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A Note on Spliced Sequences and A-density of Points with respect to a Non-negative Matrix

Kumardipta Bose* and Sayan Sengupta
Department of Mathematics, Jadavpur University, Kolkata 700032, India
e-mail : gabuaktafaltu6ele@hotmail.com
and mathematics.sayan1729@gmail.com

Abstract. For $y \in \mathbb{R}$, a sequence $x = (x_n) \in \ell^\infty$, and a non-negative regular matrix A, Bartoszewicz et. al., in 2015, defined the notion of the A-density $\delta_A(y)$ of the indices of those x_n that are close to y. Their main result states that if the set of limit points of (x_n) is countable and density $\delta_A(y)$ exists for any $y \in \mathbb{R}$ where A is a non-negative regular matrix, then $\lim_{n\to\infty}(Ax)_n = \sum_{y\in\mathbb{R}} \delta_A(y) \cdot y$. In this note we first show that the result can be extended to a more general class of matrices and then consider a conjecture which naturally arises from our investigations.

1. Introduction

We start by recalling the definition of natural density. For $n, m \in \mathbb{N}$ with $n < m$, let $[n, m]$ denote the set $\{n, n + 1, n + 2, \ldots, m\}$. Let $A \subset \mathbb{N}$. Define

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{n} \quad \text{and} \quad \underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap [1, n]|}{n}.$$

The numbers $\overline{d}(A)$ and $\underline{d}(A)$ are called the upper natural density and the lower natural density of A, respectively. If $\overline{d}(A) = \underline{d}(A)$, then this common value is called the natural density of A and we denote it by $d(A)$. Let $\mathcal{I}_d$ be the family of all subsets of $\mathbb{N}$ which have natural density 0. This $\mathcal{I}_d$ is a proper nontrivial admissible ideal of subsets of $\mathbb{N}$. The notion of natural density was used by Fast [5] and Scoenberg [18] to define the notion of statistical convergence. Details of statistical convergence and later on, ideal convergence are thoroughly described in [1, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15, 17, 19].

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* Corresponding Author.
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In [16] Osikiewicz had developed the ideas of finite and infinite splices. Let $E_1, E_2, E_3, \ldots, E_k, \ldots$ be a partition of $\mathbb{N}$ into countable number of sequences. Let $y_1, y_2, y_3, \ldots, y_k, \ldots$ be distinct real numbers. Let (x_n) be such that

$$\lim_{n\to\infty, n\in E_i} x_n = y_i.$$

Then (x_n) is called an infinite-splice (In the same way Osikiewicz defined an finite splice taking finite number of sequences and finite number of distinct real numbers). He proved the following:

Theorem 1.1.([16, Simplified version of Osikiewicz Theorem]) *Assume that (x_n) is a splice over a partition $\{E_i\}$. Let $y_i = \lim_{n\to\infty, n\in E_i} x_n$. Assume that $d(E_i)$ exists for each i and*

$$\sum_i d(E_i) = 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k = \sum_i y_i \cdot d(E_i).$$

In fact Osikiewicz considered a more general case, namely a regular matrix summability method A and A-density the details of which are presented in the preliminaries. Very recently in [2] a new approach was made to study the general version of Osikiewicz Theorem by defining the notion of the A-density of a point and an alternative version of the same result was established. In fact it was shown that the assumptions of Osikiewicz Theorem imply those of the following Theorem:

Theorem 1.2. *Suppose that $x = (x_n)$ is a bounded sequence, $\delta_A(y)$ exists for every $y \in \mathbb{R}$ and $\sum_{y\in D} \delta_A(y) = 1$. Then*

$$\lim_{n\to\infty} (Ax)_n = \sum_{y\in D} \delta_A(y) \cdot y.$$

and consequently Osikiewicz result is a consequence of Theorem 1.2.

A natural question arises whether the result is only true for regular matrices. Note that one of the main condition of regularity of a non-negative matrix $A = (a_{n,k})$ is that $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$. In this note we first show that actually the result can be extended to a larger class of matrices $A = (a_{n,k})$ satisfying $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} < \infty$ using almost the same arguments used in [2]. The main observation which leads to Theorem 2 is that if $(x_n) \in \ell^\infty$ and $\delta_A(y)$ exists for all $y \in \mathbb{R}$, then $D := \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is countable. We produce an example of a matrix $A = (a_{n,k})$ with

$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \infty$ and a bounded sequence (x_n) for which $\delta_A(y)$ exists for all $y \in \mathbb{R}$ but $D := \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is uncountable. This surprising example naturally gives rise to the following conjecture:

Conjecture $(\star)$**.** For any matrix $A = (a_{n,k})$ with $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \infty$ there exists a bounded sequence (x_n) for which $\delta_A(y)$ exists for all $y \in \mathbb{R}$ and $D := \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is uncountable.

In the last section of the note we deal with this conjecture essentially showing that it is false however showing that with imposition of certain conditions on the matrix the conjecture becomes true.

2. Preliminaries

We first present the necessary definitions and notations which will form the background of this article.

If $x = (x_n)$ is a sequence and $A = (a_{n,k})$ is a summability matrix, then by Ax we denote the sequence $((Ax)_1, (Ax)_2, (Ax)_3, \dots)$ where $(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k$. The matrix A is called regular if $\lim_{n\to\infty} x_n = L$ implies $\lim_{n\to\infty} (Ax)_n = L$. The well-known Silverman-Toeplitz theorem characterizes regular matrices in the following way. A matrix A is regular if and only if

(i) $\lim_{n\to\infty} a_{n,k} = 0$,

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1$,

(iii) $\sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |a_{n,k}| < \infty$.

