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Positive Solutions of Nonlinear Neumann Boundary Value Problems with Sign-Changing Green's Function

Mohammed Elnagi M. Elsanosi
Department of Mathematics, Faculty of Educations, University of Khartoum , Omdurman, Sudan
e-mail : mohdtoum@yahoo.com

Abstract. This paper is concerned with the existence of positive solutions of the nonlinear Neumann boundary value problems

$$\begin{cases} u'' + a(t)u = \lambda b(t)f(u), & t \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases}$$

where $a, b \in C[0,1]$ with $a(t) > 0, b(t) \geq 0$ and the Green's function of the linear problem

$$\begin{cases} u'' + a(t)u = 0, & t \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

may change its sign on $[0,1] \times [0,1]$. Our analysis relies on the Leray-Schauder fixed point theorem.

1. Introduction

Let $\lambda > 0$ be a parameter. We study the existence of positive solutions of the following nonlinear Neumann boundary value problems (NBVPs)

$$\begin{cases} u'' + a(t)u = \lambda b(t)f(u), & t \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $a, b \in C[0,1]$ with $a(t) > 0, b(t) \geq 0$, $f : [0,\infty) \to [0,\infty)$ is continuous with $f(0) > 0$.

In the past few years, several methods have been used to study the nonlinear second-order NBVPs

$$\begin{cases} u'' + m^2 u = f(t,u), & t \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.2}$$

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where $m \in (0, \frac{\pi}{2})$. See, for example, the fixed point theorem in cones [8, 10, 11, 12], Leray-Schauder alternative principle with truncation technique [2], topological degree [8], shooting method [1], sub-supersolution method [6] and the references therein.

It is worth remarking that the key condition used in these papers is $0 < m < \frac{\pi}{2}$, which guarantees the Green's function $K(t,s)$ is greater than 0 on $[0,1] \times [0,1]$, where

$$K(t,s) = \begin{cases} \frac{\cos m(1-t)\cos ms}{m \sin m}, & 0 \le s \le t \le 1, \\ \frac{\cos m(1-s)\cos mt}{m \sin m}, & 0 \le t \le s \le 1 \end{cases} \tag{1.3}$$

is the Green's function of the linear problem

$$\begin{cases} u'' + m^2 u = 0, & t \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.4}$$

Meanwhile, let

$$c := \min_{(t,s)\in[0,1]\times[0,1]} K(t,s), \quad C := \max_{(t,s)\in[0,1]\times[0,1]} K(t,s).$$

Define a cone

$$P := \{u \in C[0,1] : \min_{t\in[0,1]} u(t) \ge \frac{c}{C}\|u\|\}, \tag{1.5}$$

where $\|u\| = \max_{t\in[0,1]} u(t)$. Now, Krasnoselskii's fixed point theorem [3, 7] can be used to prove the existence and multiplicity of positive solutions of the nonlinear problem (1.2).

However, if $m = \frac{\pi}{2}$, then it is easy to check $K(t,s)$ is at least 0 and may attain zeros at some $t \in [0,1] \times [0,1]$. Thus we can not define the cone P as (1.5) since $c = 0$. In 2008, Graef, Kong and Wang [4], defined a new cone of the form

$$P_0 := \left\{ u \in C[0,1] : u(t) \ge 0 \text{ on } [0,1], \int_0^1 u(t)dt \ge C_0\|u\| \right\}$$

(where C_0 is some positive constant) to prove the existence of positive solutions. Motivated by above papers, the purpose here is to determine the values of λ for which there exists a positive solution of NBVPs (1.1) with sign-changing Green's function.

Our proof is based on the following Leray-Schauder fixed point theorem.

Lemma 1.1.([3, Leray-Schauder fixed point theorem]) *Let X be a Banach space and $T : X \to X$ a completely continuous operator. Suppose that there exists a constant $M > 0$, such that each solution $(x, \sigma) \in X \times [0,1]$ of*

$$x = \sigma T x, \qquad \sigma \in [0,1], \quad x \in X$$

satisfies $\|x\|_X \le M$. Then T has a fixed point.

Remark 1.1. For some results on the second-order periodic boundary value problems with sign-changing Green's function, we refer the readers to Ma [9].

2. Main Results

Let $C[0,1]$ be the Banach space composed of all continuous real functions defined on $[0,1]$, which is equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

We assume that:

(H1) $b \in C[0,1]$ with $b(t) \ge 0,\ t \in [0,1]$ and $a \in C[0,1]$ satisfies

$$\left(\frac{\pi}{2}\right)^2 \le \min_{t\in[0,1]} a(t) < \max_{t\in[0,1]} a(t) < \pi^2;$$

Remark 2.1. Condition (H1) implies that the Green's function $G(t,s)$ of the linear problem

$$\begin{cases} u'' + a(t)u = 0, & t \in (0,1), \\ u'(0) = u'(1) = 0 \end{cases}$$

exists and may change its sign (Notice that there exist a lot of functions of a such that (H1) holds; see Example 3.1 below.)

