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CORRIGENDUM TO "TRANSLATION SURFACES IN THE 3-DIMENSIONAL GALILEAN SPACE SATISFYING $\Delta^{II} x_i = \lambda_i x_i$" [BULL. KOREAN MATH. SOC. 54 (2017), NO. 4, PP. 1241–1254]

Ali Çakmak, Murat Kemal Karacan, Sezai Kiziltug, and Dae Won Yoon

In [1], there are some mistakes in calculations and solutions of differential equations and theorems that appered in the paper. We here provide correct solutions and theorems.

4. Translation surfaces of type 1 satisfying $\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$

In this section, we classify translation surfaces with non-degenerate second fundamental form in $\mathbb{G}_3$ satisfying the equation

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{4.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x} = \left(\Delta^{\mathbf{II}}\mathbf{x}_1, \Delta^{\mathbf{II}}\mathbf{x}_2, \Delta^{\mathbf{II}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = u,\ \mathbf{x}_2 = v,\ \mathbf{x}_3 = f(u) + g(v).$$

For the translation surface given by (3.1), the coefficients of the second fundamental form are given by

$$L_{11} = L = -\frac{f''}{\sqrt{\left(1+g'^2\right)}},\quad L_{22} = N = -\frac{g''}{\sqrt{\left(1+g'^2\right)}},\ L_{12} = M = 0. \tag{4.2}$$

The Gaussian curvature $\mathbf{K}$ is

$$\mathbf{K} = \frac{f''(u)g''(v)}{\left(1+g'^2\right)^2}.$$

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Suppose that the surface has non zero Gaussian curvature, so $f''(u)g''(v) \neq 0, \forall\, u, v \in I$. By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.2) and (2.2) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x} = \begin{pmatrix} -\frac{\sqrt{1+g'^2}f'''}{2f''^2}, \\ -\frac{\sqrt{1+g'^2}g'''}{2g''^2}, \\ 2\sqrt{1+g'^2} - f'\frac{\sqrt{1+g'^2}f'''}{2f''^2} - g'\frac{\sqrt{1+g'^2}g'''}{2g''^2} \end{pmatrix}. \tag{4.3}$$

The equation (4.1) by means of (4.3) gives rise to the following system of ordinary differential equations

$$-\frac{\sqrt{1+g'^2}f'''}{2f''^2} = \lambda_1 u, \tag{4.4}$$

$$-\frac{\sqrt{1+g'^2}g'''}{2g''^2} = \lambda_2 v, \tag{4.5}$$

$$2\sqrt{1+g'^2} - f'\frac{\sqrt{1+g'^2}f'''}{2f''^2} - g'\frac{\sqrt{1+g'^2}g'''}{2g''^2} = \lambda_3\left(f(u)+g(v)\right), \tag{4.6}$$

where $\lambda_i \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 3-types. Combining equations (4.4), (4.5) and (4.6), we have

$$f'\lambda_1 u - \lambda_3 f = \lambda_3 g - g'\lambda_2 v - 2\sqrt{1+g'^2}. \tag{4.7}$$

Here u and v are independent variables, so each side of (4.8) is equal to a constant, call it p. Hence, the two equations

$$\begin{aligned} f'\lambda_1 u - \lambda_3 f &= p, \\ \lambda_3 g - g'\lambda_2 v - 2\sqrt{1+g'^2} &= p. \end{aligned} \tag{4.8}$$

The solution of the function $f(u)$ is given by

$$f(u) = -\frac{p}{\lambda_3} + c_1 u^{\frac{\lambda_3}{\lambda_1}}, \tag{4.9}$$

where $p, c_1 \in \mathbb{R}$. The differential equation of $g(v)$ cannot be solved analytically. For the best case, i.e., $g' = 0$, so we have $g = \frac{2+p}{\lambda_3}$.

We discuss eight cases according to constants $\lambda_1, \lambda_2, \lambda_3$. We summarize the solutions of ordinary differential equations (4.8) in the following table.