For a non-negative regular matrix A and $E \subset \mathbb{N}$, following Freedman and Sember [9], the A-density of E, denoted by $\delta_A(E)$, is defined as follows

$$\overline{\delta_A}(E) = \limsup_{n\to\infty} \sum_{k\in E} a_{n,k} = \limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \mathbf{1}_E(k) = \limsup_{n\to\infty} (A\mathbf{1}_E)_n,$$

$$\underline{\delta_A}(E) = \liminf_{n\to\infty} \sum_{k\in E} a_{n,k} = \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} \mathbf{1}_E(k) = \liminf_{n\to\infty} (A\mathbf{1}_E)_n$$

where $\mathbf{1}_E$ is a 0-1 sequence such that $\mathbf{1}_E(k) = 1 \iff k \in E$. If $\overline{\delta_A}(E) = \underline{\delta_A}(E)$ then we say that the A-density of E exists and it is denoted by $\delta_A(E)$. Clearly, if A is the Cesàro matrix i.e.

$$a_{n,k} = \begin{cases} \frac{1}{n} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

then δ_A coincides with the natural density.

Throughout by ℓ^∞ we denote the set of all bounded sequences of reals.

In [2] in a new approach, the authors had defined for a sequence (x_n) a density $\delta_A(y)$ of indices of those x_n which are close to y which was not dealt with till then in the literature. This was a more general approach than that of Osikiewicz [16].

Fix $(x_n) \in \ell^\infty$. For $y \in \mathbb{R}$ let

$$\overline{\delta_A}(y) = \lim_{\varepsilon \to 0^+} \overline{\delta_A}(\{n : |x_n - y| \le \varepsilon\})$$

and

$$\underline{\delta_A}(y) = \lim_{\varepsilon \to 0^+} \underline{\delta_A}(\{n : |x_n - y| \le \varepsilon\}).$$

If $\overline{\delta_A}(y) = \underline{\delta_A}(y)$, then the common value is denoted by $\delta_A(y)$.

The main result of [2] was the following.

Theorem 2.1. *Let $x = (x_n) \in \ell^\infty$. Suppose that the set of limit points of (x_n) is countable and $\delta_A(y)$ exists for any $y \in \mathbb{R}$ where A is a non-negative regular matrix. Then*

$$\lim_{n \to \infty} (Ax)_n = \sum_{y \in \mathbb{R}} \delta_A(y) \cdot y.$$

Now recall that a non-empty family $\mathfrak{I}$ of subsets of $\mathbb{N}$ is an ideal in $\mathbb{N}$ if for $A, B \subset \mathbb{N}$, (i)$A, B \in \mathfrak{I} \Rightarrow A \cup B \in \mathfrak{I}$; (ii)$A \in \mathfrak{I}, B \subset A \Rightarrow B \in \mathfrak{I}$. Further if $\bigcup_{A \in \mathfrak{I}} A = \mathbb{N}$ i.e. $\{n\} \in \mathfrak{I}\ \forall\ n \in \mathbb{N}$, then $\mathfrak{I}$ is called admissible or free. An ideal $\mathfrak{I}$ is called a P-ideal if for any sequence of sets (D_n) from $\mathfrak{I}$, there is another sequence of sets (C_n) in $\mathfrak{I}$ such that $D_n \vartriangle C_n$ is finite for all n and $\bigcup_n C_n \in \mathfrak{I}$. Equivalently if for each sequence (A_n) of sets from $\mathfrak{I}$ there exists $A_\infty \in \mathfrak{I}$ such that $A_n \setminus A_\infty$ is finite for all $n \in \mathbb{N}$ then $\mathfrak{I}$ becomes a P-ideal.

For a bounded sequence (x_n), we now recall the following definitions (see [13]):

(i) (x_n) is $\mathfrak{I}$ convergent to y if for any $\varepsilon > 0$, $\{n : |x_n - y| \ge \varepsilon\} \in \mathfrak{I}$.

(ii) A point y is called an $\mathfrak{I}$-cluster point of (x_n) if $\{n : |x_n - y| \le \varepsilon\} \notin \mathfrak{I}$ for any $\varepsilon > 0$.

(iii) y is called an $\mathfrak{I}$-limit point of (x_n) if there is a set $B \subset \mathbb{N}$, $B \notin \mathfrak{I}$, such that $\lim_{n \in B} x_n = y$.

In this note we primarily consider non-negative matrices $A = (a_{n,k})$ satisfying

(i) $a_{n,k} \ge 0$ for all n, k;

(ii) $\lim_{n \to \infty} a_{n,k} = 0$, for all k;

(iii) $\lim_n \sum_{k=1}^{\infty} a_{n,k} < \infty$.

One should note that one can not replace the $\lim\limits_n$ in (iii) above by $\sup\limits_n$ for the simple fact that in that case if one considers a matrix A having infinitely many zero rows, like the matrix $A = (a_{n,k})$ where

$$a_{n,k} = \begin{cases} 1 & \text{if } n = k \text{ and } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

then even $\delta_A(\mathbb{N})$ does not exist.

Throughout the next section by a non-negative matrix A we will always mean a matrix satisfying the above three conditions. It should be noted that the notion A-density can be similarly defined as in the case of regular matrices and all the axiomatic conditions, as stated in [9] are satisfied here except for the fact that the density of the set of natural numbers $\mathbb{N}$ is now a finite real number, not necessarily 1. Further it is easy to check that $\{E \subset \mathbb{N} : \delta_A(E) = 0\}$ forms a P-ideal of $\mathbb{N}$. The proof being very similar to the case of regular matrices (see [2]) is omitted here.

