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Bounding the Search Number of Graph Products

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Abstract. In this paper, we provide results for the search number of the Cartesian product of graphs. We consider graphs on opposing ends of the spectrum: paths and cliques. Our main result determines the pathwidth of the product of cliques and provides a lower bound for the search number of the product of cliques. A consequence of this result is a bound for the search number of the product of arbitrary graphs G and H based on their respective clique numbers.

1. Introduction

Imagine that a security system has indicated the existence of a camouflaged, mobile intruder in some physical or computer network. How can a set of guards, or *searchers*, locate this intruder? Such a question can be considered using a *graph searching* model. In this type of model, an intruder can, at any time, move infinitely fast from vertex u to vertex v along any path that contains no searchers. To search a graph, it is necessary to formulate and execute a *search strategy*: a sequence of actions designed so that, upon their completion, all edges (and therefore vertices) of the graph have been *cleared* of the invisible intruder. In such strategies, three actions are permitted and each action may occur multiple times:

- place a searcher on a vertex;
- move a single searcher along an edge uv, starting at u and ending at v;

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- remove a searcher from a vertex.

An edge uv can be cleared of the invisible intruder in one of two ways: (i) at least two searchers are located at vertex u, and one of these searchers traverses uv to vertex v; (ii) at least one searcher is located at u, all edges incident with u, other than uv, have already been cleared of the intruder, and the searcher traverses the edge uv to vertex v. Naturally, the fundamental question is: what is the fewest number of searchers for which a search strategy exists? Using the terminology of [21], we call this parameter the *search number* of G and denote it by $s(G)$. The parameter has also been referred to as the edge-search number $es(G)$ (see [10], for example) and the sweep number $sw(G)$ (see [1], for example). In the literature, searching has been related to pebbling and thus to computer memory usage; it also has applications to assuring privacy when using bugged channels, to VLSI circuit design, and to clearing networks with brushes (see [1, 8, 9, 12, 15, 21]). The field of graph searching is rapidly expanding and in recent years new models, motivated by applications and foundational issues in computer science, have appeared.

Although the associated decision problem is NP-complete [14], the search number is known for many classes of graphs and bounds exist for graphs with particular properties (see [1, 5, 21], for example). However, very little is known about the search number of Cartesian products. The *Cartesian product* of graphs G and H, denoted $G \,\square\, H$, has vertex set $V(G \,\square\, H) = V(G) \times V(H)$, and $(u_1, v_1)(u_2, v_2) \in E(G \,\square\, H)$ if and only if (1) $u_1u_2 \in E(G)$ and $v_1 = v_2$, or (2) $u_1 = u_2$ and $v_1v_2 \in E(H)$. In 1987, Tošić [20] provided an upper bound for the search number of $G \,\square\, H$ based on the respective cardinalities and search numbers of G and H. In 1992, Kinnersley [11] showed $pw(G) = vs(G)$, where $pw(G)$ denotes the pathwidth (defined below) and $vs(G)$ the vertex separation number of a graph G. In 1994, Ellis et al. [6] showed $vs(G) \le s(G) \le vs(G) + 2$. For the Cartesian product $G \,\square\, H$, these results imply

$$pw(G \,\square\, H) \le s(G \,\square\, H) \le pw(G \,\square\, H) + 2. \tag{1.1}$$

However, as the associated decision problem for pathwidth is NP-complete, the lower bound is not necessarily useful in practice.

In this paper, we consider input graphs at opposing ends of the spectrum: paths and cliques. In Section 2, we determine $s(P_m \,\square\, P_n)$ and $s(K_m \,\square\, P_n)$. In Section 3, we determine $pw(K_m \,\square\, K_n)$ and exploit the relationship between the search number and pathwidth to show

$$s(G \,\square\, H) \ge s(K_m \,\square\, K_n) \ge pw(K_m \,\square\, K_n) = \begin{cases} \frac{m}{2}n + \frac{m}{2} - 1 & \textit{if } m \textit{ even} \\ \lceil \frac{m}{2} \rceil n - 1 & \textit{if } m \textit{ odd} \end{cases} \tag{1.2}$$

where m, n are the clique numbers of G, H, respectively. Inequality (1.2) is given by Corollary 3.12 and results from applying Corollary 3.1, Lemma 3.2, and Corollary 3.10.

To conclude this section, we define the pathwidth of a graph G and state a simple, but useful, lemma.

Definition 1.1. A *path decomposition* of a graph G is a sequence of subsets of vertices $(B_1, B_2, \dots, B_r)$ such that

(i) $\bigcup_{1 \le i \le r} B_i = V(G)$;

(ii) For all edges $vw \in E(G)$, $\exists\, i \in \{1, 2, \dots, r\}$ with $v \in B_i$ and $w \in B_i$;

(iii) For all $i, j, k \in \{1, 2, \dots, r\}$, if $i \le j \le k$ then $B_i \cap B_k \subseteq B_j$.

The *width* of a path decomposition $(B_1, B_2, \dots, B_r)$ is $\max_{1 \le i \le r} |B_i| - 1$, and the *pathwidth* of G, denoted $pw(G)$, is the minimum width over all possible path decompositions of G.

See the survey [2] for more on pathwidth; the convention is to refer to subsets $B_1, B_2, \dots, B_r$ as *bags*. It can easily be seen that an equivalent statement of (iii) is:

Fact 1.2. For each $v \in V(G)$, the set of bags $\{B_i \mid v \in B_i \text{ and } 1 \le i \le r\}$ must form a subpath in the decomposition;

i.e. the indices of the set of bags are an interval. (The important point here is that the subpath is, by definition, connected.)

To avoid confusion between a path of vertices in a graph and a path of bags in a path decomposition, we will refer to a path of bags as a *bag-path*. The next result will be used in Section 3 with respect to products of cliques. Though the original results are stated for tree decompositions, they obviously apply to path decompositions. A short proof of the result for tree decompositions exists in [4], but the authors state that earlier proofs exist in [3, 18].

Lemma 1.3.([4]) *Consider a path decomposition $(B_1, B_2, \dots, B_r)$ of graph G, for some positive integer r. Let $W \subseteq V(G)$ be a clique in G. Then $W \subseteq B_i$, for some $1 \le i \le r$.*

2. Search Number of $P_m \,\square\, P_n$ and $P_m \,\square\, K_n$

Ellis and Warren [7] proved that for $m \ge n$, $pw(P_m \,\square\, P_n) = n$ which by Inequality (1.1) implies $s(P_m \,\square\, P_n) \in \{n, n+1, n+2\}$. In this section, we determine $s(P_m \,\square\, P_n)$ exactly. The notion of a search strategy was described in Section 1 as a sequence of actions designed so that once completed, all edges (and therefore vertices) of the graph have been cleared of the invisible intruder. We note that during the search strategy, recontamination of cleared edges may occur. However, if a search strategy exists for a connected graph, once every searcher has been placed on the graph, only the action of moving a searcher along an edge is required for the remainder of the search strategy (i.e. instead of removing a searcher from a vertex x and placing it on a vertex y, the searcher could move along a path from x to y). Thus, if a search strategy exists for a connected graph, then the graph can be cleared by placing the searchers at a set of vertices and then, at each time step, moving one searcher along an edge. This approach is sometimes called internal searching in the literature and we use it in the proof of Lemma 2.1. Additionally, at a given time step, any edge that is not clear is considered to be *dirty* and, if it can be recontaminated, then it is recontaminated instantly.