No	$(\lambda_1, \lambda_2, \lambda_3)$	$f(u)$	$g(v)$
1	$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$	$f(u)$	$c_1 \pm i\sqrt{p}v$
2	$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$	$f(u)$	A
3	$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$	$-\frac{p}{\lambda_3}$	B
4	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$	$c_1 + \frac{p\ln u}{\lambda_1}$	$c_1 \pm \frac{\sqrt{-4+p^2}}{2}v$
5	$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	$-\frac{p}{\lambda_3}$	$\frac{2+p}{\lambda_3}$
6	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$	$c_1 + \frac{p\ln u}{\lambda_1}$	A
7	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0$	$-\frac{p}{\lambda_3} + c_1 u^{\frac{\lambda_3}{\lambda_1}}$	B
8	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	$-\frac{p}{\lambda_3} + c_1 u^{\frac{\lambda_3}{\lambda_1}}$	$\frac{2+p}{\lambda_3}$

where

$$\begin{aligned} A = c_1 &+ \frac{\mp 4\ln\left(\lambda_2 v + \sqrt{-4+p^2+\lambda_2^2 v^2}\right)}{2\lambda_2} \\ &+ \frac{p\left(-2\ln(2-\lambda_2 v) \pm \ln\left[4 - 2\lambda_2 v - p\left(p + \sqrt{-4+p^2+\lambda_2^2 v^2}\right)\right]\right)}{2\lambda_2} \\ &+ \frac{p\left(\pm\ln\left[4 + 2\lambda_2 v - p\left(p + \sqrt{-4+p^2+\lambda_2^2 v^2}\right)\right]\right)}{2\lambda_2}, \end{aligned}$$

$$B = \frac{e^{-\frac{1}{2}(2c_1+v)\lambda_3}\left(4 + e^{(2c_1+v)\lambda_3}\right) + 2p}{2\lambda_3}$$

or

$$B = \frac{5\cosh\left(\frac{1}{2}(v-2c_1)\lambda_3\right) + 3\sinh\left(\frac{1}{2}(v-2c_1)\lambda_3\right) + 2p}{2\lambda_3}$$

and $p, c_i \in \mathbb{R}$.

The function $f(u)$ in the first and the second rows of the above table can be any differentiable function. But we get contradictions for $p \in \mathbb{R}/\{0\}$. In the first, the third, the fourth, the fifth and the eighth rows of the above table, we have $L = 0$ or $N = 0$. So the second fundamental form in these cases are degenerate, we get contradictions with assumption. Substituting the sixth and the seventh rows into (4.4), (4.5) and (4.6), respectively, we can easily see that they do not satisfy these equations.

Definition. A surface in the three dimensional Galilean space is said to be **II**-harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 4.1. *Let* $\mathbf{M}$ *be a non-degenerate translation surface given by* (3.1) *in the three dimensional Galilean space* $\mathbb{G}_3$. *Then, there is no surface satisfying the condition* $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 4.2. *Let* $\mathbf{M}$ *be a non* **II***-harmonic translation surface with non-degenerate second fundamental form given by* (3.1) *in the three dimensional*

Galilean space $\mathbb{G}_3$*. Then, there is no surface satisfying the condition*

$$\Delta^{\mathbf{II}}\mathbf{x}_i=\lambda_i\mathbf{x}_i,$$

where $\lambda_i, p \in \mathbb{R}$*,* $i=1,2,3$.

5. Translation surfaces of type 2 satisfying $\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i = \boldsymbol{\lambda}_i\mathbf{x}_i$

In this section, we classify translation surfaces with non-degenerate second fundamental form in $\mathbb{G}_3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i= \lambda_i\mathbf{x}_i, \tag{5.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x} = \left(\Delta^{\mathbf{II}}\mathbf{x}_1,\ \Delta^{\mathbf{II}}\mathbf{x}_2,\ \Delta^{\mathbf{II}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = u+v,\ \ \mathbf{x}_2 = g(v),\ \ \mathbf{x}_3 = f(u).$$

For the translation surface given by (3.2), the coefficients of the second fundamental form are given by

$$L_{11} = L = \frac{f''g'}{\sqrt{f'^2+g'^2}},\ \ L_{22} = \ N = \frac{f'g''}{\sqrt{f'^2+g'^2}},\ L_{12} = M = 0. \tag{5.2}$$

The Gaussian curvature $\mathbf{K}$ is

$$\mathbf{K} = \frac{f'f''g'g''}{\left(f'^2+g'^2\right)^2}.$$

Suppose that the surface has non zero Gaussian curvature, so

$$f'(u)f''(u)g'(v)g''(v) \neq 0, \forall\ u,v \in I.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (5.2) and (2.2) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x} = \begin{pmatrix} -\frac{\sqrt{f'^2+g'^2}}{f'g'} + f'''\frac{\sqrt{f'^2+g'^2}}{2g'f''^2} + g'''\frac{\sqrt{f'^2+g'^2}}{2f'g''^2}, \\ \frac{\sqrt{f'^2+g'^2}}{2f'g''^2}\left(-3g''^2+g'g'''\right), \\ \frac{\sqrt{f'^2+g'^2}}{2g'f''^2}\left(-3f''^2+f'f'''\right) \end{pmatrix}. \tag{5.3}$$