3. Results for Non-negative Matrices

The main result which we are going to establish in this paper is the following.

Theorem 3.1. *Let $x = (x_n) \in \ell^\infty$, and the set of limit points of (x_n) is countable. Let A be a non-negative matrix as defined before (necessarily with $\sup\limits_n \sum\limits_{k=1}^{\infty} a_{n,k} < \infty$). Suppose $\delta_A(y)$ exists for all $y \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} (Ax)_n = \sum_{y\in\mathbb{R}} \delta_A(y) \cdot y$$

As in [2] we start with the following observation.

Lemma 3.2. *Let $(x_n) \in \ell^\infty$ and $\delta_A(y)$ exists for all $y \in \mathbb{R}$. Then $D := \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is countable and*

$$\sum_{y\in D} \delta_A(y) \leq M.$$

where $\sup\limits_n \sum\limits_{k=1}^{\infty} a_{n,k} = M$.

Proof. Let (r_n) be a strictly decreasing sequence converging to M. For $m \in \mathbb{N}$ let

$$D_m := \{y \in \mathbb{R} \mid \delta_A(y) \geq \frac{1}{m}\}.$$

Note that $D_1 \subset D_2 \subset ... \subset D_m \subset ...$ and $D = \bigcup\limits_m D_m$. Now if $y_1, y_2, ..., y_l \in D_m$ be distinct, let us choose $\varepsilon = \min\limits_{i\neq j} \frac{|y_i - y_j|}{3} > 0$. Consequently the sets $E_i = \{n : x_n \in$

$(y_i - \varepsilon, y_i + \varepsilon)\}$ are pairwise disjoint. Moreover

$$\underline{\delta_A}(E_i) \geq \delta_A(y_i) \geq \frac{1}{m} \Rightarrow \liminf \sum_{k \in E_i} a_{n,k} \geq \frac{1}{m}.$$

Then for any $\tau > 0$, $\exists\ n_1 \in \mathbb{N}$ such that $\sum\limits_{k \in E_i} a_{n,k} > \frac{1}{m} - \tau\ \forall\ n \geq n_1$. Again $\limsup\limits_{n} \sum\limits_{k=1}^{\infty} a_{n,k} < r_n\ \forall n \in \mathbb{N}$. So for any fixed r_p, we get $n_2 \in \mathbb{N}$ such that $\sum\limits_{k=1}^{\infty} a_{n,k} < M + \delta < r_p\ \forall\ n \geq n_2$ and for a suitably chosen δ. Let $n_0 = \max\{n_1, n_2\}$. As E_i's are disjoint, we have

$$\sum_{k \in \bigcup\limits_{i=1}^{l} E_i} a_{n,k} = \sum_{i=1, k \in E_i}^{l} a_{n,k} \geq \frac{l}{m} - l\tau\ \forall\ n \geq n_0.$$

But

$$\sum_{k \in \bigcup\limits_{i=1}^{l} E_i} a_{n,k} \leq \sum_{k=1}^{\infty} a_{n,k} < r_p.$$

Now note that $\frac{l}{m} - l\tau \leq r_p \Rightarrow l \leq \frac{mr_p}{1-\tau m}$. Hence choosing τ so that $1 - \tau m > 0$ we observe that l must be finite. Thus D_m is finite for each m which implies that $D = \bigcup\limits_{m} D_m$ can be at most countable.

Again

$$\begin{aligned}
\sum_{y \in D_m} \delta_A(y) \leq \sum_{i=1}^{l} \underline{\delta_A}(E_i) &= \sum_{i=1}^{l} \liminf_{n} \sum_{k \in E_i} a_{n,k} \\
&\leq \sum_{i=1}^{l} \Big(\sum_{k \in E_i} a_{n,k} + \frac{\varepsilon_0}{l}\Big) \quad \forall n \geq N \text{ (for some } N)
\end{aligned}$$

where ε_0 is arbitrary. So

$$\sum_{y \in D_m} \delta_A(y) \leq \sum_{k \in \bigcup\limits_{i=1}^{l} E_i} a_{n,k} + \varepsilon_0 \leq \sum_{k=1}^{\infty} a_{n,k} + \varepsilon_0 \leq r_p$$

for suitably chosen ε_0. Finally in view of the fact that $D = \bigcup\limits_{m} D_m$ we get

$$\sum_{y \in D} \delta_A(y) = \lim_{m \to \infty} \sum_{y \in D_m} \delta_A(y) \leq r_p.$$

Since this is true for any r_p, so letting $p \to \infty$ we get $\sum_{y \in D} \delta_A(y) \leq M$. $\square$

In [2] it was observed that generally it is not true that $D = \{y \in \mathbb{R} : \delta_A(y) > 0\}$ should be nonempty as also $\overline{D} := \{y \in \mathbb{R} : \overline{\delta_A}(y) > 0\}$ need not be countable.

The next result extends Theorem 6 [2].

