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Study of Generalized Derivations in Rings with Involution

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Abstract. Let R be a prime ring with involution of the second kind and centre $Z(R)$. Suppose R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$. The purpose of this paper is to study the commutativity of a prime ring R satisfying any one of the following identities: (i) $F(x) \circ x^* \in Z(R)$ (ii) $F([x, x^*]) \pm x \circ x^* \in Z(R)$ (iii) $F(x \circ x^*) \pm [x, x^*] \in Z(R)$ (iv) $F(x) \circ d(x^*) \pm x \circ x^* \in Z(R)$ (v) $[F(x), d(x^*)] \pm x \circ x^* \in Z(R)$ (vi) $F(x) \pm x \circ x^* \in Z(R)$ (vii) $F(x) \pm [x, x^*] \in Z(R)$ (viii) $[F(x), x^*] \mp F(x) \circ x^* \in Z(R)$ (ix) $F(x \circ x^*) \in Z(R)$ for all $x \in R$.

1. Introduction

Throughout this paper R will represent a prime ring with center $Z(R)$. An additive mapping $* : R \to R$ is called an involution if $*$ is an anti-automorphism of order 2; that is, $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution $*$ is known as ring with involution or $*$-ring. If $char(R) \neq 2$, involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case, $S(R) \cap Z(R) \neq (0)$. A ring R is said to be normal if $xx^* = x^*x$ for all $x \in R$. An example is the ring of quaternions. A description of such rings can be found in [12], where further references can be found.

A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) =$

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$d(x)y + xd(y)$ for all $x, y \in R$. Following Bresar [9], an additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Basic examples are derivations and generalized inner derivations i.e., maps of type $x \mapsto ax + xb$ for some $a, b \in R$. A map $f : R \to R$ is said to be centralizing on R if $[f(x), x] \in Z(R)$ for all $x \in R$. In a special case, when $[f(x), x] = 0$ holds for all $x \in R$, a map f is said to be commuting on R. The study of centralizing and commuting mappings on prime rings was initiated by the result of Posner [16], which states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring has to be commutative. Through the years, a lot of work has been done in this context by a number of authors (see, for example, [1, 5, 7, 8, 13, 17] and references therein).

Very recently in many papers the additive mappings like derivations, generalized derivations have been studied in the setting of rings with involution and in fact it was seen that there is a close connection between these mappings and the commutativity of the ring R. For instance in [2], it is proved that let R be a prime ring with involution $*$ such that $char(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), x^*] \in Z(R)$ for all $x \in R$ and $d(S(R) \cap Z(R)) \neq (0)$. Then R is commutative. Many other results in this direction can be found in [3, 4, 10, 15]. The goal of the present paper is to continue this line of investigation by considering certain identities involving Jordan product in the setting of generalized derivations.

2. Main Results

We begin our investigation with the following lemmas, which are essential to prove our main results.

Lemma 2.1.([15]) *Let R be a prime ring with involution $'*'$ of the second kind. Then $[x, x^*] \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 2.2.([15]) *Let R be a prime ring with involution $'*'$ of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Theorem 2.3. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $F(x) \circ x^* \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. By the given hypothesis, we have

$$F(x) \circ x^* \in Z(R) \quad for \quad all \quad x \in R. \tag{2.1}$$

A Linearization of (2.1) yields that

$$F(x) \circ y^* + F(y) \circ x^* \in Z(R) \quad for \quad all \quad x, y \in R.$$

This can be further written as

$$[F(x) \circ y^*, r] + [F(y) \circ x^*, r] = 0 \quad for \quad all \ x, y, r \in R. \tag{2.2}$$

Replacing y by hy where $h \in H(R) \cap Z(R)$, we get

$$[F(x) \circ (hy)^*, r] + [F(hy) \circ x^*, r] = 0 \quad for \quad all \ x, y, r \in R.$$

On solving, we obtain

$$[F(x) \circ y^*, r]h + [F(y) \circ x^*, r]h + [y \circ x^*, r]d(h) = 0 \quad for \quad all \quad x, y, r \in R$$

and $h \in H(R) \cap Z(R)$. Using (2.2), we get $[y \circ x^*, r]d(h) = 0$ for all $x, y, r \in R$ and $h \in H(R) \cap Z(R)$. Using the primeness of R we have either $[y \circ x^*, r] = 0$ for all $x, y, r \in R$ or $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. First if we consider $[y \circ x^*, r] = 0$ for all $x, y, r \in R$. Replacing y by x we get $[x \circ x^*, r] = 0$ for all $x, r \in R$. Thus in view of Lemma 2.2, we get R is commutative. Now consider the second case $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This intern implies $d(k) = 0$ for all $k \in S(R) \cap Z(R)$ and hence $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ky where $k \in S(R) \cap Z(R)$ in (2.2) and using $d(z) = 0$ for all $z \in Z(R)$, We obtain

$$-[F(x) \circ y^*, r]k + [F(y) \circ x^*, r] = 0 \quad for \quad all \quad x, y, r \in R \quad and \ k \in S(R) \cap Z(R).$$