(H2) $f : [0,\infty) \to [0,\infty)$ is continuous with $f(0) > 0$;

(H3) There exists $h > 1$ such that

$$\int_0^1 G^+(t,s)b(s)ds \ge h \int_0^1 G^-(t,s)b(s)ds,$$

where G^+ and G^- are the positive and negative parts of G, respectively.

Throughout the paper, we assume that

$$f(u) = f(0), \qquad \text{for } u \le 0.$$

Lemma 2.1. *Suppose that (H1), (H2) and (H3) hold. Let $0 < \delta < 1$. Then there exists a positive number $\overline{\lambda}$ such that for $0 < \lambda < \overline{\lambda}$, the integral equation*

$$u(t) = \lambda \int_0^1 G^+(t,s)b(s)f(u(s))ds \tag{2.1}$$

has a positive solutions $\overline{u}_\lambda$ with $\|\overline{u}_\lambda\| \to 0$ as $\lambda \to 0$, and $\overline{u}_\lambda(t) \ge \lambda\delta f(0)p(t)$, where

$$p(t) = \int_0^1 G^+(t,s)b(s)ds.$$

Proof. The proof is motivated by Hai [5]. Let $A : C[0,1] \to C[0,1]$ defined by

$$(Au)(t) = \lambda \int_0^1 G^+(t,s)b(s)f(u(s))ds, \qquad t \in [0,1].$$

Then $A : C[0,1] \to C[0,1]$ is completely continuous and the fixed points of A are solutions of (2.1).

We shall apply Lemma 1.1 to prove that A has a fixed point for λ small. Let $\varepsilon > 0$ be such that

$$f(u) \geq \delta f(0) \qquad \text{for } \ 0 \leq u \leq \varepsilon. \tag{2.2}$$

Suppose that $\lambda < \frac{\varepsilon}{2\|p\|\overline{f}(\varepsilon)}$, thus

$$\frac{\overline{f}(\varepsilon)}{\varepsilon} < \frac{1}{2\lambda\|p\|}, \tag{2.3}$$

where $\overline{f}(t) = \max\limits_{0 \leq s \leq t} f(s)$.

It follows from (H2) that

$$\lim_{t \to 0^+} \frac{\overline{f}(t)}{t} = +\infty,$$

which together with (2.3) implies that there exists $A_\lambda \in (0,\varepsilon)$ such that

$$\frac{\overline{f}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda\|p\|}. \tag{2.4}$$

Now, let $u \in C[0,1]$ and $\theta \in (0,1)$ be such that $u = \theta A u$. Then we have

$$|u(t)| = \left|\theta\lambda \int_0^1 G^+(t,s)b(s)f(u(s))ds\right| \leq \lambda p(t)\overline{f}(\|u\|), \quad t \in [0,1] \tag{2.5}$$

and therefore

$$\frac{\overline{f}(\|u\|)}{\|u\|} \geq \frac{1}{\lambda\|p\|},$$

which implies that $\|u\| \neq A_\lambda$. Note that $A_\lambda \to 0$ as $\lambda \to 0$. By Lemma 1.1, A has a fixed point $\tilde{u}_\lambda$ with $\|\tilde{u}_\lambda\|_\infty \leq A_\lambda < \varepsilon$. Consequently, $\tilde{u}_\lambda(t) \geq \lambda\delta f(0)p(t), \ t \in [0,1]$, and the proof is completed. □

Theorem 2.2. *Let (H1), (H2) and (H3) hold. Then there exists a positive number λ^* such that (1.1) has a positive solution for $\lambda \in (0,\lambda^*)$.*

Proof. Let $q(t) = \int_0^1 G^-(t,s)b(s)ds$. (H3) implies that there exist positive numbers $\alpha, \gamma \in (0,1)$ such that

$$q(t)|f(s)| \leq \gamma p(t) f(0) \tag{2.6}$$

for $s \in [0, \alpha]$, $t \in [0, 1]$. Fix $\delta \in (\gamma, 1)$ and let λ^* be such that

$$\|\tilde{u}_\lambda\|_\infty + \lambda\delta f(0)\|p\| \le \alpha \tag{2.7}$$

for $\lambda < \lambda^*$, where $\tilde{u}_\lambda$ is given by Lemma 2.1, and

$$|f(x) - f(y)| \le f(0)\frac{\delta - \gamma}{2} \tag{2.8}$$

for $x, y \in [-\alpha, \alpha]$ with $|x - y| \le \lambda^* \delta f(0)\|p\|$.