Theorem 3.3. *Let (x_n) be a bounded sequence and A be a non-negative matrix such that $\delta_A(y)$ exists for every $y \in \mathbb{R}$ and moreover $\sum_{y \in D} \delta_A(y) = M$. Then*

$$\lim_{n \to \infty} (Ax)_n = \sum_{y \in D} \delta_A(y) \cdot y.$$

Proof. Since (x_n) is bounded, there exists a $K > 0$ such that $|x_n| \leq K$ for every $n \in \mathbb{N}$. Let $D = \{y_i : i = 1, 2, ...\}$ where y_i's are distinct. Let $\varepsilon > 0$ be given and let $r \in \mathbb{N}$ be such that

$$\sum_{i=1}^{r} \delta_A(y_i) > M - \varepsilon \quad \text{and} \quad |\sum_{i=r+1}^{\infty} y_i \cdot \delta_A(y_i)| < \varepsilon.$$

Let $N \in \mathbb{N}$ be such that

$$\frac{1}{3} \min_{i \neq j} |y_i - y_j| > \frac{1}{N}$$

$\forall\ i, j \in 1, 2, ..., r$ and such that the sets $E_i = \{j : |x_j - y_i| < \frac{1}{N}\}$ have the following property

$$\delta_A(y_i) - \frac{\varepsilon}{r(k+1)} \leq \underline{\delta_A}(E_i) \leq \overline{\delta_A}(E_i) \leq \delta_A(y_i) + \frac{\varepsilon}{r(k+1)}$$

for $i = 1, 2, ..., r$. Obviously $E_1, ..., E_r$ are pairwise disjoint. Now we can choose a $m_0 \in \mathbb{N}$ such that

$$\underline{\delta_A}(E_i) - \frac{1}{N} < \sum_{k \in E_i} a_{n,k} < \overline{\delta_A}(E_i) + \frac{1}{N}$$

$$| \sum_{k \in E_i} a_{n,k} - \delta_A(y_i) | < \frac{1}{N} + \frac{\varepsilon}{r(k+1)} \tag{3.1}$$

for every $n \geq m_0$ and $i = 1, 2, ..., r$. Then for $n \geq m_0$ we have

$$\begin{aligned}(Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k \leq \sum_{k \in E_1} a_{n,k}(y_1 + \frac{1}{N}) + \sum_{k \in E_2} a_{n,k}(y_2 + \frac{1}{N}) + \dots \\ + \sum_{k \in E_r} a_{n,k}(y_r + \frac{1}{N}) + K. \sum_{k \in (E_1 \cup \dots \cup E_r)^c} a_{n,k}.\end{aligned}$$

Now we can choose $m_1 > m_0$ such that $\forall n \geq m_1$

$$\sum_{k=1}^{\infty} a_{n,k} < M + \varepsilon.$$

Then

$$M + \varepsilon > \sum_{k=1}^{\infty} a_{n,k} = \sum_{k \in \bigcup_{i=1}^{r} E_i} a_{n,k} + \sum_{k \in (\bigcup_{i=1}^{r} E_i)^c} a_{n,k}$$

and consequently

$$\sum_{k \in \bigcup_{i=1}^{r} E_i} a_{n,k} = \sum_{i=1}^{r} \sum_{k \in E_i} a_{n,k} > \sum_{i=1}^{r} \delta_A(y_i) - \frac{r}{N} - \frac{\varepsilon}{K+1}.$$

Therefore for $n \geq m_1$ we have

$$\sum_{k \in (\bigcup_{i=1}^{r} E_i)^c} a_{n,k} < (M + \varepsilon) - (M - \frac{r}{N} - (1 + \frac{1}{K+1})\varepsilon) = \frac{r}{N} + (2 + \frac{1}{K+1})\varepsilon.$$

Subsequently we get for $n \geq m_1$,

$$\begin{aligned}(Ax)_n \leq \sum_{k \in E_1} a_{n,k}(y_1 + \frac{1}{N}) + \sum_{k \in E_2} a_{n,k}(y_2 + \frac{1}{N}) + \dots \\ + \sum_{k \in E_r} a_{n,k}(y_r + \frac{1}{N}) + \frac{Kr}{n} + (2 + \frac{1}{K+1})\varepsilon K.\end{aligned}$$

Analogously

$$\begin{aligned}(Ax)_n \geq \sum_{k \in E_1} a_{n,k}(y_1 - \frac{1}{N}) + \sum_{k \in E_2} a_{n,k}(y_2 - \frac{1}{N}) + \dots \\ + \sum_{k \in E_r} a_{n,k}(y_r - \frac{1}{N}) - \frac{Kr}{n} - (2 + \frac{1}{K+1})\varepsilon K.\end{aligned}$$

Thus

$$(Ax)_n - \sum_{i=1}^{r} \sum_{k \in E_i} a_{n,k}(y_i + \frac{1}{N}) \leq \frac{Kr}{n} + (2 + \frac{1}{K+1})\varepsilon K \tag{3.2}$$

and

$$(Ax)_n - \sum_{i=1}^{r} \sum_{k \in E_i} a_{n,k}(y_i - \frac{1}{N}) \geq -\frac{Kr}{n} - (2 + \frac{1}{K+1})\varepsilon K. \tag{3.3}$$

Hence using (1) and (2) we obtain

$$\begin{aligned}
&(Ax)_n - \sum_{i=1}^{\infty} \delta_A(y_i).y_i = (Ax)_n - \sum_{i=1}^{r} \delta_A(y_i).y_i - \sum_{i=r+1}^{\infty} \delta_A(y_i).y_i \\
&\leq (Ax)_n - \sum_{i=1}^{r} \delta_A(y_i).y_i + |\sum_{i=r+1}^{\infty} \delta_A(y_i).y_i| \\
&\leq (Ax)_n - \sum_{i=1}^{r} \delta_A(y_i).y_i + \varepsilon \\
&= [(Ax)_n - \sum_{i=1}^{r} \sum_{k\in E_i} a_{n,k}(y_i + \frac{1}{N})] + \sum_{i=1}^{r}[\sum_{k\in E_i} a_{n,k}(y_i + \frac{1}{N}) - \delta_A(y_i).y_i] + \varepsilon \\
&\leq \sum_{i=1}^{r}[(\sum_{k\in E_i} a_{n,k} - \delta_A(y_i))(y_i + \frac{1}{N})] + \frac{1}{N}\sum_{i+1}^{r} \delta_A(y_i) + \frac{Kr}{N} + (2K + \frac{K}{K+1} + 1)\varepsilon \\
&\leq \sum_{i=1}^{r}[|(\sum_{k\in E_i} a_{n,k} - \delta_A(y_i))|(|y_i| + \frac{1}{N})] + \frac{M}{n} + \frac{Kr}{N} + (2K + \frac{K}{K+1} + 1)\varepsilon \\
&\leq r(\frac{1}{N} + \frac{\varepsilon}{r(K+1)}).(K + 1 + \frac{1}{N}) + \frac{M}{n} + \frac{Kr}{N} + (2K + \frac{K}{K+1} + 1)\varepsilon.
\end{aligned}$$