Lemma 2.1. *For $n \geq 3$, $s(P_n \,\square\, P_n) \geq n+1$.*

Proof. Let $n \geq 3$ and label the vertices of $P_n \,\square\, P_n$ as $v_{i,j}$ for $1 \leq i,j \leq n$. For a contradiction, suppose there exists a search strategy for $P_n \,\square\, P_n$ that uses n searchers. Let R_i be the subgraph induced by $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$ and C_j be the subgraph induced by $\{v_{1,j}, v_{2,j}, \ldots, v_{n,j}\}$; we informally refer to the subgraphs R_i and C_j as *row* i and *column* j, respectively. Let t be the last time step for which

(i) at the end of step $t-1$, at least one edge of each of R_i and C_i is dirty for all $i \in [n]$, and

(ii) at the end of step t, every edge of C_k is clear for some $k \in [n]$. (Note that once C_k is cleared, it never becomes dirty again.)

Certainly there must be such t in order for there to exist a search strategy of $P_n \,\square\, P_n$. Suppose that for some $x \in [n]$, R_x does not contain a searcher at the end of step t. Then as R_x contains a dirty edge, it is recontaminated, and so the edge of C_x incident with $v_{x,k}$ is too, contradicting (i). Therefore, at the end of step t, every row contains at least one searcher.

From (i) and (ii), we conclude that a searcher moves wlog from $v_{i+1,k}$ to $v_{i,k}$ during step t for some $i \in [n-1]$. If $i > 1$, a searcher must be located at $v_{i,k}$ immediately prior to step t because edge $(v_{i,k}, v_{i+1,k})$ was dirty but edge $(v_{i-1,k}, v_{i,k})$ was clean. Therefore, at the end of step t, there are two searchers located at $v_{i,k}$ and all other rows contain at least one searcher: $s(P_n \,\square\, P_n) \geq n+1$. To complete the proof, we assume $i = 1$ and let $t' > t$ be the time step during which a second row or column is cleared.

Claim 1: After step t and before step t', no searcher can move from one row to another.

Since only n searchers are available, there is exactly one searcher in each row at the end of step t. Suppose that after step t and before step t', a searcher moves from row j to row $j+1$ or $j-1$. Since R_j contains a dirty edge (by (i) and (ii)) but no searcher, any clear edges in R_j become recontaminated along with the two edges of C_k incident with $v_{j,k} \in R_j \cap C_k$. Claim 1 has been proven.

Claim 2: At the end of step t, every edge in R_1 is dirty.

During step t, a searcher moves from $v_{k,2}$ to $v_{k,1}$ and at the end of step t, there is exactly one searcher in each row. Then at the end of step $t-1$, there is no searcher in R_1 (else there are $n+1$ searchers) and, by (i), edge $(v_{1,k}v_{2,k})$ is dirty. Thus, every edge in R_1 is dirty at the end of step $t-1$ and also at step t. Claim 2 has been proven.

To conclude the proof, we consider two cases: $k \in \{2, 3, \ldots, n-1\}$ and $k \in \{1, n\}$.

Case 1: Suppose $k \in \{2, 3, \ldots, n-1\}$. For $j \in [n]$, let s_j be the searcher in R_j at the end of step t. At the end of step t, s_1 is located at $v_{1,k}$ and by Claim 2, every edge of R_1 is dirty. Since every edge in R_1 is dirty, every vertex of $R_2 \backslash \{v_{2,k}\}$ is incident with a dirty edge. As there is only one searcher in R_2, s_2 must be located at $v_{2,k}$ (otherwise C_k is recontaminated via $v_{2,k}$). By repeating this argument, we find that s_i must be located at $v_{j,k}$ for each $j \in [n]$. Then each searcher is located at a vertex incident with at least two dirty edges, and so no searcher can move at step $t+1$ without said move resulting in recontamination of at least two edges of C_k. Therefore, $s(P_n \,\square\, P_n) \geq n+1$.

Case 2: Suppose $k \in \{1, n\}$ and wlog assume $k = 1$. Then during step t, searcher s_1 moves from $v_{2,1}$ to $v_{1,1}$. By Claim 2, at the end of step t, edge $(v_{1,2}, v_{1,3})$ is dirty. Then adjacent edge $(v_{1,2}, v_{2,2})$ is also dirty at the end of step $t-1$. Thus, after step t and before step t', s_1 may move to $v_{1,2}$, but cannot move elsewhere by Claim 1 (and because $v_{1,2}$ has at least two incident dirty edges). Thus, at the end of step $t'-1$, s_1 is located at either $v_{1,1}$ or $v_{1,2}$ and edge $(v_{1,2}, v_{2,2})$ is dirty. Similarly, at the end of step $t'-1$ for $j \in \{2, 3, \ldots, n-2\}$, if s_j is located at a vertex of $\{v_{j,1}, v_{j,2}, \ldots, v_{j,i}\}$ then edges $(v_{j,j}, v_{j,j+1})$ and $(v_{j,j}, v_{j-1,j})$ are dirty. To prevent recontamination of the edges in C_1, searcher s_{j+1} must be located at a vertex of $\{v_{j+1,1}, v_{j+1,2}, \ldots, v_{j+1,j+1}\}$ at the end of step $t'-1$.

Note that searcher s_n cannot be located at $v_{n,n}$ at the end of step $t'-1$; otherwise R_n would be clear before step t'. Thus, s_n is located on one of $\{v_{1,n}, v_{2,n}, \ldots, v_{n-1,n}\}$ at the end of step $t'-1$. As no searcher is located in C_n at the end of step $t'-1$ and C_n contains at least one dirty edge, every edge of C_n is dirty at the end of step $t'-1$.

For R_j to be clear by the end of step t', some searcher s_j must move from $v_{j,n-1}$ to $v_{j,n}$. Thus $j = n-1$ or $j = n$ since, for $j < n-1$, s_j cannot be located at $v_{j,n-1}$ at step $t'-1$. Since edges $(v_{n-1,n-1}, v_{n-2,n-1})$ and $(v_{n-1,n-1}, v_{n-1,n})$ are both dirty at step $t'-1$, we note that $j \neq n-1$ (otherwise, edges of C_1 are recontaminated). Therefore, at step t', s_n must move from $v_{n,n-1}$ to $v_{n,n}$, and R_n is clear at the end of step t'. This implies that at the end of step $t'-1$, searcher s_j must be located at $v_{j,j}$, for $2 \leq j \leq n-1$ (otherwise, edges of C_1 are recontaminated). Recall that $(v_{j,j}, v_{j-1,j})$, $(v_{j,j}, v_{j,j+1})$ are both dirty for $j \in \{2, 3, \ldots, n-1\}$ at the end of step $t'-1$ (and therefore t'). So none of $s_2, s_3, \ldots, s_n$ can move at step $t'+1$ without recontamination of some edges of C_1.

