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On a Symbolic Method for Fully Inhomogeneous Boundary Value Problems

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Abstract. This paper presents a symbolic method for solving a boundary value problem with inhomogeneous Stieltjes boundary conditions over integro-differential algebras. The proposed symbolic method includes computing the Green's operator as well as the Green's function of the given problem. Examples are presented to illustrate the proposed symbolic method.

1. Introduction

For the last five decades, many researchers and engineers have been actively developing applications of general boundary value problems (BVPs) of higher order ordinary differential equations with general boundary conditions. Symbolic analysis of boundary value problems, and the formulation of Green's operator and Green's function of semi-inhomogeneous boundary value problems, were first attempted by Markus Rosenkranz et al. in 2004 [1], also see [3, 4, 5, 6, 7]. In this paper, we extend the idea of the symbolic method for semi-inhomogeneous boundary value problems (inhomogeneous differential equation with homogeneous boundary conditions) to fully inhomogeneous boundary value problems (inhomogeneous differential equation with inhomogeneous boundary conditions) over integro-differential algebras.

The paper is layed out as follows. In Section 1.1 we recall the algebra of integro-differential operators, and in Section 1.2 we present the outline of the symbolic method for semi-inhomogeneous boundary value problems. Section 2, describes the proposed symbolic method for fully inhomogeneous boundary value problems. Sample computations are provided in Section 3, using the proposed algorithm to illustrate the symbolic method.

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1.1. Algebra of Integro-differential Operators

To express the boundary value problem, Green's operator and Green's function in operator based notations, the basic concepts of integro-differential algebras and the algebra of integro-differential operators are recalled, see [1] or [3, 4, 5, 6, 7] for further details. Throughout this section $\mathbb{K}$ denotes the field of characteristic zero and $\mathcal{F} = C^\infty[a,b]$ for simplicity.

Definition 1.1.([1, 7]) An algebraic structure $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ is called an *integro-differential algebra* over $\mathbb{K}$ if $\mathcal{F}$ is a commutative $\mathbb{K}$-algebra with $\mathbb{K}$-linear operators $\mathtt{D}$ and $\mathtt{A}$ such that the following conditions are satisfied

(1) $\mathtt{D}(\mathtt{A}f) = f$,

(2) $\mathtt{D}(fg) = (\mathtt{D}f)g + f(\mathtt{D}g)$,

(3) $(\mathtt{AD}f)(\mathtt{AD}g) + \mathtt{AD}(fg) = (\mathtt{AD}f)g + f(\mathtt{AD}g)$.

Here $\mathtt{D} : \mathcal{F} \to \mathcal{F}$ and $\mathtt{A} : \mathcal{F} \to \mathcal{F}$ are two maps defined by $\mathtt{D} = \frac{d}{dx}$, a derivation, and $\mathtt{A} = \int_a^x dx$, a $\mathbb{K}$-linear right inverse of $\mathtt{D}$, i.e. $\mathtt{D} \circ \mathtt{A} = 1$ (the identity map). The map $\mathtt{A}$ is called an *integral* for $\mathtt{D}$ and $\mathtt{A} \circ \mathtt{D} = 1 - \mathtt{E}$, where $\mathtt{E}$ is called the *evaluation* operator of $\mathcal{F}$ defined as $\mathtt{E} : f \mapsto f(a)$, evaluates at initial point a. An integro-differential algebra over $\mathbb{K}$ is called *ordinary* if $\mathrm{Ker}(\mathtt{D}) = \mathbb{K}$.

For an ordinary integro-differential algebra, the evaluation can be treated as a multiplicative linear functional $\mathtt{E} : \mathcal{F} \to \mathbb{K}$, i.e., $\mathtt{E}(fg) = (\mathtt{E}f)(\mathtt{E}g)$, for all $f, g \in \mathcal{F}$. Let $\Phi \subseteq \mathcal{F}^*$ be a set of all multiplicative linear functionals including $\mathtt{E}$. To specify a BVP, we also need a collection of "point evaluations" as new generators. For example, the boundary conditions $u(2) = 1, u'(1) = 5$, $\int_0^2 u \; dx = 0$ on a function $u \in \mathcal{F} = C^\infty[a,b]$ gives rise to the functional $\mathtt{E}_2 u = 1$, $\mathtt{E}_1\mathtt{D}u = 5$, $\mathtt{E}_2\mathtt{A}u = 0 \in \mathcal{F}^*$.