Analogously from (1) and (3) we get

$$\begin{aligned}
(Ax)_n - \sum_{i=1}^{\infty} \delta_A(y_i).y_i \geq &- r(\frac{1}{N} + \frac{\varepsilon}{r(K+1)}).(K + 1 + \frac{1}{N}) \\
&- \frac{M}{n} - \frac{Kr}{N} - (2K + \frac{K}{K+1} + 1)\varepsilon.
\end{aligned}$$

Since N can be chosen arbitrarily large, we obtain

$$|(Ax)_n - \sum_{i=1}^{\infty} \delta_A(y_i).y_i| \leq (2K + \frac{K}{K+1} + 1)\varepsilon$$

for every $\varepsilon > 0$. Therefore we can conclude that

$$\lim_{n\to\infty} (Ax)_n = \sum_{i=1}^{\infty} \delta_A(y_i) \cdot y_i. \qquad \square$$

Next we prove the following result which is a variant of the corresponding result of [2] forming the basis of a necessary condition for the existence of a limit of $\{(Ax)_n\}$.

Proposition 3.4. *Suppose $x = (x_n) \in l^{\infty}$. If $\delta_A(y) = M$, then My is a limit point of the sequence $\{(Ax)_n\}$.*

Proof. Since (x_n) is bounded, there is $K > 0$ such that $|x_n| \le K \ \forall N \in \mathbb{N}$. Let y be such that $\overline{\delta_A}(y) = M$. Let $N \in \mathbb{N}$ and let $E_N = \{j \in \mathbb{N} : |x_j - y| < \frac{1}{N}\}$. Then there exists $k_N \ge N$ such that

$$\sum_{k \in E_N} a_{k_N,k} > \overline{\delta_A}(E_N) - \frac{1}{N} = M - \frac{1}{N}.$$

Again as we have $\limsup_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = M$, we get

$$\sum_{k=1}^{\infty} a_{k_N,k} < M + \frac{1}{N}.$$

Since $y - \frac{1}{N} < x_k < y + \frac{1}{N} \ \forall x_k \in E_N$ and $-K \le x_k \le K \ \forall x_k \notin E_N$, so we have

$$\begin{aligned} &\sum_{k \in E_N} a_{k_N,k}(y - \frac{1}{N}) - \sum_{k \in (E_N)^c} a_{k_N,k} \cdot K \\ &\le (Ax)_{k_N} \le \sum_{k \in E_N} a_{k_N,k}(y + \frac{1}{N}) + \sum_{k \in (E_N)^c} a_{k_N,k} \cdot K. \end{aligned}$$

Thus

$$\begin{aligned} &y(\sum_{k=1}^{\infty} a_{k_N,k} - M) - \frac{1}{N} \sum_{k \in E_N} a_{k_N,k} - \sum_{k \in (E_N)^c} a_{k_N,k}(K + y) \\ &\le (Ax)_{k_N} - My \le y(\sum_{k=1}^{\infty} a_{k_N,k} - M) + \frac{1}{N} \sum_{k \in E_N} a_{k_N,k} + \sum_{k \in (E_N)^c} a_{k_N,k}(K - y) \end{aligned}$$

and consequently

$$| \ (Ax)_{k_N} - My \ | \le | \sum_{k \in E_N} a_{k_N,k} . \frac{1}{N} \ | + | \sum_{k \in (E_N)^c} a_{k_N,k}(K + |y|) \ | + \frac{1}{N}|y|.$$

Since

$$\sum_{k \in (E_N)^c} a_{k_N,k} = \sum_{k=1}^{\infty} a_{k_N,k} - \sum_{k \in E_N} a_{k_N,k} < M + \frac{1}{N} - (M - \frac{1}{N}) = \frac{2}{N}$$

so

$$\begin{aligned} &|(Ax)_{k_N} - My| \le (\frac{M}{N} + \frac{1}{N^2}) + \sum_{k \in (E_N)^c} a_{k_N,k}(K + |y|) + \frac{1}{N}|y| \\ &\le (\frac{M}{N} + \frac{1}{N^2}) + \frac{2}{N}(K + |y|) + \frac{1}{N}|y|. \end{aligned}$$

Therefore

$$\lim_{N\to\infty}(Ax)_{k_N} = My. \qquad \square$$

Corollary 3.5. *Let (x_n) be a bounded sequence. Suppose that there are y and z $(y \neq z)$ with $\delta_A(y) = \delta_A(z) = M$. Then the limit $\lim_{n\to\infty}(Ax)_n$ does not exist.*

Now we recall some important results from [2] which will be useful in the sequel.

Lemma 3.6.([2]) *Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$. Assume that $X := \{n : x_n \in [a,b]\} \notin \mathcal{I}$. Suppose that*

$$\{n : a \leq x_n \leq t-\varepsilon\} \in \mathcal{I} \text{ or } \{n : t+\varepsilon \leq x_n \leq b\} \in \mathcal{I}$$

for any $t \in (a,b)$ and any $\varepsilon > 0$ such that $\varepsilon < \min\{t-a, b-t\}$. Then there is $y \in [a,b]$ such that $\{n : |x_n - y| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.