Using (2.2) in the previous equation, we get $2[F(y) \circ x^*, r]k = 0$ for all $x, y, r \in R$ and $k \in S(R) \cap Z(R)$. Since $char(R) \neq 2$, we have $[F(y) \circ x^*, r]k = 0$ for all $x, y, r \in R$ and $k \in S(R) \cap Z(R)$. Now using the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we have $[F(y) \circ x^*, r] = 0$ for all $x, y, r \in R$. That is, $[F(y) \circ x, r] = 0$ for all $x, y, r \in R$. Taking $x = z$ where $z \in Z(R)$, we get $[2F(x)z, r] = 0$ for all $x, r \in R$ and $z \in Z(R)$. Since $char(R) \neq 2$, we get $[F(x)z, r] = 0$ for all $x, r \in R$ and $z \in Z(R)$. Further since $S(R) \cap Z(R) \neq (0)$ and using the primeness of R, we have $[F(y), r] = 0$ for all $y, r \in R$. Replace y by r, we get $[F(r), r] = 0$ for all $r \in R$. Thus in view of [18, Theorem 3.1], R is commutative. □

Theorem 2.4. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $F([x, x^*]) \pm x \circ x^* \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. We first consider the case

$$F([x, x^*]) + x \circ x^* \in Z(R) \quad for \quad all \quad x \in R. \tag{2.3}$$

If F is zero, then we get $x \circ x^* \in Z(R)$. Then by Lemma 2.2, we get R is commutative. Now consider F is non zero. Linearizing (2.3), we get

$$F([x, y^*]) + F([y, x^*]) + x \circ y^* + y \circ x^* \in Z(R) \quad for \quad all \quad x, y \in R. \tag{2.4}$$

This can be further written as

$$[F([x, y^*]), r] + [F([y, x^*]), r] + [x \circ y^*, r] + [y \circ x^*, r] = 0 \quad for \quad all \quad x, y, r \in R.$$

Replacing y by hy where $h \in H(R) \cap Z(R)$, we have

$$[F([x, y^*]h), r] + [F([y, x^*]h), r] + [x \circ y^*, r]h + [y \circ x^*, r]h = 0$$

for all $x, y, r \in R$ and $h \in H(R) \cap Z(R)$. This further implies that

$$(2.5)\quad [F([x,y^*]),r]h+[[x,y^*],r]d(h)+[F([y,x^*]),r]h+[[y,x^*],r]d(h)+[x\circ y^*,r]h$$

$+[y\circ x^*,r]h=0$ for all $x,y,r\in R$ and $h\in H(R)\cap Z(R)$. Using (2.4) in (2.5), we get $([[x,y^*],r]+[[y,x^*],r])d(h)=0$ for all $x,y,r\in R$ and $h\in H(R)\cap Z(R)$. Using the primeness of R we have either $([[x,y^*],r]+[[y,x^*],r])=0$ for all $x,y,r\in R$ or $d(h)=0$ for all $h\in H(R)\cap Z(R)$. If we consider $([[x,y^*],r]+[[y,x^*],r])=0$ for all $x,y,r\in R$. This implies that $[x,y^*]+[y,x^*]\in Z(R)$ for all $x,y\in R$. Replacing y by x we get $[x,x^*]\in Z(R)$ for all $x\in R$. Thus in view of Lemma 2.1, we get R is commutative. Now suppose that $d(h)=0$ for all $h\in H(R)\cap Z(R)$. This further implies that $d(z)=0$ for all $z\in Z(R)$. Replacing y by ky in (2.4) where $k\in S(R)\cap Z(R)$, we get

$$-F([x,y^*]k)+F([y,x^*]k)-(x\circ y^*)k+(y\circ x^*)k\in Z(R)\quad for\ \ all\ \ x,y\in R$$