Note that s_n could move from $v_{n,n}$ to $v_{n-1,n}$ at step $t'+1$ (or $t'+2$). However, this results in s_n becoming incident with two dirty edges $(v_{n-1,n-1}, v_{n-1,n})$, $(v_{n-2,n}, v_{n-1,n})$. If s_1 is located at $v_{1,1}$, then s_1 can now move to $v_{1,2}$ at step $t'+1$ (or $t'+2$). However, this results in s_1 being incident with two dirty edges and consequently, all searchers are incident with at least two dirty edges. So no searcher can move after step $t'+2$ without recontaminating C_1.

Therefore, $s(P_n \,\square\, P_n) \geq n+1$. □

Lemma 2.2. *For $n \geq 3$ and a connected finite graph G, $s(G \,\square\, P_n) \leq |V(G| + 1$.*

Proof. Let G be a connected finite graph and label the vertex set of $G \,\square\, P_n$ as

$v_{i,j}$, for $1 \le i \le |V(G)|$ and $1 \le j \le n$. Place one searcher on each vertex of $\{v_{i,1} \ : \ 1 \le i \le |V(G)|\}$; we will refer to these searchers as "the first $|V(G)|$ searchers". The $|V(G)| + 1^{th}$ searcher clears the edges of the subgraph induced by $\{v_{i,1} \ : \ 1 \le i \le |V(G)|\}$. Then the first $|V(G)|$ searchers move from $v_{i,1}$ to $v_{i,2}$ for each $1 \le i \le |V(G)|$ and the $|V(G)| + 1^{th}$ searcher clears the edges of the subgraph induced by $\{v_{i,2} \ : \ 1 \le i \le |V(G)|\}$. Continuing in this manner, we find $|V(G)|+1$ searchers sufficient to clear $G \,\square\, K_n$. $\square$

In [21], it was observed that if H is a minor of G, then $s(G) \ge s(H)$. Since K_m is a minor of $K_m \,\square\, P_n$, we observe $s(K_m \,\square\, P_n) \ge s(K_m) = m+1$. Let $\alpha = \min\{m,n\}$. As $P_\alpha \,\square\, P_\alpha$ is a minor of $P_m \,\square\, P_n$, we observe $s(P_m \,\square\, P_n) \ge s(P_\alpha \,\square\, P_\alpha) = \alpha + 1 = \min\{m,n\} + 1$ by Lemma 2.1. Applying Lemma 2.2 to achieve the upper bounds, the following theorem is immediate.

Theorem 2.3. *For $m, n \ge 3$, $s(P_m \,\square\, P_n) = \min\{m,n\} + 1$ and $s(K_m \,\square\, P_n) = m+1$.*

3. Pathwidth of the Product of Cliques

With respect to the search number of products of cliques, it was shown in [21] that $s(K_n \,\square\, K_2) = n+1$ for $n \ge 3$ and that, for $n \ge 1$, $m \ge 2$,

$$s(K_m \,\square\, K_n) \le n(m-1)+1. \tag{3.1}$$

In this section, we improve the above bound by a factor of a half. To do this, we consider the pathwidth of $K_m \,\square\, K_n$. Robertson and Seymour introduced the concepts of pathwidth [16] and treewidth [17] which played a fundamental role in their work on graph minors. Pathwidth is of interest to researchers because many intractable problems can be solved efficiently on graphs of bounded pathwidth.

Let $\omega(G)$, $\omega(H)$ denote the clique numbers of G, H, respectively. It was shown in [21] that $s(G) \ge s(H)$ when H is a minor of G; thus the following corollary is immediate.

Corollary 3.1. *For any graphs G and H,*

(a) $s(G \,\square\, H) \ge \max\{s(G \,\square\, K_{\omega(H)}), s(H \,\square\, K_{\omega(G)})\}$, *and*

(b) $s(G \,\square\, H) \ge s(K_{\omega(G)} \,\square\, K_{\omega(H)})$.

Corollary 3.1 with Inequality (1.1) yields the following relationship with pathwidth.

Lemma 3.2.

(a) *For any graphs G and H, $s(G \,\square\, H) \ge pw(K_{\omega(G)} \,\square\, K_{\omega(H)})$.*

(b) *For $n \ge 1$, $m \ge 2$, $pw(K_m \,\square\, K_n) \le s(K_m \,\square\, K_n) \le pw(K_m \,\square\, K_n) + 2$.*

For Lemma 3.2(a) to be useful, $pw(K_{\omega(G)} \square K_{\omega(H)})$ must be known. The remainder of this section is devoted to proving that for $n \geq m \geq 2$,

$$pw(K_m \square K_n) = \begin{cases} \frac{m}{2}n + \frac{m}{2} - 1 & \text{if } m \text{ even} \\ \lceil \frac{m}{2} \rceil n - 1 & \text{if } m \text{ odd.} \end{cases}$$

We first note that the treewidth of the product of two cliques of order $n \geq 3$ was determined in [13]: $tw(K_n \square K_n) = \frac{n^2}{2} + \frac{n}{2} - 1$. As treewidth forms a lower bound for pathwidth, the result of [13] provides a lower bound for $pw(K_n \square K_n)$, for $n \geq 3$. Seymour and Thomas [19] showed that construction of a bramble of size k proves $tw(G) \geq k - 1$ and, to determine the lower bound for $tw(K_n \square K_n)$, Lucena [13] constructed a bramble of order $\frac{n^2}{2} + \frac{n}{2}$. Although it seems a generalization of the bramble construction in [13] could be used to obtain a lower bound for $tw(K_m \square K_n)$, this would still only yield a lower bound for $pw(K_m \square K_n)$. Instead, we consider a direct approach to providing a lower bound for $pw(K_m \square K_n)$, without introducing brambles. In Section 3.2, we prove the upper bound for $pw(K_m \square K_n)$ and in Section 3.3, we state conclusions and implications of the upper and lower bounds.

The following notation is used in the remainder of this section: label the vertex set of $K_m \square K_n$ as $v_{i,j}$ for $1 \leq i \leq m$, $1 \leq j \leq n$. For any $i \in [m]$, the subgraph of $K_m \square K_n$ induced by vertices $\{v_{1,i}, v_{2,i}, \ldots, v_{m,i}\}$ is called an m-clique as it is a subgraph isomorphic to K_m. Similarly, for any $j \in [n]$, the subgraph of $K_m \square K_n$ induced by vertices $\{v_{j,1}, v_{j,2}, \ldots, v_{j,n}\}$ is called an n-clique.