Definition 1.2.([1, 7]) Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra over $\mathbb{K}$ and $\Phi \subseteq \mathcal{F}^*$. The *integro-differential operators* $\mathcal{F}[\mathtt{D}, \mathtt{A}]$ are defined as the $\mathbb{K}$-algebra generated by the symbols $\mathtt{D}$ and $\mathtt{A}$, the functions $f \in \mathcal{F}$ and the characters (functionals) $\mathtt{E}_c, \phi, \chi \in \Phi$, modulo the Noetherian and confluent rewrite system given in Table 1.

Table 1: Rewrite rules for integro-differential operators

$fg \to f \cdot g$	$\mathtt{D}f \to f\mathtt{D} + f'$	$\mathtt{A}f\mathtt{A} \to (\mathtt{A}f)\mathtt{A} - \mathtt{A}(\mathtt{A}f)$
$\chi\phi \to \phi$	$\mathtt{D}\phi \to 0$	$\mathtt{A}f\mathtt{D} \to f - \mathtt{A}f' - (\mathtt{E}f)\mathtt{E}$
$\phi f \to (\phi f)\phi$	$\mathtt{DA} \to 1$	$\mathtt{A}f\phi \to (\mathtt{A}f)\phi$

For an integro-differential algebra $\mathcal{F}$, a fully inhomogeneous BVP is given by a monic *differential operator* $L = \mathtt{D}^n + a_{n-1}\mathtt{D}^{n-1} + \cdots + a_1\mathtt{D} + a_0$ and the *boundary conditions* $b_1, \ldots, b_n \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ with *boundary data* $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Given a forcing function $f \in \mathcal{F}$ and a set of boundary data $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, we want to find $u \in \mathcal{F}$ such that

$$\begin{aligned} Lu &= f, \\ b_1 u = \alpha_1, &\ldots, b_n u = \alpha_n. \end{aligned} \tag{1.1}$$

The quantities $\{f; \alpha_1, \ldots, \alpha_n\}$ are known collectively as the *data* for the BVP. We are not only interested to solve the BVP (1.1) for a specific data but also finding a suitable form of the solution that will exhibit its dependence on the data. The feature of (1.1) that enables us to achieve this goal is its linearity. If u_1 is the solution of the data $\{f_1; \alpha_{11}, \ldots, \alpha_{n1}\}$ and u_2 is the solution of the data $\{f_2; \alpha_{12}, \ldots, \alpha_{n2}\}$, then $\lambda_1 u_1 + \lambda_2 u_2$ is the solution of the data $\{\lambda_1 f_1 + \lambda_2 f_2; \lambda_1 \alpha_{11} + \lambda_2 \alpha_{12}, \ldots, \lambda_1 \alpha_{n1} + \lambda_2 \alpha_{n2}\}$. Hence one can decompose the data as

$$\{f; \alpha_1, \ldots, \alpha_n\} = \{f; 0, \ldots, 0\} + \{0; \alpha_1, \ldots, \alpha_n\}. \tag{1.2}$$

The BVP with data $\{f; 0, \ldots, 0\}$ is an inhomogeneous differential equation with homogeneous boundary conditions; the BVP with data $\{0; \alpha_1, \ldots, \alpha_n\}$ is a homogeneous differential equation with inhomogeneous boundary conditions. Symbolically, we can write the solution u of (1.1) as

$$u = F(f, \alpha_1, \ldots, \alpha_n),$$

where F is a linear operator that transforms the data into the solution. Hence we regard F as the inverse operator of L.

In [1], Rosenkranz et al. presented a symbolic solution $F(f, 0, \ldots, 0)$ of a BVP with data $\{f; 0, \ldots, 0\}$. In this paper we find the solution of fully inhomogeneous BVP with data $\{f; \alpha_1, \ldots, \alpha_n\}$. To motivate the solution, we briefly recall the symbolic solution $F(f, 0, \ldots, 0)$ in Section 1.2. In Section 2, we give the solution $F(f, \alpha_1, \ldots, \alpha_n)$ of fully inhomogeneous BVP.