and $k\in S(R)\cap Z(R)$. This further implies that

$$-F([x,y^*])k-[x,y^*]d(k)+F([y,x^*])k+[y,x^*]d(k)-(x\circ y^*)k+(y\circ x^*)k\in Z(R)$$

for all $x,y\in R$ and $k\in S(R)\cap Z(R)$. Using (2.4) and the fact that $d(z)=0$ for all $z\in Z(R)$, we get $(F([y,x^*])+y\circ x^*)k\in Z(R)$ for all $x,y\in R$ and $k\in S(R)\cap Z(R)$. Now using the primeness and the fact that $S(R)\cap Z(R)\neq (0)$, we obtain $(F([y,x^*])+y\circ x^*)\in Z(R)$ for all $x,y\in R$. Replace x by x^*, we get $(F([y,x])+y\circ x)\in Z(R)$ for all $x,y\in R$. Taking $y=x$, we get $x^2\in Z(R)$ for all $x\in R$. Replacing x by $x+y$ and using $x^2\in Z(R)$ for all $x\in R$, we get $xy+yx\in Z(R)$ for all $x,y\in R$. This can be further written as $[xy+yx,r]=0$ for all $x,y,r\in R$. Replace y by z where $z\in Z(R)$, we get $2[x,r]z=0$ for all $x,r\in R$ and $z\in Z(R)$. Finally using the facts that $char(R)\neq 2$, $S(R)\cap Z(R)\neq (0)$ and the primeness of R, we obtain R is commutative.
The second case can be proved in a similar manner with necessary variations. □

Theorem 2.5. *Let R be a prime ring with involution of the second kind such that $char(R)\neq 2$. If R admits a generalized derivation $F:R\to R$ associated with a derivation $d:R\to R$, such that $F(x\circ x^*)\pm[x,x^*]\in Z(R)$ for all $x\in R$. Then R is commutative.*

Proof. We first consider the positive sign case. If F is zero then we have $[x,x^*]\in Z(R)$ for all $x\in R$. Thus in view of Lemma 2.1, we get R is commutative. If F is nonzero then we have

$$(2.6)\qquad F(x\circ x^*)+[x,x^*]\in Z(R)\quad for\ \ all\ \ x\in R.$$

Linearizing (2.6), we get

$$(2.7)\qquad F(x\circ y^*)+F(y\circ x^*)+[x,y^*]+[y,x^*]\in Z(R)\quad for\ \ all\ \ x,y\in R.$$

Replacing y by hy where $h \in H(R) \cap Z(R)$ and using (2.7), we get

$$(x \circ y^* + y \circ x^*)d(h) \in Z(R) \quad for \ \ all \ \ x, y \in Z(R) \ \ and \ \ h \in H(R) \cap Z(R).$$

Now using the primeness of R, we have $(x \circ y^* + y \circ x^*) \in Z(R)$ for all $x, y \in R$ or $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replace y by x, we get $x \circ x^* \in Z(R)$ for all $x \in R$. Thus by Lemma 2.2, we get R is commutative. Now consider $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This further implies that $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ky in (2.7) where $k \in S(R) \cap Z(R)$ and using (2.7), we have $2(F(y \circ x^*) + [y, x^*])k \in Z(R)$ for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. This further implies that $F(y \circ x^*) + [y, x^*] \in Z(R)$ for all $x, y \in R$. Replacing x by x^*, we get $F(y \circ x) + [y, x] \in Z(R)$ for all $x, y \in R$. Taking $y = x$, we get $F(x^2) \in Z(R)$ for all $x \in R$. Linearizing this and using $F(x^2) \in Z(R)$ for all $x \in R$, we get $F(x \circ y) \in Z(R)$ for all $x, y \in R$. Replacing y by z, where $z \in Z(R)$ we get $2F(xz) \in Z(R)$ for all $x \in R$ and $z \in Z(R)$. This can be further written as $2[F(xz), r] = 0$ for all $x, r \in R$ and $z \in Z(R)$. Since $char(R) \neq 2$ this implies that $[F(xz), r] = 0$ for all $x, r \in R$ and $z \in Z(R)$. Since $d(z) = 0$ for all $z \in Z(R)$, we finally arrive at $[F(x), r]z = 0$ for all $x, r \in R$ and $z \in Z(R)$. Now using the primeness and the fact that $S(R) \cap Z(R) \neq (0)$ we have $[F(x), r] = 0$ for all $x, r \in R$. Replace r by x we get $[F(x), x] = 0$ for all $x \in R$. Then by [18, Theorem 3.1], we get R is commutative. Following the same steps, we get R is commutative in the second case as well. □