3.1. Lower Bound for the Pathwidth of the Product of Cliques

Lemma 3.3. *For $n \geq 2$, $pw(K_2 \square K_n) \geq n$.*

Proof. For a contradiction, suppose $(B_1, B_2, \ldots, B_r)$ is a path decomposition where $\max_{1 \leq i \leq r} |B_i| \leq n$ for some $n \geq 2$. By Lemma 1.3, there exists $i \in [r], j \in [r]$ such that bag B_i contains the n-clique $\{v_{1,1}, v_{1,2}, \ldots, v_{1,n}\}$ and B_j contains the n-clique $\{v_{2,1}, v_{2,2}, \ldots, v_{2,n}\}$. Certainly, $i \neq j$ (else $|B_i| \geq 2n$), so wlog assume $i < j$.

Let B_x be the lowest-indexed bag that contains a pair of vertices of the form $v_{1,\alpha}, v_{2,\alpha}$, for any $\alpha \in [n]$. Clearly $i < x < j$ (else one of B_i, B_j contains $n+1$ vertices). As B_x contains at most $n-2$ vertices other than $v_{1,\alpha}, v_{2,\alpha}$, we observe $v_{1,\beta} \notin B_x$, for some $\beta \in [n]$. Therefore, the pair $v_{1,\beta}, v_{2,\beta}$ must appear together in a bag with higher index than B_x (by Definition 1.1(ii), $v_{1,\beta}, v_{2,\beta}$ must appear in some bag together). But then we do not have a path decomposition as the set of bags containing $v_{1,\beta}$ does not form a bag-path: $v_{1,\beta} \in B_i$, $v_{1,\beta} \notin B_x$, and $v_{i,\beta}$ is in a bag with higher index than B_x. $\square$

We next prove a simple, but useful, lemma.

Lemma 3.4. *Let S be a set containing $m \geq 3$ elements. Consider an ordered partition of S into at least three non-empty subsets, each of which contains strictly fewer than $\lceil \frac{m}{2} \rceil$ elements, and label the subsets of the ordered partition $S_1, S_2, \ldots, S_r$, for* *some integer $r \geq 3$. Then, for some $t \in \mathbb{N}$,*

$$1 \leq |S_1 \cup S_2 \cup \ldots S_{t-1}| \leq \left\lfloor \frac{m}{2} \right\rfloor, \ 1 \leq |S_t| \leq \left\lfloor \frac{m}{2} \right\rfloor, \ 1 \leq |S_{t+1} \cup S_{t+2} \cup \ldots S_r| \leq \left\lfloor \frac{m}{2} \right\rfloor.$$

Proof. Let t be the smallest integer for which $|S_1 \cup S_2 \cup \cdots \cup S_t| > \lfloor \frac{m}{2} \rfloor$. Then $1 \leq |S_1 \cup S_2 \cup \cdots \cup S_{t-1}| \leq \lfloor \frac{m}{2} \rfloor$. By the hypothesis, $1 \leq |S_t| < \lceil \frac{m}{2} \rceil$. As a result, $1 \leq |S_t| \leq \lfloor \frac{m}{2} \rfloor$ as desired.

As $|S_1 \cup S_2 \cup \cdots \cup S_t| > \lfloor \frac{m}{2} \rfloor$, we know $|S_{t+1} \cup S_{t+2} \cup \cdots \cup S_r| \leq \lfloor \frac{m}{2} \rfloor$. It remains to show that $1 \leq |S_{t+1} \cup S_{t+2} \cup \cdots \cup S_r|$. If m is even, then $|S_t| \leq \frac{m}{2} - 1 < \lceil \frac{m}{2} \rceil$, so $|S_1 \cup S_2 \cup \cdots \cup S_t| \leq \frac{m}{2} + \frac{m}{2} - 1 < m$. If m is odd, then $|S_t| \leq \lfloor \frac{m}{2} \rfloor$, so $|S_1 \cup S_2 \cup \cdots \cup S_t| \leq \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor < m$. Thus, $1 \leq |S_{t+1} \cup S_{t+2} \cup \cdots \cup S_r|$. □

In the remaining 2 proofs of this subsection, we will repeatedly apply the result of Lemma 1.3 to observe that in a path decomposition, every n-clique (and m-clique) must be contained in some bag.

Theorem 3.5. *For $n \geq m \geq 4$, $pw(K_m \square K_n) \geq \lceil \frac{m}{2} \rceil n - 1$.*

Proof. For a contradiction, suppose $(B_1, \ldots, B_r)$ is a path decomposition where $\max_{1 \leq i \leq r} |B_i| \leq \lceil \frac{m}{2} \rceil n - 1$. Let S be the set of m n-cliques in $K_m \square K_n$. Bags $B_1, B_2, \ldots, B_r$ form an ordered partition of S into non-empty subsets, each bag containing fewer than $\lceil \frac{m}{2} \rceil$ n-cliques (as each bag contains at most $\lceil \frac{m}{2} \rceil n - 1$ vertices). By Lemma 3.4, $X = B_1 \cup B_2 \cup \cdots \cup B_{t-1}$ contains i n-cliques, for some $i \in [\lfloor \frac{m}{2} \rfloor]$, $B = B_t$ contains j n-cliques, for some $j \in [\lfloor \frac{m}{2} \rfloor]$, and $Y = B_{t+1} \cup B_{t+2} \cup \cdots \cup B_r$ contains k n-cliques, for some $k \in [\lfloor \frac{m}{2} \rfloor]$.

Suppose wlog that $i \geq k$ and pair each n-clique in Y with a distinct n-clique in X. For instance, if $\{v_{b,1}, v_{b,2}, \ldots, v_{b,n}\}$ is an n-clique in Y, it is paired with some n-clique $\{v_{a,1}, v_{a,2}, \ldots, v_{a,n}\}$ in X. Every bag on the bag-path between X and Y must contain at least one of $v_{a,\ell}$, $v_{b,\ell}$ for each $\ell \in [n]$ (otherwise we contradict Fact 1.2 (or equivalently Definition 1.1(iii))). Since there are k pairings, there are at least kn vertices in B in addition to the jn vertices from the j n-cliques in B. So, $|B| \geq (j+k)n = (m-i)n$ as $i + j + k = m$ (the number of n-cliques). Note that $(m-i)n \leq |B| \leq \lceil \frac{m}{2} \rceil n - 1$ (the upper bound being the initial hypothesis) implies $i > \lfloor \frac{m}{2} \rfloor$, which contradicts the fact that $i \in [\lfloor \frac{m}{2} \rfloor]$. Therefore, B contains at least $\lceil \frac{m}{2} \rceil n$ vertices and $pw(K_m \square K_n) \geq \lceil \frac{m}{2} \rceil n - 1$. □

Given a minimum width path decomposition $(B_1, B_2, \ldots, B_r)$ of graph G, the *length* of the decomposition is r. The next result will be used to increase the lower bound of $pw(K_m \square K_n)$ for m even.