Proposition 3.7.([2]) *Let $\mathcal{I}$ be a P-ideal. Assume that $(x_n) \in \ell^\infty$ does not have any $\mathcal{I}$-limit points. Then the set of limit points of (x_n), i.e. the set*

$$\{y \in \mathbb{R} : x_{n_k} \to y \text{ for some increasing sequence } (n_k) \text{ of natural numbers}\},$$

is uncountable and closed.

Corollary 3.8.([2]) *Let $[a,b]$ be a fixed interval and $\mathcal{I}$ be a P-ideal. Assume that $\{n : x_n \in [a,b]\} \notin \mathcal{I}$ and any point $y \in (a,b)$ is not an $\mathcal{I}$-limit point of (x_n). Then the set of limit points of (x_n) in $[a,b]$, i.e. the set*

$$\{y \in (a,b) : x_{n_k} \to y \text{ for some increasing sequence } (n_k) \text{ of natural numbers}\},$$

is uncountable and closed.

Corollary 3.9.([2]) *Let $(x_n) \in \ell^\infty$. Assume that the set of limit points of (x_n) is countable. Then the sequence (x_n) has at least one $\mathcal{I}$-limit point for every P-ideal $\mathcal{I}$.*

We can easily prove the following results analogous to the results of [2] which will help us to reach our final goal.

Lemma 3.10. *Let $r \in (0,1)$, $r_1 \geq r_2 \geq r_3 \geq \dots$, $\lim_{n\to\infty} r_n = r$ and let (E_n) be a decreasing sequence of subsets of $\mathbb{N}$.*

(i) *If $\underline{\delta_A}(E_n) = r_n$, $n \in \mathbb{N}$, then there is a subset E of $\mathbb{N}$ with $\underline{\delta_A}(E) = r$ and such that $E \subset^* E_n$, $n \in \mathbb{N}$, i.e. $E \setminus E_n$ is finite for every $n \in \mathbb{N}$. Moreover, if $\overline{\delta_A}(E_n) \to r$, then $\delta_A(E) = r$.*

(ii) *If $\overline{\delta_A}(E_n) = r_n$, $n \in \mathbb{N}$, then there is a subset E of $\mathbb{N}$ with $\overline{\delta_A}(E) = r$ and such that $E \subset^* E_n$, $n \in \mathbb{N}$.*

Theorem 3.11. *Let $(x_n) \in \ell^\infty$. A point $y \in \mathbb{R}$ is an A-statistical limit point of (x_n) if and only if $\overline{\delta_A}(y) > 0$. Moreover if $\delta_A(y) > 0$, then there is $E \subset \mathbb{N}$ with $\delta_A(E) = \delta_A(y)$ and $\lim\limits_{n \in E} x_n = y$.*

Corollary 3.12. *Let $(x_n) \in \ell^\infty$. A point $y \in \mathbb{R}$ is an A-statistical cluster point of (x_n) and it is not an A-statistical limit point if and only if*

(1) $\overline{\delta_A}(\{j : |x_j - y| \leq 1/n\}) > 0$ *for every* n;

(2) $\delta_A(y) = \lim\limits_{n\to\infty} \overline{\delta_A}(\{j : |x_j - y| \leq 1/n\}) = 0$.

Proposition 3.13. *Let $(x_n) \in \ell^\infty$. Assume that $y_1, y_2, \dots$ are the only distinct real numbers such that $\delta_A(y_i) > 0\ \forall i$. Then there exists a partition $E_1, E_2, \dots$ such that $\delta_A(E_i) = \delta_A(y_i)$, $i = 1, 2, \dots$ and $\lim\limits_{n \in E_i} x_n = y_i$.*

Lemma 3.14. *Assume that $\{E_n : n = 1, 2, \dots\}$ is a partition of $\mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} \delta_A(E_n) < M$. Then there is a partition $\{F_n : n = 0, 1, 2, \dots\}$ of $\mathbb{N}$ such that*

(i) $F_n \subset E_n$;

(ii) $\delta_A(F_n) = \delta_A(E_n)$;

(iii) $\delta_A(F_0) = M - \sum\limits_{n=1}^{\infty} \delta_A(E_n)$.

Finally we prove a sufficient condition for a bounded sequence (x_n) to have the property that $\sum\limits_{y\in\mathbb{R}} \delta_A(y) = M$.

Theorem 3.15. *Let (x_n) be a bounded sequence. Suppose that the set of limit points of (x_n) is countable and $\delta_A(y)$ exists for all $y \in D$. Then $\sum\limits_{y\in D} \delta_A(y) = M$.*

Proof. Suppose that, on the contrary,

$$\sum_{y\in D} \delta_A(y) < M.$$

Then by Corollary 14 in [2] the set D is non-empty. By Lemma 3.2 the set D is countable. Enumerate D as $\{y_1, y_2, \dots\}$. By Proposition 3.13 there is a partition $\{E_k : k = 1, 2, \dots\}$ of $\mathbb{N}$ such that $\delta_A(E_k) = \delta_A(y_k)$ and $\lim\limits_{n\to\infty, n\in E_k} x_n = y_k$. By Lemma 3.14 there is a partition $\{F_k : k = 0, 1, 2, \dots\}$ such that $F_k \subset E_k$, $\delta_A(F_k) = \delta_A(E_k)$ for every $k \geq 1$, $F_0 = \mathbb{N} \setminus \bigcup\limits_{i=1}^{\infty} F_i$ so that $\delta_A(F_0) = M - \sum\limits_{k=1}^{\infty} \delta_A(F_k)$. Thus $\delta_A(F_0) > 0$. Consider the sequence $(x_n)_{n\in F_0}$ and the ideal $\mathcal{I}_A|_{F_0} = \{E \subset F_0 : E \in \mathcal{I}_A\}$. Since $\delta_A(y) = 0$ for every $y \notin D$, so by Theorem 3.11 y can not be an A-statistical limit point of (x_n) and so can not be an $\mathcal{I}_A|_{F_0}$-limit point of $(x_n)_{n\in F_0}$