Theorem 2.6. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $F(x) \circ d(x^*) \pm x \circ x^* \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. We first consider the case

$$F(x) \circ d(x^*) + x \circ x^* \in Z(R) \quad for \ \ all \ \ x \in Z(R). \tag{2.8}$$

If either F or d or both are zero then we get $x \circ x^* \in Z(R)$ for all $x \in Z(R)$. Then by Lemma 2.2, we get R is commutative. Now consider the case in which both F and d are nonzero. Linearizing (2.8), we get

$$F(x) \circ d(y^*) + F(y) \circ d(x^*) + x \circ y^* + y \circ x^* \in Z(R) \quad for \ \ all \ \ x, y \in R. \tag{2.9}$$

Replacing y by hy where $h \in H(R) \cap Z(R)$ and using (2.9), we get

$$(F(x) \circ y^* + y \circ d(x^*))d(h) \in Z(R) \quad for \ \ all \ \ x, y \in R \ \ and \ \ h \in H(R) \cap Z(R).$$

Using the primeness condition, we get either $F(x) \circ y^* + y \circ d(x^*) \in Z(R)$ for all $x, y \in R$ or $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. First consider $F(x) \circ y^* + y \circ d(x^*) \in Z(R)$ for all $x, y \in R$. This can be further written as

$$[F(x) \circ y^*, r] + [y \circ d(x^*), r] = 0 \quad for \ \ all \ \ x, y, r \in R. \tag{2.10}$$

Replacing y by ky where $k \in S(R)\cap Z(R)$ and using (2.10), we get $2[y\circ d(x^*), r]k = 0$ for all $x, y, r \in R$ and $k \in S(R) \cap Z(R)$. Using the primeness and the facts that $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get $[y \circ d(x^*), r] = 0$ for all $x, y, r \in R$. Replacing x by x^* we have $[y\circ d(x), r] = 0$ for all $x, y, r \in R$. Replacing y by z where $z \in Z(R)$, we get $2[d(x), r]z = 0$ for all $x, r \in R$ and $z \in Z(R)$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $[d(x), r] = 0$ for all $x, r \in R$. Then by posner's result [16], we get R is commutative. Now we consider the case $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ky in (2.9) where $k \in S(R) \cap Z(R)$ and using (2.9), we obtain

$$(2.11)\quad 2(F(y) \circ d(x^*) + y \circ x^*)k \in Z(R) \;\; for \;\; all \;\; x, y \in R \;\; and \;\; k \in S(R) \cap Z(R).$$

This further implies that $F(y) \circ d(x^*) + y \circ x^* \in Z(R)$ for all $x, y \in R$. That is, $F(y) \circ d(x) + y \circ x \in Z(R)$ for all $x, y \in R$. Taking $x = z$ where $z \in Z(R)$ and using $d(z) = 0$ for all $z \in Z(R)$, we get $2yz \in Z(R)$ for all $y \in R$ and $z \in Z(R)$. Since $char(R) \neq 2$, we get that $yz \in Z(R)$ for all $y \in R$ and $z \in Z(R)$. This can be written as $[yz, r] = 0$ for all $y, r \in R$ and $z \in Z(R)$. This further implies that $[y, r] = 0$ for all $y, r \in R$. That is, R is commutative.

Similarly we can prove the second part. □

Theorem 2.7. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $[F(x), d(x^*)] \pm x \circ x^* \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. The proof is on the similar lines as in the above theorem. □

Theorem 2.8. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $F(x) \pm x \circ x^* \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. we first consider the case

$$(2.12)\qquad F(x) + x \circ x^* \in Z(R) \;\; for \;\; all \;\; x \in R.$$

If F is zero then we have $x \circ x^* \in Z(R)$ for all $x \in R$. Then by Lemma 2.2, we get R is commutative. Consider F is nonzero. Linearizing (2.12), we get

$$(2.13)\qquad x \circ y^* + y \circ x^* \in Z(R) \;\; for \;\; all \;\; x, y \in R.$$

Replacing y by ky where $k \in S(R) \cap Z(R)$ and using (2.13), we get $(y \circ x^*)k \in Z(R)$ for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. Using primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we have $y \circ x^* \in Z(R)$ for all $x, y \in R$. Taking $y = x$, we get $x \circ x^* \in Z(R)$ for all $x \in R$. Thus in view of Lemma 2.2, we get R is commutative.