Lemma 3.6. *For even m and $n \geq m \geq 4$, suppose $pw(K_m \square K_n) \leq \frac{m}{2} n + \frac{m}{2} - 2$ and of the path decompositions of minimum width, let $(B_1, B_2, \ldots, B_r)$ be a decomposition of minimum length. Then for each $i \in [r]$, B_i contains fewer than $\frac{m}{2}$ n-cliques.*

Proof. For m even and $n \geq m \geq 4$, let $(B_1, B_2, \ldots, B_r)$ be a minimum length path decomposition for which $\max_{1 \leq i \leq r} |B_i| \leq \frac{m}{2} n + \frac{m}{2} - 1$. We first observe that every bag in the decomposition contains at most $\frac{m}{2}$ n-cliques; otherwise, some bag contains at least $(\frac{m}{2}+1)n = \frac{m}{2}n + n \geq \frac{m}{2}n + m > \frac{m}{2}n + \frac{m}{2} - 1$ vertices, which yields a contradiction.

Next, assume that for some $j \in [r]$, bag B_j contains exactly $\frac{m}{2}$ n-cliques. First, suppose there exists $i < j < k$ such that bags B_i, B_k each contain at least one n-clique that does not appear in B_j. Let $\{v_{\alpha,1}, v_{\alpha,2}, \ldots, v_{\alpha,n}\}$ be such an n-clique in B_i and $\{v_{\beta,1}, v_{\beta,2}, \ldots, v_{\beta,n}\}$ such an n-clique in B_k. Then, for each pair $v_{\alpha,s}, v_{\beta,s}$ with $s \in [n]$, at least one vertex of the pair must be in B_j (else we contradict Definition 1.1(ii) and (iii)). Then $|B_j| \geq \frac{m}{2}n + n > \frac{m}{2}n + \frac{m}{2} - 1$ which yields a contradiction.

Thus, wlog no bag of lower index than j contains an n-clique not already contained in B_j. However, then no bag of lower index than j contains an m-clique not already contained in B_j. Otherwise, for some $x < j$, B_x contains an m-clique and each of these m vertices must appear in a bag as part of its associated n-clique. Thus, each of the m vertices (of the m-clique of B_x) must appear in B_j. Since exactly $\frac{m}{2}$ of them already appear in B_j in an n-clique, this means $|B_j| \geq \frac{m}{2}n + \frac{m}{2}$, which yields a contradiction.

Thus B_j contains $\frac{m}{2}$ n-cliques, and no lower-indexed bag contains an n-clique or an m-clique. As each vertex must appear in a bag with its associated m-clique, every vertex in B_j must appear in B_{j+1}. We now have a contradiction as the minimum width decomposition is not of minimum length. Consequently, every bag in the decomposition contains strictly fewer than $\frac{m}{2}$ n-cliques. □

Theorem 3.7. *For $n \geq m \geq 4$ and m even, $pw(K_m \square K_n) \geq \frac{m}{2}n + \frac{m}{2} - 1$.*

Proof. Suppose $n \geq m \geq 4$ and m is even. For a contradiction, let $(B_1, B_2, \ldots, B_r)$ be a minimum length path decomposition for which $\max_{1 \leq i \leq r} |B_i| \leq \frac{m}{2}n + \frac{m}{2} - 1$. As a result of Lemma 3.6, we can apply Lemma 3.4; let S be the set of m n-cliques in $K_m \square K_n$. Let $X = B_1 \cup B_2 \cup \cdots \cup B_{t-1}$ contain i n-cliques for some $i \in [\frac{m}{2}]$, $B = B_t$ contain j n-cliques for some $j \in [\frac{m}{2}]$, and $Y = B_{t+1} \cup B_{t+2} \cup \cdots \cup B_r$ contain k n-cliques for some $k \in [\frac{m}{2}]$.

We now show that X and Y each must contain at least one m-clique that does not appear in B. To see this, suppose that X contains no m-clique: all m-cliques appear in $B \cup Y$. As each vertex must appear in a bag with its associated m-clique, it is clear that any vertex of X must also appear in B (else we contradict Definition 1.1(iii)). Then X is unnecessary in the path decomposition, which contradicts the assumption of having a minimum width path decomposition that is of minimum length. Clearly the same argument ensures X does not contain all the m-cliques. Consequently, X, Y each contain at least one m-clique.

Suppose wlog that $i \geq k$. We pair the k n-cliques in Y with k n-cliques in X. If $\{v_{a,1}, v_{a,2}, \ldots, v_{a,n}\}$ in X is paired with $\{v_{b,1}, v_{b,2}, \ldots, v_{b,n}\}$ in Y, then B must contain at least one of $v_{a,\ell}, v_{b,\ell}$, for each $\ell \in [n]$ (else we contradict Definition 1.1(iii)). Thus, $|B| \geq (j+k)n$.

Recall that X, Y must each contain at least one m-clique that does not appear in B; let $\{v_{1,x}, v_{2,x}, \ldots, v_{m,x}\}$ be such an m-clique in X and $\{v_{1,y}, v_{2,y}, \ldots, v_{m,y}\}$

such an m-clique in Y. At least one vertex from each pair $v_{\ell,x}, v_{\ell,y}$, for $\ell \in [m]$, must appear in B, and at most $j+k$ of these m vertices already appear in B. This leaves an additional $m-(j+k)=i$ vertices. Thus, $|B| \geq (j+k)n+i = (m-i)n+i = mn-in+i \geq \frac{m}{2}n+\frac{m}{2}$, as $1 \leq i \leq \frac{m}{2}$. However, this contradicts the initial assumption $pw(K_m \,\Box\, K_n) \leq \frac{m}{2}n+\frac{m}{2}-2$. □

3.2. Upper Bound for the Pathwidth of the Product of Cliques

We now provide upper bounds for the pathwidth of the product of cliques. Theorem 3.8 provides a general upper bound and Theorem 3.9 provides an improved upper bound for odd m. To illustrate the decomposition used in Theorem 3.8, we refer the reader to Figure 1, in which the bags B_1, B_2, B_3, and B_n are illustrated. We note that for simplicity, the edges of $K_m \,\Box\, K_n$ have been omitted in Figures 1, 2, and 3.

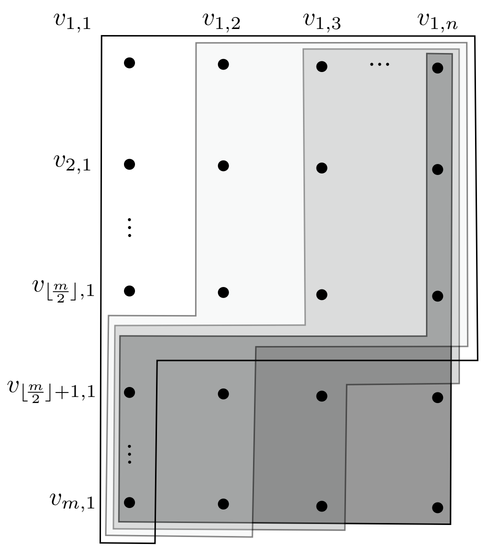

Figure 1: An illustration highlighting bags B_1, B_2, B_3, and B_n, following the decomposition of Theorem 3.8.

