that $\{\alpha_2\} \cup \{\gamma_j \mid j \in [a_2(r) - 2]\}$ is closed, and $(X, E_{\alpha_2} \cup \bigcup_i E_{\gamma_i})$ is the disjoint union of at least three complete graphs, so that $\alpha_1\alpha_1\alpha_1 \in A_3(r)$, a contradiction to $a_2(r) = a_3(r)$.

By the claim,

$$\gamma_k\alpha_1\alpha_2 \in A_3(r). \tag{4.2}$$

Since $\alpha_1, \alpha_2 \in M_2(r)$, $\alpha_i\alpha_i\beta_i \in A_3(r)$ for some $\beta_i \in A_2(r)$. Since these elements are non-isometric in (X, r), it follows from $a_2(r) = a_3(r)$ that each γ_i appears only once in $A_3(r)$. Therefore, we conclude from (4.2) that

$$\gamma_j\alpha_1\alpha_2 \in A_3(r) \text{ for each } j \in [a_2(r) - 2]. \tag{4.3}$$

Notice that

$$|\{\alpha_i\alpha_i\beta_i \mid i \in [2]\} \cup \{\gamma_j\alpha_1\alpha_2, \mid j \in [a_2(r) - 2]\}| = a_2(r).$$

It follows from (4.3) that

$$\beta_1, \beta_2 \in \{\alpha_1, \alpha_2\}.$$

Since $9 \leq n$, it follows from (4.3) that there exists $W \in \binom{R_{\alpha_i}(x)}{3}$ for some $i \in [2]$ such that $A_2(r_W) \subseteq \{\alpha_1, \alpha_2\}$. This implies that

$$\alpha_i\alpha_i\alpha_i \in A_3(r) \text{ for some } i \in [2].$$

Notice that $\{\alpha_i\}$ is closed, so that (X, E_{α_i}) is the disjoint union of complete graphs by Lemma 2.1, and it has exactly two connected components, otherwise, the following non-isometric elements would induce a contradiction:

$$\alpha_j\alpha_j\alpha_i,\ \alpha_j\alpha_j\alpha_j,\ \alpha_i\alpha_i\alpha_i \text{ where } i \neq j.$$

Therefore, (X, r) is a partition given in Theorem 1.3(ii). □

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