Similarly we can prove the other case. □

On similar lines we can prove the following result.

Theorem 2.9. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $F(x) \pm [x, x^*] \in Z(R)$ for all $x \in R$. Then R is commutative.*

Theorem 2.10. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $[F(x), x^*] \mp F(x) \circ x^* \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. We first consider the case

$$[F(x), x^*] - F(x) \circ x^* \in Z(R) \quad for \ \ all \ \ x \in R. \tag{2.14}$$

Linearizing (2.14), we get

$$[F(x), y^*] + [F(y), x^*] - F(x) \circ y^* - F(y) \circ x^* \in Z(R) \quad for \ \ all \ \ x, y \in Z(R). \tag{2.15}$$

Replacing y by hy where $h \in H(R) \cap Z(R)$ and using (2.15), we get

$$([y, x^*] - y \circ x^*)d(h) \in Z(R) \quad for \ \ all \ \ x, y \in R \ \ and \ \ h \in H(R) \cap Z(R).$$

Using the primeness condition we have $[y, x^*] - y \circ x^* \in Z(R)$ for all $x, y \in R$ or $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. First suppose $[y, x^*] - y \circ x^* \in Z(R)$ for all $x, y \in R$. This further implies that $[y, x] - y \circ x \in Z(R)$ for all $x, y \in R$. Replacing y by x we get $x^2 \in Z(R)$ for all $x \in R$. Replacing x by $x+y$, we obtain $x \circ y \in Z(R)$ for all $x, y \in R$. That is, $[x \circ y, r] = 0$ for all $x, y, r \in R$. Taking $y = z$, where $z \in Z(R)$, we get $2[x, r]z = 0$ for all $x, r \in R$ and $z \in Z(R)$. Since $char(R) \neq 2$, $S(R) \cap Z(R) \neq (0)$ and using the primeness of R, we have $[x, r] = 0$ for all $x, r \in R$. This implies that R is commutative. Now we consider the case $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This intern implies that $d(z) = 0$ for all $z \in Z(R)$. Replacing y by ky where $k \in S(R) \cap Z(R)$ in (2.15) and using (2.15), we get

$$2([F(y), x^*] - F(y) \circ x^*)k \in Z(R) \quad for \ \ all \ \ x, y \in R \ \ and \ \ k \in S(R) \cap Z(R). \tag{2.16}$$

Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we obtain

$$[F(y), x^*] - F(y) \circ x^* \in Z(R) \quad for \ \ all \ \ x, y \in R. \tag{2.17}$$

Replacing x by x^*, we get

$$[F(y), x] - F(y) \circ x \in Z(R) \quad for \ \ all \ \ x, y \in R.$$

This can be further written as $F(y)x - xF(y) - F(y)x - xF(y) \in Z(R)$ for $x, y \in R$. This implies that $-2xF(y) \in Z(R)$ for all $x, y \in R$. That is, $xF(y) \in Z(R)$ for all $x, y \in R$, since $char(R) \neq 2$. That is, $[xF(y), r] = 0$ for all $x, y, r \in R$. Replacing r by x, we get

$$x[F(y), x] = 0 \quad for \ \ all \ \ x, y \in R. \tag{2.18}$$

Replacing y by yx we get $x[F(y),x]x + xy[d(x),x] + x[y,x]d(x) = 0$ for all $x, y \in R$. using (2.18), we get $xy[d(x),x] + x[y,x]d(x) = 0$ for all $x, y \in R$. Taking $y = x$, we have

$$x^2[d(x),x] = 0 \quad for \quad all \quad x \in R. \tag{2.19}$$

Replacing x by $x + z$ where $z \in Z(R)$ and using $d(z) = 0$ for all $z \in Z(R)$, we obtain $(z^2 + 2xz)[d(x),x] = 0$ for all $x \in R$ and $z \in Z(R)$. Left multiplying by x we get $z^2x[d(x),x] + x^2[d(x),x]2z = 0$ for all $x \in R$ and $z \in Z(R)$. Using (2.19), we get $z^2x[d(x),x] = 0$ for all $x \in R$ and $z \in Z(R)$. Now using the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$x[d(x),x] = 0 \quad for \quad all \quad x \in R. \tag{2.20}$$

Replacing x by $x + z$ and using $d(z) = 0$ for all $z \in Z(R)$, we get $z[d(x),x] = 0$ for all $x \in R$ and $z \in Z(R)$. Using primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we finally arrive at $[d(x),x] = 0$ for all $x \in R$. Thus by the result of Posner [16], we get R is commutative.