Theorem 3.8. *For $n \geq m \geq 2$, $pw(K_m \,\Box\, K_n) \leq \lceil \frac{m}{2} \rceil n + \lfloor \frac{m}{2} \rfloor - 1$.*

Proof. For $k \in [n]$, let $B_k = \bigcup_{i=1}^{\lfloor m/2 \rfloor} \{v_{i,k}, v_{i,k+1}, \ldots, v_{i,n}\} \,\cup \bigcup_{i=\lfloor m/2 \rfloor+1}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$.
Observe that each bag B_k contains at most $\lceil \frac{m}{2} \rceil n + \lfloor \frac{m}{2} \rfloor$ vertices. We now verify that $(B_1, B_2, \ldots, B_n)$ is a path decomposition (see Definition 1.1). Consider arbitrary vertex $v_{x,y} \in V(K_m \,\Box\, K_n)$. Clearly $v_{x,y} \in B_y$, so $(B_1, B_2, \ldots, B_n)$ satisfies condition (i) of Definition 1.1.

Let $v_{s,t}$ be a vertex adjacent to $v_{x,y}$. From the definition of the Cartesian product, either $s = x$ or $t = y$. If $t = y$ then obviously $v_{s,t} \in B_y$. Wlog $y < t$. If $s = x \geq \lfloor \frac{m}{2} \rfloor + 1$, then

$$v_{s,t} \in \bigcup_{i=\lfloor \frac{m}{2} \rfloor+1}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,y}\} \subseteq B_y.$$

If $s = x \leq \lfloor \frac{m}{2} \rfloor$, then

$$v_{s,t} \in \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{v_{i,y}, v_{i,y+1}, \ldots, v_{i,n}\} \subseteq B_y.$$

Thus, $(B_1, B_2, \ldots, B_n)$ satisfies condition (ii) of Definition 1.1.

To verify condition (iii) of Definition 1.1, we assume $v_{x,y} \in B_p$, $v_{x,y} \notin B_q$, and $v_{x,y} \in B_r$, for $1 \leq p < q < r \leq n$, and seek a contradiction. If $x \geq \lfloor \frac{m}{2} \rfloor + 1$, then because $p < q$,

$$v_{x,y} \in \bigcup_{i=\lfloor \frac{m}{2} \rfloor}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,p}\} \subseteq B_p \Rightarrow v_{x,y} \in \bigcup_{i=\lfloor \frac{m}{2} \rfloor}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,q}\} \subseteq B_q.$$

If $x \leq \lfloor \frac{m}{2} \rfloor$, then because $q < r$,

$$v_{x,y} \in \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{v_{i,r}, v_{i,r+1}, \ldots, v_{i,n}\} \subseteq B_r \Rightarrow v_{x,y} \in \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{v_{i,q}, v_{i,q+1}, \ldots, v_{i,n}\} \subseteq B_q.$$

Therefore, $(B_1, B_2, \ldots, B_n)$ satisfies condition (iii) of Definition 1.1. □

The final result of this subsection improves the upper bound for odd m, but the bags are not all formed in the same way. The bags are described by (3.2)–(3.4) in Theorem 3.9, but we illustrate the formation of the bags in Figures 2 and 3. For $k \in \{1, \ldots, \lceil \frac{n}{2} \rceil\}$, B_k is described by (3.2) and B_1, B_2, B_3 and $B_{\lceil \frac{n}{2} \rceil}$ are highlighted by different shades of grey in Figure 2. For $k \in \{\lceil \frac{n}{2} \rceil + 1, \ldots \lceil \frac{n}{2} \rceil + \lfloor \frac{m}{2} \rfloor\}$, B_k is described by (3.3) and for $k = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$, B_k is described by (3.4). Bags $B_{\lceil \frac{n}{2} \rceil+1}$, $B_{\lceil \frac{n}{2} \rceil+2}$, and $B_{\lceil \frac{n}{2} \rceil+\lfloor \frac{m}{2} \rfloor}$ are highlighted (in dark, medium, and light grey, respectively) in Figure 3, while $B_{\lceil \frac{n}{2} \rceil+\lceil \frac{m}{2} \rceil}$ is indicated with a dotted line.

Theorem 3.9. *For $n \geq m \geq 3$ and m odd, $pw(K_m \Box K_n) \leq \lceil \frac{m}{2} \rceil n - 1$.*

Proof. For $1 \leq k \leq \lceil \frac{n}{2} \rceil$, let

$$B_k = \bigcup_{i=1}^{\lfloor \frac{m}{2} \rfloor} \{v_{i,k}, v_{i,k+1}, \ldots, v_{i,n}\} \ \cup \bigcup_{i=\lceil \frac{m}{2} \rceil}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}, \tag{3.2}$$

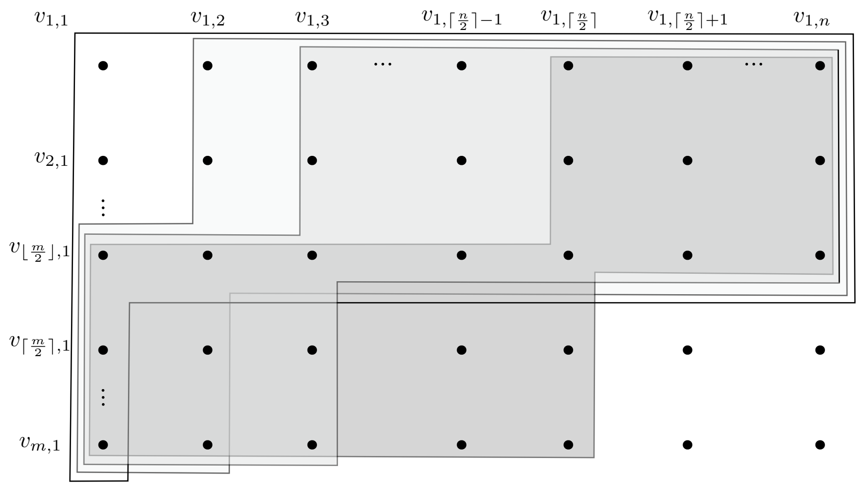

Figure 2: An illustration highlighting some bags described by (3.2) following the decomposition of Theorem 3.9.

for $\lceil\frac{n}{2}\rceil + 1 \leq k \leq \lceil\frac{n}{2}\rceil + \lfloor\frac{m}{2}\rfloor$, let