1.2. Solution of Semi-inhomogeneous Boundary Value Problems

The algorithm for computing the solution, $F(f, 0, \ldots, 0)$, of a semi-inhomogeneous BVP is described in [1] with details, and also [6]. Consider a semi-inhomogeneous BVP

$$\begin{aligned} Lu &= f, \\ b_1 u = 0, &\ldots, b_n u = 0, \end{aligned} \tag{1.3}$$

where L is a surjective linear map and $B = \{b_1, \ldots, b_n\} \subseteq \mathcal{F}^*$ is a closed subspace of the dual space. We call $F(f, 0, \ldots, 0) \in \mathcal{F}$ a solution of (1.3) for a given forcing function $f \in \mathcal{F}$, if $LF(f, 0, \ldots, 0) = f$ and $F(f, 0, \ldots, 0) \in B^\perp$. In operator notations $LF = 1$ and $BF = 0$, and the operator F is called *Green's operator*. The Green's operator maps each f to its unique solution $F(f, 0, \ldots, 0)$. The BVP (1.3) is called *regular* if and only if $B^\perp$ is complement of $\mathrm{Ker}(L)$ so that $\mathcal{F} = \mathrm{Ker}(L) \oplus B^\perp$

as a direct sum. The regularity of a BVP can be tested algorithmically [1, p. 30] as follows: If $u_1, \dots, u_n$ is a basis for $\mathrm{Ker}(L)$ and $\{b_1, \dots, b_n\}$ is a basis for B, then the BVP is regular if and only if the *evaluation matrix*

$$b(u) = \begin{pmatrix} b_1(u_1) & \dots & b_1(u_n) \\ \vdots & \ddots & \vdots \\ b_n(u_1) & \dots & b_n(u_n) \end{pmatrix} \tag{1.4}$$

is regular.

The differential operator L is always surjective and the $\dim \mathrm{Ker}(L) = n < \infty$. Moreover, we can find the right inverse of L using variation of parameters as follows: Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra and let $L \in \mathcal{F}[\mathtt{D}]$ be monic with regular fundamental system $u_1, \dots, u_n$. Then the *fundamental right inverse* of L is [1, Corollary 29] given by

$$M = \sum_{i=1}^{n} u_i \mathtt{A} d^{-1} d_i \in \mathcal{F}[\mathtt{D}, \mathtt{A}], \tag{1.5}$$

where d is the determinant of Wronskian matrix W for $u_1, \dots, u_n$ and d_i the determinant of W_i obtained from W by replacing the i-th column by the n-th unit vector. If $\{b_1, \dots, b_n\}$ and $\{u_1, \dots, u_n\}$ are bases for B and $\mathrm{Ker}(L)$ respectively with b_i biorthogonal to u_i, then the projector operator P is [1, p. 26] determined by

$$P = \sum_{i=1}^{n} u_i b_i. \tag{1.6}$$

The main steps [1] to determine the solution $F(f, 0, \dots, 0)$ of a semi-inhomogeneous BVP and the corresponding Green's operator F are:

I. Compute the fundamental right inverse $M \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ from a given fundamental system as in (1.5).

II. Compute the projector $P \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ onto $\mathrm{Ker}(L)$ along $B^{\perp}$ as in (1.6).

III. Now the Green's operator F is computed as $F = M - PM$, and the solution is $u = F(f, 0, \dots, 0) = (M - PM)f$.

2. Solution of Fully-inhomogeneous Boundary Value Problems

In this section, we present a method/algorithm to compute the solution $u = F(f, \alpha_1, \dots, \alpha_n)$ and the corresponding Green's operator F for a BVP with the data $\{f; \alpha_1, \dots, \alpha_n\}$. From equation (1.2), one can decompose the solution as

$$F(f, \alpha_1, \dots, \alpha_n) = F(f, 0, \dots, 0) + F(0, \alpha_1, \dots, \alpha_n).$$

In Section 1.2, we computed the solution $F(f, 0, \ldots, 0)$. In this section, we present a method for computing the solution $F(0, \alpha_1, \ldots, \alpha_n)$ and then $F(f, \alpha_1, \ldots, \alpha_n)$ as a composition of two solutions $F(f, 0, \ldots, 0)$ and $F(0, \alpha_1, \ldots, \alpha_n)$.