for any $y \in D^c$. If y_i was $\mathcal{I}_A|_{F_0}$-limit point of $(x_n)_{n\in F_0}$, then there would be a set $B \subset F_0$ such that $B \notin \mathcal{I}_A|_{F_0}$, and $\lim_{n\in B} x_n = y_i$. But then $B \notin \mathcal{I}_A$ and since $B \cap F_i = \emptyset$ and $\lim_{n\in B\cup F_i} x_n = y_i$, we would have $\delta_A(y) = \overline{\delta_A}(y) \geq \overline{\delta_A}(B \cup F_i) > \overline{\delta_A}(F_i) = \delta_A(y)$, which gives a contradiction. Therefore the sequence $(x_n)_{n\in F_0}$ has no $\mathcal{I}_A|_{F_0}$-limit points.

Note that $\mathcal{I}_A|_{F_0}$ is a P-ideal. To see this assume that $A_1, A_2, \cdots \in \mathcal{I}_A|_{F_0}$. Then $A_1, A_2, \cdots \in \mathcal{I}_A$ and as $\mathcal{I}_A$ is a P-ideal, we can find $A_\infty \in \mathcal{I}_A$ such that $A_n \setminus A_\infty$ is finite for every n. Since $A_\infty \cap F_0 \in \mathcal{I}_A|_{F_0}$ and each $A_n \subset F_0$, then $A_n \setminus (A_\infty \cap F_0)$ is finite for every n. Now by Proposition 14 in [2] applied to the sequence $(x_n)_{n\in F_0}$ and P-ideal $\mathcal{I}_A|_{F_0}$, we obtain that the sequence $(x_n)_{n\in F_0}$ has uncountably many limit points which are, in turn, limit points of (x_n). This contradicts the assumption. □

Finally combining Theorem 3.3 with Theorem 3.15 we get the desired proof of our main result.

4. Conjecture ($\star$)

Throughout the paper we have considered the matrices with the following properties.

(i) $a_{n,k} \geq 0$ for all n, k;

(ii) $\lim_{n\to\infty} a_{n,k} = 0$, for all k;

(iii) $\lim_n \sum_{k=1}^{\infty} a_{n,k} < \infty$.

Now the natural question arises whether the results discussed above remain true if any one of the conditions for the matrix can be relaxed. The first condition is clearly essential.

Coming to the second condition, if $\lim_{n\to\infty} a_{n,k} > 0$ for some k, then note that the axiomatic condition in [9] namely the condition 'for $E_1, E_2 \subset \mathbb{N}$ such that $E_1 \triangle E_2$ is finite,$\delta_A(E_1) = \delta_A(E_2)$',which is very crucial in defining a density function does not hold anymore. For example, let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & \ldots & & \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \ldots \\ . & . & . & & & & \\ . & . & . & & & & \\ 1 & \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} & 0 & \ldots \\ . & . & & & & & \\ . & . & & & & & \end{bmatrix}$$

Here $a_{n,1} \to 1$ as $n \to \infty$. We take $E_1 = \mathbb{N}$,$E_2 = \mathbb{N} \setminus \{1\}$. Then $E_1 \triangle E_2 = \{1\}$,but $\delta_A(E_1) = 2$ whereas $\delta_A(E_2) = 1$.

Finally let us relax condition (iii) and assume that $\lim_n \sum_{k=1}^{\infty} a_{n,k} = \infty$. Observe that Lemma 3.6 forms the backbone of our main result as only then the sequence can be partitioned into countably infinitely many splices. Below we produce an example of a matrix A with $\lim_n \sum_{k=1}^{\infty} a_{n,k} = \infty$ and a bounded real sequence (x_n) for which $\{y \in \mathbb{R} : \delta_A(y) > 0\}$ is uncountable,i.e. Lemma 3.2 is not true when $M = \infty$.

Proposition 4.1. *Define the matrix $A = (a_{n,k})$ in the following way*

$$a_{n,k} = \begin{cases} \frac{1}{g(n)} & \text{if } n \geq k \\ 0 & \text{otherwise} \end{cases}$$

where $g : \mathbb{N} \to (0, \infty)$ is defined as $g(k) = n + 1$ if $k \in [2^n, 2^{n+1})$. Then there is a bounded sequence (x_n) such that $\{y \in \mathbb{R} : \delta_A(y) > 0\}$ is uncountable.