The equation (5.1) by means of (5.2) gives rise to the following system of ordinary differential equations

$$-\frac{\sqrt{f'^2+g'^2}}{f'g'} + f'''\frac{\sqrt{f'^2+g'^2}}{2g'f''^2} + g'''\frac{\sqrt{f'^2+g'^2}}{2f'g''^2} = \lambda_1\left(u+v\right), \tag{5.4}$$

$$\frac{\sqrt{f'^2+g'^2}}{2f'g''^2}\left(-3g''^2+g'g'''\right) = \lambda_2 g(v), \tag{5.5}$$

$$\frac{\sqrt{f'^2+g'^2}}{2g'f''^2}\left(-3f''^2+f'f'''\right) = \lambda_3 f(u), \tag{5.6}$$

where $\lambda_i \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 3-types. Combining equations (5.4), (5.5) and (5.6), we have

$$-(u+v) f'g'\lambda_1 + f'g\lambda_2 + fg'\lambda_3 + 2\sqrt{f'^2+g'^2} = 0. \tag{5.7}$$

In the above differential equation, for the best case, i.e., $f'=0$ and $g'=0$. We summarize the solutions of ordinary differential equations (5.7) in the following table.

No	$\lambda_1, \lambda_2, \lambda_3$	$f(u)$	$g(v)$
1	$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$	$c_1 \pm \sqrt{p}u$	$c_2 \pm i\sqrt{p}v$
2	$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$	$c_1 \pm \sqrt{p}u$	B
3	$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$	A	$c_2 \pm \sqrt{p}v$
4	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$	c_1	c_2
5	$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	c_1	c_2
6	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$	c_1	c_2
7	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0$	c_1	c_2
8	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	c_1	c_2

where

$$A = \frac{e^{-\frac{1}{2}(\sqrt{p}u+2c_1)\lambda_3}\left(4e^{\sqrt{p}u\lambda_3}+e^{2c_1\lambda_3}\right)}{2\lambda_3}, \frac{e^{-\frac{1}{2}(\sqrt{p}u+2c_1)\lambda_3}\left(4+e^{(\sqrt{p}u+2c_1)\lambda_3}\right)}{2\lambda_3},$$

$$B = \frac{e^{-\frac{1}{2}(\sqrt{p}v+2c_1)\lambda_2}\left(4e^{\sqrt{p}v\lambda_2}+e^{2c_1\lambda_2}\right)}{2\lambda_2}, \frac{e^{-\frac{1}{2}(\sqrt{p}v+2c_1)\lambda_2}\left(4+e^{(\sqrt{p}v+2c_1)\lambda_2}\right)}{2\lambda_2}$$

and $p, c_i \in \mathbb{R}$.

In the all rows of the above table, we have $L=0$ or $N=0$. So the second fundamental form in these cases are degenerate, we get contradictions with assumption.

Theorem 5.1. *Let* $\mathbf{M}$ *be a non-degenerate translation surface given by* (3.2) *in the three dimensional Galilean space* $\mathbb{G}_3$*. Then, there is no surface satisfying the condition* $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 5.2. *Let* $\mathbf{M}$ *be a non* $\mathbf{II}$*-harmonic translation surface with non-degenerate second fundamental form given by* (3.2) *in the three dimensional Galilean space* $\mathbb{G}_3$*. Then, there is no surface satisfying the condition*

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i,$$

where $\lambda_i, p \in \mathbb{R}$, $i=1,2,3$.

References

[1] A. Çakmak, M. K. Karacan, S. Kiziltug, and D. W. Yoon, *Translation surfaces in the 3-dimensional Galilean space satisfying* $\Delta^{II} x_i = \lambda_i x_i$, Bull. Korean Math. Soc. **54** (2017), no. 4, 1241–1254.

Ali ÇAKMAK
Department of Mathematics
Faculty of Arts and Sciences
Bitlis Eren University
13000 Bitlis, Turkey
Email address: acakmak@beu.edu.tr

Murat Kemal Karacan
Department of Mathematics
Faculty of Arts and Sciences
Usak University 1 Eylul Campus
64200 Usak, Turkey
Email address: murat.karacan@usak.edu.tr

Sezai Kiziltug
Department of Mathematics
Faculty of Arts and Sciences
Erzincan University
24000 Erzincan, Turkey
Email address: skiziltug@erzincan.edu.tr

Dae Won Yoon
Department of Mathematics
Education and RINS
Gyeongsang National University
Jinju 660-701, Korea
Email address: dwyoon@gnu.ac.kr