Consider a semi-homogeneous boundary value problem

$$\begin{aligned} Lu &= 0, \\ b_1 u &= \alpha_1, \ldots, b_n u = \alpha_n. \end{aligned}$$

Let H be any function (not necessarily satisfying the differential operator L) such that $b_i H = \alpha_i$, for $i = 1, \ldots, n$. Set

$$u = H + v,$$

then one can observe that v satisfies the semi-inhomogeneous BVP

$$\begin{aligned} Lv &= -LH, \\ b_1 v &= 0, \ldots, b_n v = 0, \end{aligned}$$

and the solution of semi-inhomogeneous BVP is computed as

$$v = F(-LH, 0, \ldots, 0) = (M - PM)(-LH).$$

Since $u = H + v$, we have

$$u = F(0, \alpha_1, \ldots, \alpha_n) = (M - PM)(-LH) + H, \tag{2.1}$$

On simplification of (2.1), we have

$$u = F(0, \alpha_1, \ldots, \alpha_n) = PH,$$

where $P \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ is the projector onto $\mathrm{Ker}(L)$ along $B^{\perp}$ as given in equation (1.6) and $H \in \mathcal{F}$ is an irrespective operator of L such that $b_i H = \alpha_i$, for $i = 1, \ldots, n$. We call H as a *right inverse* of B such that $b_i H = \alpha_i$ and it is computed as in the following lemma.

Since H is depending only on the boundary data, this amounts to an *interpolation problem* with Stieltjes boundary conditions, presented in the next lemma. The rest of paper, right inverse of B means the right inverse of each element of B such that $b_i H = \alpha_i$.

Lemma 2.1. *Let $\{u_1, \ldots, u_n\} \subset \mathcal{F}$ and $\{b_1, \ldots, b_n\} \subset \mathcal{F}^*$ be bases for Ker(L) and B respectively, and $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}$ be boundary data. Then there exists a unique right inverse H of B such that $b_i H = \alpha_i$ is given by*

$$H = u_v^T b(u)^{-1} \alpha_v, \tag{2.2}$$

where $u_v = (u_1, \ldots, u_n)$ and $\alpha_v = (\alpha_1, \ldots, \alpha_n)$ are column vectors and $b(u)$ is evaluation matrix as in equation (1.4).

Proof. From given data, we have conditions $b_1u = \alpha_1, \ldots, b_nu = \alpha_n$. Suppose

$$H = c_1u_1 + \cdots + c_nu_n \tag{2.3}$$

is an interpolating function satisfying the given conditions, i.e., $b_iH = \alpha_i$, where c_i are the unknown coefficients to be determined, for $i = 1, \ldots, n$. From the given boundary conditions $\{b_1, \ldots, b_n\}$ with $\{\alpha_1, \ldots, \alpha_n\}$, one can write equation (2.3) as

$$b(u)(c_1, \ldots, c_n)^T = (\alpha_1, \ldots, \alpha_n)^T,$$

where $b(u)$ is the evaluation matrix. Since the evaluation matrix is regular, there exists inverse of $b(u)$, and we have

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = b(u)^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Since H is the linear combination of $u_1, \ldots, u_n$ with coefficients $c_1, \ldots, c_n$, we have

$$H = u_v^T b(u)^{-1} \alpha_v.$$

It completes the proof. □

Now the generalization of the above results is presented in the following theorem.

Theorem 2.2. *Let $(\mathcal{F}, \mathtt{D}, \mathtt{A})$ be an ordinary integro-differential algebra and $B = \{b_1, \ldots, b_n\} \subset \mathcal{F}^*$. Given a boundary data $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}$, the BVP*

$$\begin{aligned} Lu &= 0, \\ b_1u &= \alpha_1, \ldots, b_nu = \alpha_n. \end{aligned}$$

has the unique solution

$$u = F(0, \alpha_1, \ldots, \alpha_n) = PH,$$

where $P \in \mathcal{F}[\mathtt{D}, \mathtt{A}]$ is a projector onto Ker(L) along $B^\perp$ as in equation (1.6) *and $H \in \mathcal{F}$ such that $b_iH = \alpha_i$ as in equation* (2.2).

Finally, we compute the solution of fully inhomogeneous BVP with data $\{f; \alpha_1, \ldots, \alpha_n\}$ as composition of two solutions

$$F(f, \alpha_1, \ldots, \alpha_n) = (M - PM)(f) + PH$$

and the corresponding Green's operator is

$$F = (M - PM) + PH.$$

One can easily check that the Green's operator F satisfies $LF = 1$ and $BF = \alpha$, and the solution $F(f, \alpha_1, \ldots, \alpha_n)$ satisfies $LF(f, \alpha_1, \ldots, \alpha_n) = f$ and

$BF(f, \alpha_1, \ldots, \alpha_n) = \alpha$.