(3.3)
$$B_k = \bigcup_{i=k-\lceil\frac{n}{2}\rceil+\lfloor\frac{m}{2}\rfloor}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lceil\frac{n}{2}\rceil}\} \cup \bigcup_{i=1}^{k-\lceil\frac{n}{2}\rceil+\lfloor\frac{m}{2}\rfloor} \{v_{i,\lceil\frac{n}{2}\rceil+1}, v_{i,\lceil\frac{n}{2}\rceil+2}, \ldots, v_{i,n}\}$$

and, for $k = \lceil\frac{n}{2}\rceil + \lceil\frac{m}{2}\rceil$, let

$$B_k = \{v_{m,1}, v_{m,2}, \ldots, v_{m,\lceil\frac{n}{2}\rceil}\} \cup \bigcup_{i=1}^{m} \{v_{i,\lceil\frac{n}{2}\rceil+1}, v_{i,\lceil\frac{n}{2}\rceil+2}, \ldots, v_{i,n}\}. \tag{3.4}$$

We now verify that $(B_1, B_2, \ldots, B_{\lceil\frac{n}{2}\rceil+\lceil\frac{m}{2}\rceil})$ is a path decomposition. Consider an arbitrary vertex $v_{x,y} \in V(K_m \,\square\, K_n)$. If $1 \leq y \leq \lceil\frac{n}{2}\rceil$ then from (3.2), $v_{x,y} \in B_y$. If $\lceil\frac{n}{2}\rceil + 1 \leq y \leq n$ then from (3.4), $v_{x,y} \in B_{\lceil\frac{n}{2}\rceil+\lceil\frac{m}{2}\rceil}$. Thus, $(B_1, B_2, \ldots, B_{\lceil\frac{n}{2}\rceil+\lceil\frac{m}{2}\rceil})$ satisfies condition (i) of Definition 1.1.

Let $v_{s,t}$ be a vertex adjacent to $v_{x,y}$. From the definition of the Cartesian product, either $s = x$ or $t = y$. First, suppose $t = y$ and wlog $s < x$. If $1 \leq y \leq \lceil\frac{n}{2}\rceil$ then from (3.2), $v_{s,t}$, $v_{x,y}$ are both in B_y. If $\lceil\frac{n}{2}\rceil + 1 \leq y \leq n$, then $v_{s,y}, v_{x,y}$ are both in $B_{\lceil\frac{n}{2}\rceil+\lceil\frac{m}{2}\rceil}$. Second, suppose $s = x$ and wlog $t < y$. We consider the possible cases:

(i) Assume $x = m$. Then clearly $v_{x,y}, v_{s,t}$ are both in $B_{\lceil\frac{n}{2}\rceil+\lceil\frac{m}{2}\rceil}$ by (3.4).

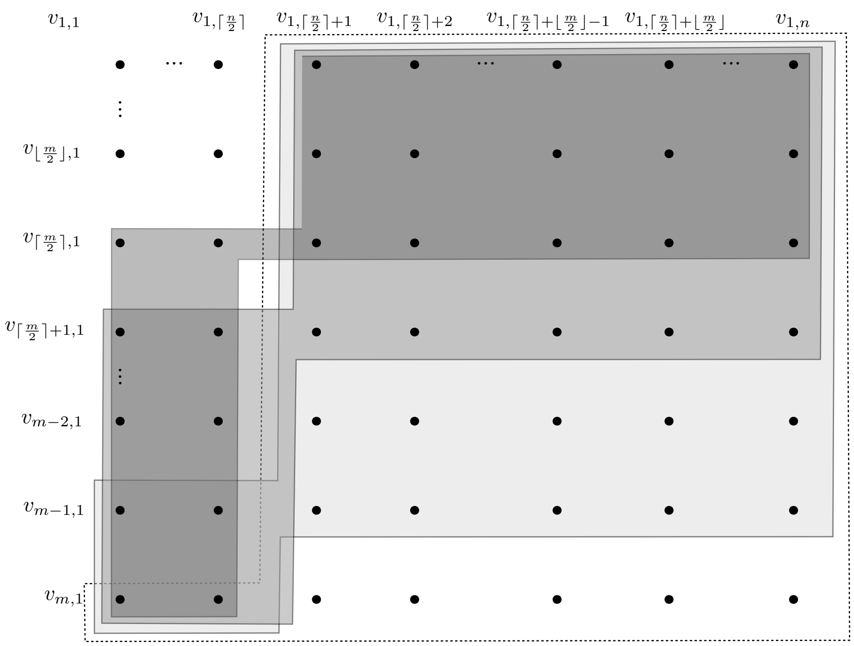

Figure 3: An illustration highlighting some bags described by (3.3) and (3.4) following the decomposition of Theorem 3.9.

(ii) Assume $\lfloor \frac{m}{2} \rfloor \leq x < m$. Then $v_{x,y}, v_{s,t}$ are both in $B_{x+\lceil \frac{n}{2}\rceil - \lfloor \frac{m}{2}\rfloor}$ by (3.3).

(iii) Assume $1 \leq x \leq \lfloor \frac{m}{2} \rfloor - 1$. If $1 \leq t \leq \lceil \frac{n}{2} \rceil$, then $v_{x,y}, v_{s,t}$ are both in B_t by (3.2). If $t > \lceil \frac{n}{2} \rceil$ then $v_{x,y}, v_{s,t}$ are both in $B_{\lceil \frac{n}{2}\rceil + \lceil \frac{m}{2}\rceil}$ by (3.4).

Therefore, $(B_1, B_2, \ldots, B_{\lceil \frac{n}{2}\rceil + \lceil \frac{m}{2}\rceil})$ satisfies condition (ii) of Definition 1.1.

To verify condition (iii) of the definition, we assume $v_{x,y} \in B_p$, $v_{x,y} \notin B_{p+1}$, and $v_{x,y} \in B_r$, for $1 \leq p < p+1 < r \leq n$, and seek a contradiction.

(a) Suppose $1 \leq p < p+1 \leq \lceil \frac{n}{2} \rceil$. Then by (3.2),

$$B_p \backslash B_{p+1} = \{v_{1,p}, v_{2,p}, \ldots, v_{\lfloor \frac{m}{2}\rfloor, p}\}.$$

If $r \leq \lceil \frac{n}{2} \rceil$ then, by (3.2), clearly no vertex in the set $\{v_{1,p}, v_{2,p}, \ldots, v_{\lfloor \frac{m}{2}\rfloor, p}\}$ is in B_r. Thus, $r \geq \lceil \frac{n}{2} \rceil + 1$. In this case, consider

$$B_{\lceil \frac{n}{2}\rceil+1} = \bigcup_{i=\lceil \frac{m}{2}\rceil}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lceil \frac{n}{2}\rceil}\} \cup \bigcup_{i=1}^{\lceil \frac{m}{2}\rceil} \{v_{i,\lceil \frac{n}{2}\rceil+1}, v_{i,\lceil \frac{n}{2}\rceil+2}, \ldots, v_{i,n}\}.$$

Clearly, for $1 \le p < p+1 \le \lceil \frac{n}{2} \rceil$, no vertex in the set $\{v_{1,p}, v_{2,p}, \ldots, v_{\lfloor \frac{m}{2} \rfloor,p}\}$ is in $B_{\lceil \frac{n}{2} \rceil+1}$.