Now we consider the case

$$[F(x),x^*] + x \circ x^* \in Z(R) \quad for \quad all \quad x \in R. \tag{2.21}$$

Using the same steps as we did in the above case, we get

$$[F(y),x^*] + F(y) \circ x^* \in Z(R) \quad for \quad all \quad x, y \in R.$$

Replacing x by x^*, we have

$$[F(y),x] + F(y) \circ x \in Z(R) \quad for \quad all \quad x, y \in R.$$

This can be further written as

$$F(y)x - xF(y) + F(y)x + xF(y) \in Z(R) \quad for \quad all \quad x, y \in R.$$

That is, $2F(y)x \in Z(R)$ for all $x, y \in R$. Since $char(R) \neq 2$, this implies that $F(y)x \in Z(R)$ for all $x, y \in R$. That is, $[F(y)x, r] = 0$ for all $x, y, r \in R$. This intern implies that $F(y)[x,r] + [F(y),r]x = 0$ for all $x, y, r \in R$. Taking $r = x$, we get

$$[F(y),x]x = 0 \; for \; all \; x, y \in R. \tag{2.22}$$

Replacing y by yx, we arrive at

$$[F(y),x]x^2 + y[d(x),x]x + [y,x]d(x)x = 0 \quad for \quad all \quad x, y \in R.$$

Using (2.22), we get

$$y[d(x),x]x + [y,x]d(x)x = 0 \quad for \quad all \quad x, y \in R.$$

Replacing y by z where $z \in Z(R)$. We obtain

$$z[d(x),x]x = 0 \quad for \ \ all \ \ x \in R \ \ and \ \ z \in Z(R). \tag{2.23}$$

Using the primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we have $[d(x),x]x = 0$ for all $x \in R$. Replacing x by $x+z$, where $z \in Z(R)$, using (2.23) and the fact that $d(z) = 0$ for all $z \in Z(R)$, we get $[d(x),x]z = 0$ for all $x \in R$ and $z \in Z(R)$. This further implies that $[d(x),x] = 0$ for all $x \in R$. Hence in view of Posner's result [16], we get R is commutative. □

Theorem 2.11. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a generalized derivation $F : R \to R$ associated with a derivation $d : R \to R$, such that $F(x \circ x^*) \in Z(R)$ for all $x \in R$. Then R is commutative.*

Proof. Proceeding on the similar lines as we did in the previous result, we get $F(x \circ y^*) \in Z(R)$ for all $x, y \in R$ (where $d(Z(R)) = (0)$). Replacing y by z^* where $z \in Z(R)$, we get $2F(x)z \in Z(R)$ for all $x \in R$ and $z \in Z(R)$. Using the conditions that R is prime, $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we finally arrive at $F(x) \in Z(R)$ for all $x \in R$. This implies that $[F(x), r] = 0$ for all $x, r \in R$. Then by the result [18], we get R is commutative. □

At the end of paper, let us write an example which shows that the restriction of the second kind involution in our results is not superfluous

Example 2.21. Let $R = \left\{ \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \middle| \ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{Z}_2 \right\}$. Of course R under matrix addition and matrix multiplication is a noncommutative prime ring. Define mappings $F, D, * : R \longrightarrow R$ such that

$$F\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix},$$
$$D\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix},$$
$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}^* = \begin{pmatrix} \alpha_4 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{pmatrix}.$$

Obviously,

$$Z(R) = \left\{ \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1 \end{pmatrix} \middle| \ \alpha_1 \in \mathcal{Z}_2 \right\}.$$

Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, F, D are nonzero generalized derivation and derivation such that the following conditions hold

(i) $F(x) \circ x^* \in Z(R)$,

(ii) $F([x, x^*]) \pm x \circ x^* \in Z(R)$,

(iii) $F(x \circ x^*) \pm [x, x^*] \in Z(R)$,

(iv) $F(x) \circ D(x^*) \pm x \circ x^* \in Z(R)$,

(v) $[F(x), D(x^*)] \pm x \circ x^* \in Z(R)$,

(vi) $F(x) \pm x \circ x^* \in Z(R)$,

(vii) $F(x) \pm [x, x^*] \in Z(R)$,

(viii) $[F(x), x^*] \mp F(x) \circ x^* \in Z(R)$,

(ix) $F(x \circ x^*) \in Z(R)$,

for all $x \in R$. However, R is not commutative. Hence, the hypothesis of second kind involution is crucial in our results.

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