3. Sample Computations

Example 3.1. Consider the one-dimensional problem of a thin rod occupying the interval $(0, a)$ on the x-axis. This is one of the classical examples of the ordinary linear BVPs [2]. We solve

$$\frac{d^2u}{dx^2} = f, \quad 0 < x < 1; \quad u(0) = \alpha, \quad u(1) = \beta, \tag{3.1}$$

for the temperature $u \in C^\infty[0, 1]$, where $f \in C^\infty[0, 1]$ is the prescribed source density (per unit length of the rod) of heat and α, β are the prescribed end temperatures.

The operator representation of the given BVP (3.1) is

$$\begin{aligned} &Lu = f \\ &\mathtt{E}_0 u = \alpha, \mathtt{E}_1 u = \beta, \end{aligned}$$

where the differential operator $L = \mathtt{D}^2$ with $\mathrm{Ker}(L) = \{1, x\}$, and the set of boundary operators is $B = \{\mathtt{E}_0, \mathtt{E}_1\}$ with boundary data $\{\alpha, \beta\}$. The null space projector P is computed as in equation (1.6), and it is given by

$$P = (1 - x)\mathtt{E}_0 + x\mathtt{E}_1$$

The fundamental right inverse of L, computed as described in equation (1.5), and it is given by

$$M = x\mathtt{A} - \mathtt{A}x.$$

The right inverse H of B is computed as follows: For a given fundamental system $\{1, x\}$, boundary operators $\{\mathtt{E}_0, \mathtt{E}_1\}$ with boundary data $\{\alpha, \beta\}$, the operator H calculated as

$$H = (1, x)\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha(1 - x) + \beta x.$$

The Green's operator of BVP (3.1) can be computed using the proposed algorithm as

$$\begin{aligned} F &= (1 - P)M + PH \\ &= x\mathtt{A} - \mathtt{A}x - x\mathtt{E}_1\mathtt{A} + x\mathtt{E}_1\mathtt{A}x + \alpha(1 - x) + \beta x. \end{aligned}$$

By translating the symbols, the Green's operator is

$$F = x\int_0^x - \int_0^x \xi - x\int_0^1 + x\int_0^1 \xi + \alpha(1 - x) + \beta x,$$

and integration can be performed in closed term using elementary integration techniques as

$$F = (1 - x)\int_0^x \xi + x\int_x^1 (1 - \xi) + \alpha(1 - x) + \beta x.$$

Now the solution of the given BVP (3.1) is

$$u = F(f, \alpha, \beta) = \int_0^1 g(x,\xi) f(\xi)\ d\xi + \alpha(1-x) + \beta x, \tag{3.2}$$

where the Green's function $g(x,\xi)$ is

$$g(x,\xi) = \begin{cases} (x-1)\xi & \text{if } \ 0 \le \xi \le x \le 1, \\ x(\xi - 1) & \text{if } \ 0 \le x \le \xi \le 1. \end{cases}$$

For specific $f(x)$, the integration in (3.2) can be expressed as

$$u = (1-x)\int_0^x \xi f(\xi)\ d\xi + x\int_x^1 (1-\xi) f(\xi)\ d\xi + \alpha(1-x) + \beta x.$$

Example 3.2. Consider the following BVP

$$\begin{aligned} &\frac{d^3u}{dx^3} - 3\frac{d^2u}{dx^2} + 3\frac{du}{dx} - u = x^2 e^x, \\ &u(0) = 0, u'(0) = 0, u(1) = 0. \end{aligned}$$

Operator notation of the given BVP is

$$\begin{aligned} &Lu = f \\ &\mathtt{E}_0 u = \alpha_1, \mathtt{E}_0\mathtt{D} u = \alpha_2, \mathtt{E}_1 u = \alpha_3, \end{aligned} \tag{3.3}$$

where $L = \mathtt{D}^3 - 3\mathtt{D}^2 + 3\mathtt{D} - 1$, $\mathtt{E}_0 u = u(0), \mathtt{E}_0\mathtt{D}u = u'(0), \mathtt{E}_1 u = u(1)$ and $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Fundamental right inverse of L is

$$M = \frac{1}{2}e^x \mathtt{A} e^{-x} x^2 - e^x x \mathtt{A} e^{-x} x + \frac{1}{2} e^x x^2 \mathtt{A} e^{-x},$$

and H is computed as

$$H = \begin{pmatrix} e^x & xe^x & x^2e^x \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ e & e & e \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