(b) Suppose $1 \le p \le \lceil \frac{n}{2} \rceil < p+1 \le \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$. Then we may assume $p = \lceil \frac{n}{2} \rceil$ and $v_{x,y} \in B_{\lceil \frac{n}{2} \rceil}$, but $v_{x,y} \notin B_{\lceil \frac{n}{2} \rceil+1}$. In this case, using (3.2) and (3.3), we find

$$B_{\lceil \frac{n}{2} \rceil} \backslash B_{\lceil \frac{n}{2} \rceil+1} = \{v_{1,\lceil \frac{n}{2} \rceil}, v_{2,\lceil \frac{n}{2} \rceil}, \ldots, v_{\lfloor \frac{m}{2} \rfloor,\lceil \frac{n}{2} \rceil}\}.$$

It suffices to consider $r = \lceil \frac{n}{2} \rceil + 2$ to obtain a contradiction. If $m \ge 4$ then, from (3.3),

$$B_{\lceil \frac{n}{2} \rceil+2} = \bigcup_{i=\lceil \frac{m}{2} \rceil+1}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lceil \frac{n}{2} \rceil}\} \cup \bigcup_{i=1}^{\lceil \frac{m}{2} \rceil+1} \{v_{i,\lceil \frac{n}{2} \rceil+1}, v_{i,\lceil \frac{n}{2} \rceil+2}, \ldots, v_{i,n}\}$$

and, if $m = 3$, then from (3.4),

$$B_{\lceil \frac{n}{2} \rceil+2} = \{v_{m,1}, v_{m,2}, \ldots, v_{m,\lceil \frac{n}{2} \rceil}\} \cup \bigcup_{i=1}^{m} \{v_{i,\lceil \frac{n}{2} \rceil+1}, v_{i,\lceil \frac{n}{2} \rceil+2}, \ldots, v_{i,n}\}.$$

In either case, $(B_{\lceil \frac{n}{2} \rceil} \backslash B_{\lceil \frac{n}{2} \rceil+1}) \cap B_{\lceil \frac{n}{2} \rceil+2} = \emptyset$ and we have obtained a contradiction with $r = \lceil \frac{n}{2} \rceil + 2$. Thus, if $1 \le p \le \lceil \frac{n}{2} \rceil < q \le \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$, condition (iii) of Definition 1.1 is satisfied.

(c) Suppose $\lceil \frac{n}{2} \rceil + 1 \le p < p+1 < r \le \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$. Then $v_{x,y} \in B_p$ and $v_{x,y} \notin B_{p+1}$ and by (3.4),

$$B_p \backslash B_{p+1} = \{v_{p-\lceil \frac{n}{2} \rceil+\lfloor \frac{m}{2} \rfloor,1}, v_{p-\lceil \frac{n}{2} \rceil+\lfloor \frac{m}{2} \rfloor,2}, \ldots, v_{p-\lceil \frac{n}{2} \rceil+\lfloor \frac{m}{2} \rfloor,\lceil \frac{n}{2} \rceil}\}.$$

However, $(B_p \backslash B_{p+1}) \cap B_{p+2} = \emptyset$; to see this, we explicitly state B_{p+2} below.

If $p+2 < \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$ then, from (3.3),

$$B_{p+2} = \bigcup_{i=p+2-\lceil \frac{n}{2} \rceil+\lfloor \frac{m}{2} \rfloor}^{m} \{v_{i,1}, v_{i,2}, \ldots, v_{i,\lceil \frac{n}{2} \rceil}\} \cup \bigcup_{i=1}^{p+2-\lceil \frac{n}{2} \rceil+\lfloor \frac{m}{2} \rfloor} \{v_{i,\lceil \frac{n}{2} \rceil+1}, v_{i,\lceil \frac{n}{2} \rceil+2}, \ldots, v_{i,n}\}.$$

If $p+2 = \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$ then, from (3.4),

$$B_{p+2} = \{v_{m,1}, v_{m,2}, \ldots, v_{m,\lceil \frac{n}{2} \rceil}\} \cup \bigcup_{i=1}^{m} \{v_{i,\lceil \frac{n}{2} \rceil+1}, v_{i,\lceil \frac{n}{2} \rceil+2}, \ldots, v_{i,n}\}.$$

As $p < \lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil$, it is clear that no vertex of $B_p \backslash B_{p+1}$ appears in B_{p+2}. We have obtained a contradiction, using $r = p+2$. Thus, condition (iii) of Definition 1.1 is satisfied.

Therefore, $(B_1, B_2, \ldots, B_{\lceil \frac{n}{2} \rceil + \lceil \frac{m}{2} \rceil})$ forms a path decomposition.

As $n \ge m \ge 3$, counting the number of vertices in B_k for (3.2), (3.3), and (3.4) finds $|B_k| \le \lceil \frac{m}{2} \rceil n$. $\square$

3.3. The Pathwidth of the Product of Cliques

The following corollary is immediate from Lemma 3.3 and Theorems 3.5, 3.7–3.9.

Corollary 3.10. *For $n \geq m \geq 2$,*

$$pw(K_m \,\Box\, K_n) = \begin{cases} \frac{m}{2}n + \frac{m}{2} - 1 & \text{if } m \text{ even} \\ \lceil \frac{m}{2} \rceil n - 1 & \text{if } m \text{ odd.} \end{cases}$$

Our final results follow directly from Corollary 3.1(a), Lemma 3.2, and Corollary 3.10. Corollary 3.11 bounds the search number of the Cartesian product of cliques to within 2 and improves the bound of [21], given in Inequality (3.1), by half. Corollary 3.12 provides the lower bound for the search number of the product of two general graphs G and H.

Corollary 3.11. *For $n \geq m \geq 2$, if m is even, then*

$$\frac{m}{2}n + \frac{m}{2} - 1 \leq s(K_m \,\Box\, K_n) \leq \frac{m}{2}n + \frac{m}{2} + 1$$

and, if m is odd, then

$$\left\lceil \frac{m}{2} \right\rceil n - 1 \leq s(K_m \,\Box\, K_n) \leq \left\lceil \frac{m}{2} \right\rceil n + 1.$$

Corollary 3.12. *For $|V(H)| \geq |V(G)| \geq 4$, where the clique numbers of graphs G and H are m and n respectively,*

$$s(G \,\Box\, H) \geq \begin{cases} \frac{m}{2}n + \frac{m}{2} - 1 & \textit{if } m \textit{ even} \\ \lceil \frac{m}{2} \rceil n - 1 & \textit{if } m \textit{ odd.} \end{cases}$$

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