Proof. Let (x_n) be defined as follows:

$$x_1 = \frac{1}{2},$$

$$x_2 = \frac{1}{4}, x_3 = \frac{3}{4},$$

$$x_4 = \frac{1}{8}, x_5 = \frac{3}{8}, x_6 = \frac{5}{8}, x_7 = \frac{7}{8}, \quad \text{etc}$$

In general

$$x_{2^n+k} = \frac{2k+1}{2^{n+1}} \quad \text{for} \quad k \in [0, 2^n).$$

Now consider a dyadic interval $I = [\frac{k}{2^p}, \frac{k+1}{2^p})$. Then I contains 2^{n-p} elements from $\{x_{2^n+k} : k \in [0, 2^n)\}$. Therefore

$$\delta_A\{m : x_m \in I\} \geq \liminf_{m\to\infty} \frac{1}{g(m)} |\{l \in [0, 2^n) : x_{2^n+l} \in I\}| = \lim_{n\to\infty} \frac{1}{n+2} 2^{n-p} = \infty$$

where n is the largest natural number such that $2^{n+1} \leq m$. Let $y \in [0, 1]$. Note that for every $\varepsilon > 0$ there is a dyadic interval $I = [\frac{k}{2^p}, \frac{k+1}{2^p})$ contained in $(y - \varepsilon, y + \varepsilon)$. Thus $\delta_A(y) = \infty$ for every $y \in [0, 1]$ (consequently $\delta_A(y)$ exists and it equals ∞). As for each point $y \in \mathbb{R} \setminus [0, 1]$, $\delta_A(y) = 0$ so we can conclude that $\delta_A(y)$ exists for all $y \in \mathbb{R}$ and $\{y : \delta_A(y) > 0\} = [0, 1]$ is uncountable. □

In view of the above example one can naturally think of the following conjecture.

Conjecture $(\star)$**.** For any matrix $A = (a_{n,k})$ with

(i) $a_{n,k} \geq 0$ for all n, k,

(ii) $\lim_{n\to\infty} a_{n,k} = 0$ for all k,

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = \infty$.

there exists a bounded sequence (x_n) for which $\delta_A(y)$ exists for all $y \in \mathbb{R}$ and $D := \{y \in \mathbb{R} : \delta_A(y) > 0\}$ is uncountable.

Proposition 4.2. *There is a non-negative matrix $A = (a_{n,k})$ satisfying all the above properties (as prescribed in Conjecture $(\star)$ such that for any bounded sequence (x_n) for which $\delta_A(y)$ exists for all $y \in \mathbb{R}$ the set $D = \{y : \delta_A(y) > 0\}$ is a singleton.*

Proof. Let the matrix $A = (a_{n,k})$ be such that

$$a_{n,k} = \begin{cases} n & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\lim_n \sum_{k=1}^{\infty} a_{n,k} = \infty$. Let (x_n) be a bounded sequence. First note that isolated points of the sequence have density 0 with respect to matrix A. Now if (x_n) has more than one limit point, let y be one of them. Choose another limit point z of the sequence (x_n). Let we choose $0 < \varepsilon < \frac{|y-z|}{2}$ and consider the set $B_\varepsilon = \{k : x_k \in (y-\varepsilon, y+\varepsilon)\}$. Observe that the set $B_\varepsilon^c = \{k : x_k \notin (y-\varepsilon, y+\varepsilon)\}$ must be infinite, or else z would not be a limit point of (x_n). Let $B_\varepsilon^c = \{n_1, n_2, ...\}$. So $\sum_{k\in B_\varepsilon} a_{n_i,k} = 0$ for all $i = 1, 2, ...$. This implies that $\liminf_n \sum_{k\in B_\varepsilon} a_{n,k} = 0$. But it is easy to see that $\limsup_n \sum_{k\in B_\varepsilon} a_{n,k} = \infty$. So $\delta_A(y)$ does not exist. This is true for all limit points of (x_n). Finally if (x_n) has only one limit point y, say, i.e. (x_n) is convergent to y_0 then obviously $\delta_A(y_0) > 0$ and $D = \{y : \delta_A(y) > 0\} = \{y_0\}$. □

Finally one can ask for which matrices the Conjecture $(\star)$ is true ? The following result shows that the Conjecture is valid for matrices A satisfying certain conditions.

Proposition 4.3. *Let $A = (a_{n,k})$ be a non-negative matrix satisfying the conditions of Conjecture $(\star)$. In addition let there exist a sequence of positive real numbers $(\delta_k)_{k=1}^{\infty}$ with the following properties.*

(i) *For any k, $\min\{a_{11}, a_{12}, ..., a_{1k}\} \geq \delta_k$;*

(ii) $\lim_{n\to\infty} \delta_n.2^{n-p} > 0$ *, for any $p < n$.*

Then there exists a bounded sequence (x_n) such that the set $\{y : \delta_A(y) > 0\}$ is uncountable.

Proof. Let (x_n) be defined as in Proposition 4.1. Now consider a dyadic interval $I = [\frac{k}{2^p}, \frac{k+1}{2^p})$. Then I contains 2^{n-p} elements from $\{x_{2^n+k} : k \in [0, 2^n)\}$. Therefore

$$\delta_A\{m : x_m \in I\} \geq \liminf_{m\to\infty} \delta_m.|\{l \in [0, 2^n) : x_{2^n+l} \in I\}| = \lim_{n\to\infty} \delta_n 2^{n-p} = \infty$$

where n is the largest natural number such that $2^{n+1} \leq m$. Let $y \in [0, 1]$. Note that for every $\varepsilon > 0$ there is a dyadic interval $I = [\frac{k}{2^p}, \frac{k+1}{2^p})$ contained in $(y - \varepsilon, y + \varepsilon)$.

Thus $\delta_A(y) = \infty$ for every $y \in [0, 1]$ (consequently $\delta_A(y)$ exists and it equals ∞). As for each point $y \in \mathbb{R} \setminus [0, 1]$, $\delta_A(y) = 0$ so we can conclude that $\delta_A(y)$ exists for all $y \in \mathbb{R}$ and $\{y : \delta_A(y) > 0\} = [0, 1]$ is uncountable. □

Problem 1. Find the necessary and sufficient conditions or some other condition for a matrix A for the Conjecture ($\star$) to be true.

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