The Green's operator of BVP (3.3) is computed as follows

$$\begin{aligned} F &= (1-P)M + PH \\ &= \frac{1}{2}e^x \mathtt{A} e^{-x} x^2 - x e^x \mathtt{A} x e^{-x} + \frac{1}{2} x^2 e^x \mathtt{A} e^{-x} - \frac{1}{2} x^2 e^x \mathtt{E}_1 \mathtt{A} e^{-x} x^2 \\ &\quad + x^2 e^x \mathtt{E}_1 \mathtt{A} x e^{-x} - \frac{1}{2} x^2 e^x \mathtt{E}_1 \mathtt{A} e^{-x}, \end{aligned}$$

and the complete solution is

$$u = F(f, \alpha, \beta) = \int_0^1 g(x, \xi) f(\xi)\ d\xi + 0$$

where the Green's function $g(x, \xi)$ is

$$g(x, \xi) = \begin{cases} -\frac{1}{2} e^{x-\xi} \xi(-\xi + 2x + x^2\xi - 2x^2) & \text{if } 0 \le \xi \text{ and } \xi \le x \text{ and } x \le 1, \\ -\frac{1}{2} e^{x-\xi} x^2(\xi^2 - 2\xi + 1) & \text{if } 0 \le x \text{ and } x \le \xi \text{ and } \xi \le 1. \end{cases}$$

If $f(x) = x^2 e^x$, then the exact solution of the given BVP (3.3) is

$$u = \frac{1}{60} x^5 e^x - \frac{1}{60} x^2 e^x.$$

Example 3.3. A particle of mass m moves along the u axis under the influence of a force $f(t)$ directed along the axis. The motion of the particle is determined by Newton's law with initial conditions

$$m \frac{d^2 u}{dt^2} = f(t), \quad t > 0; \qquad u(0) = \alpha, \quad \frac{du}{dt}(0) = \beta. \tag{3.4}$$

Operator notation of (3.4) is

$$\begin{aligned} &m\mathtt{D}^2 u = f(t), \\ &\mathtt{E}_0 u = \alpha, \quad \mathtt{E}_0 \mathtt{D} u = \beta. \end{aligned}$$

The solution of (3.4) computed similar to Example 3.1, and it is given by

$$u = F(f, \alpha, \beta) = \int_0^\infty g(t, \xi) f(\xi)\ d\xi + \alpha + \beta t \tag{3.5}$$

where the Green's function $g(t, \xi)$ is

$$g(t, \xi) = \begin{cases} 0 & \text{if } 0 \le t \le \xi < \infty, \\ \frac{t-\xi}{m} & \text{if } 0 \le \xi \le t < \infty. \end{cases}$$

Equation (3.5) can be written now as

$$u = \frac{1}{m} \int_0^t (t - \xi) f(\xi)\ d\xi + \alpha + \beta t.$$

Example 3.4. Consider a BVP of damped oscillator. Given a forcing function $f \in C^\infty[0, \pi]$, we find $u \in C^\infty[0, \pi]$ such that

$$\frac{d^2 u}{dx^2} + 2\frac{du}{dx} + u = f, \quad u(0) = \alpha, \quad u(\pi) = \beta.$$

Following the algorithm in Section 2 similar to Example 3.1, the solution is

$$u = F(f,\alpha,\beta) = \int_0^\pi g(x,\xi)f(\xi)\,d\xi + \left(\frac{\beta e^\pi - \alpha}{\pi}\right) x e^{-x} + \alpha e^{-x}, \tag{3.6}$$

where the Green's function $g(x,\xi)$ is given by

$$g(x,\xi) = \begin{cases} \frac{1}{\pi}(\pi - x)\xi e^{\xi - x} & \text{if } 0 \le \xi \le x \le \pi, \\ \frac{1}{\pi}(\pi - \xi) x e^{\xi - x} & \text{if } 0 \le x \le \xi \le \pi. \end{cases}$$

Conclusion

In this paper, we discussed a symbolic method to solve BVPs with inhomogeneous Stieltjes boundary conditions over integro-differential algebras. We extend the idea of the symbolic method for semi-inhomogeneous BVPs to the fully inhomogeneous BVPs over algebras. Using the proposed method one can compute the Green's operator and the Green's function of the given problem. Sample computations are presented using the proposed method.

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