Let $\lambda < \lambda^*$. We look for a solution u_λ of (1.1) of the form $\tilde{u}_\lambda + v_\lambda$. Thus v_λ satisfies

$$v_\lambda(t) = \lambda \int_0^1 G(t,s)g(s)f(\tilde{u}_\lambda + v_\lambda)ds - \lambda \int_0^1 G^+(t,s)b(s)f(\tilde{u}_\lambda)ds, \ t \in [0,1].$$

For each $w \in C[0,1]$, let $v = Aw$ be the solution of

$$v(t) = \lambda \int_0^1 G(t,s)g(s)f(\tilde{u}_\lambda + w)ds - \lambda \int_0^1 G^+(t,s)b(s)f(\tilde{u}_\lambda)ds, \ t \in [0,1].$$

Then $A : C[0,1] \to C[0,1]$ is completely continuous. Let $v \in C[0,1]$ and $\theta \in (0,1)$ be such that $v = \theta A v$. Then we have

$$v(t) = \theta\lambda \int_0^1 G(t,s)b(s)f(\tilde{u}_\lambda + v)ds - \theta\lambda \int_0^1 G^+(t,s)b(s)f(\tilde{u}_\lambda)ds, \ t \in [0,1].$$

We claim that $\|v\| \ne \lambda\delta f(0)\|p\|$. Suppose on the contrary that $\|v\| = \lambda\delta f(0)\|p\|$. Then, by (2.7) and (2.8), we get

$$\|\tilde{u}_\lambda + v\| \le \|\tilde{u}_\lambda\| + \|v\| \le \alpha$$

and

$$\|f(\tilde{u}_\lambda + v) - f(\tilde{u}_\lambda)\| \le f(0)\frac{\delta - \gamma}{2},$$

which together with (2.6) implies that

$$|v(t)| \le \lambda\frac{\delta - \gamma}{2} f(0)p(t) + \lambda\gamma f(0)p(t) = \lambda\frac{\delta + \gamma}{2} f(0)p(t), \ t \in [0,1]. \tag{2.9}$$

In particular,

$$\|v\| \le \lambda\frac{\delta + \gamma}{2} f(0)\|p\| < \lambda\delta f(0)\|p\|,$$

a contradiction, and the claim is proved. By Lemma 2.1, A has a fixed point v_λ with $\|v_\lambda\| \le \lambda\delta f(0)\|p\|$, Hence, v_λ satisfies (2.9) and, using Lemma 2.1, we obtain

$$u_\lambda(t) \ge \tilde{u}_\lambda(t) - v_\lambda(t) \ge \lambda\delta f(0)p(t) - \lambda\frac{\delta + \gamma}{2} f(0)p(t) = \lambda\frac{\delta - \gamma}{2} f(0)p(t), \quad t \in [0,1],$$

i.e., u_λ is a positive solution of (1.1). This completes the proof of Theorem 2.2. □

3. Applications

Example 3.1. Let us consider the following nonlinear Neumann boundary value problem

$$\begin{cases} u'' + (\frac{2\pi}{3})^2 u = \lambda(u^3 - 3u^2 + u\sin u + 1), & t \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{3.1}$$

It is well-known that the Green's function corresponding to (3.1) is given by

$$G(t,s) = \frac{1}{A}\begin{cases} \cos(\frac{2\pi}{3}(1-t))\cos(\frac{2\pi}{3}(s)), & 0 \le s \le t \le 1, \\ \cos(\frac{2\pi}{3}(1-s))\cos(\frac{2\pi}{3}(t)), & 0 \le t \le s \le 1, \end{cases}$$

where $A = \frac{2\pi}{3}\sin\frac{2\pi}{3} = \frac{\sqrt{3}\pi}{3} > 0$, $m = \frac{2\pi}{3}$ is a constant, $\lambda > 0$ is a parameter, $b(\cdot) \equiv 1$ and $f(u) = u^3 - 3u^2 + u\sin u + 1$.

It is not difficult to check that conditions (H1) and (H2) are satisfied. Now, we need only to look for a constant $k > 1$ such that (H3) holds. In fact, by simple computation, we get

$$\int_0^1 G(t,s)dt = \left(\frac{3}{2\pi}\right)^2, \tag{3.2}$$

$G(0,0) = \frac{2}{A}\cos(\frac{2\pi}{3}) < 0$, $G(0,\frac{1}{2}) = \frac{1}{A}\cos(\frac{\pi}{3}) > 0$ and

$$\int_0^1 G^+(t,s)b(s)ds - \int_0^1 G^-(t,s)b(s)ds = \int_0^1 G(t,s)ds = \frac{1}{m^2} > 0, \quad t \in [0,1].$$

Thus, there exists a constant $\varepsilon > 0$ sufficiently small such that

$$\int_0^1 G^+(t,s)b(s)ds - \int_0^1 G^-(t,s)b(s)ds \ge \varepsilon \int_0^1 G^-(t,s)b(s)ds, \quad t \in [0,1].$$

That is,

$$\int_0^1 G^+(t,s)b(s)ds \ge k\int_0^1 G^-(t,s)b(s)ds, \quad t \in [0,1]$$

with $k = \varepsilon + 1 > 1$. And therefore condition (H3) is satisfied as well. It follows from Theorem 2.2 that the nonlinear Neumann problem (3.1) has a positive solution for λ small.

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