Bull. Korean Math. Soc. **56** (2019), No. 2, pp. 535–547
https://doi.org/10.4134/BKMS.b180417
pISSN: 1015-8634 / eISSN: 2234-3016

# ON THE INTERSECTION OF $k$-FIBONACCI AND PELL NUMBERS

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Abstract. In this paper, by using the lower bound of linear forms in logarithms of Matveev and the theory of continued fractions by means of a variation of a result of Dujella and Pethő, we find all generalized Fibonacci numbers which are Pell numbers. This paper continues a previous work that searched for Pell numbers in the Fibonacci sequence.

## 1. Introduction

For $k \geq 2$, we consider the *$k$-generalized Fibonacci sequence* or, for simplicity, the *$k$-Fibonacci sequence* $F^{(k)} := (F_n^{(k)})_{n \geq 2-k}$ given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We shall refer to $F_n^{(k)}$ as the *$n$th $k$-Fibonacci number*. We note that this generalization is in fact a family of sequences where each new choice of $k$ produces a distinct sequence. For example, the usual Fibonacci numbers are obtained for $k = 2$. For small values of $k$, these sequences are called Tribonacci ($k = 3$), Tetranacci ($k = 4$), Pentanacci ($k = 5$), Hexanacci ($k = 6$), Heptanacci ($k = 7$) and Octanacci ($k = 8$).

An interesting fact about the $k$-Fibonacci sequence is that the first $k + 1$ non–zero terms in $F^{(k)}$ are powers of two, namely

$$F_1^{(k)} = 1 \quad \text{and} \quad F_n^{(k)} = 2^{n-2} \quad \text{for all} \quad 2 \leq n \leq k+1, \tag{1}$$

while the next term is $F_{k+2}^{(k)} = 2^k - 1$. In fact, the inequality (see [6])

$$F_n^{(k)} < 2^{n-2} \quad \text{holds for all} \quad n \geq k+2.$$

Below we present the values of these numbers for the first few values of $k$ and $n$.

Received April 30, 2018; Revised October 12, 2018; Accepted October 29, 2018.
2010 *Mathematics Subject Classification.* 11B39, 11J86.
*Key words and phrases.* $k$-Fibonacci numbers, Pell numbers, linear forms in logarithms, reduction method.

| $k$ | Name | First non–zero terms ($n \geq 1$) |
|---|---|---|
| 2 | Fibonacci | 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ... |
| 3 | Tribonacci | 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ... |
| 4 | Tetranacci | 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ... |
| 5 | Pentanacci | 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, ... |
| 6 | Hexanacci | 1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ... |
| 7 | Heptanacci | 1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ... |
| 8 | Octanacci | 1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ... |
| 9 | Nonanacci | 1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, ... |
| 10 | Decanacci | 1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1023, 2045, 4088, 8172, ... |

In general, Cooper and Howard [15] proved the following nice formula:

**Lemma 1.** *For $k \geq 2$ and $n \geq k+2$,*

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j}\, 2^{n-(k+1)j-2},$$

*where*

$$C_{n,j} = (-1)^j \left[ \binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right].$$

In the above, we used the convention that $\binom{a}{b} = 0$ if either $a < b$ or if one of $a$ or $b$ is negative and denote $\lfloor x \rfloor$ the greatest integer less than or equal to $x$. For example, assuming that $k+2 \leq n \leq 2k+2$, Cooper and Howard's formula becomes the identity

$$F_n^{(k)} = 2^{n-2} - (n-k) \cdot 2^{n-k-3} \quad \text{for all} \quad k+2 \leq n \leq 2k+2.$$

Several authors have worked on problems involving generalized Fibonacci sequences. For instance, F. Luca [20] and D. Marques [22] proved that 55 and 44 are the largest repdigits (i.e., numbers with only one distinct digit in its decimal expansion) in the sequences $F^{(2)}$ and $F^{(3)}$, respectively. Moreover, D. Marques conjectured that there are no repdigits, with at least two digits, belonging to $F^{(k)}$ for $k > 3$. This conjecture was confirmed shortly afterwards by Bravo and Luca [7]. We refer to [9] and [13] for results on the largest prime factor of $F_n^{(k)}$.

Also, there is the Pell sequence, which is as important as the Fibonacci sequence and its generalizations. The Pell sequence $P := (P_n)_{n \geq 0}$ is defined by the recurrence

$$P_n = 2P_{n-1} + P_{n-2} \quad \text{for all} \quad n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$ as initial conditions. Below we present the first few elements of the Pell sequence:

$$P = \{0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, \ldots\}.$$

Further details about the Pell sequence can be found, for instance, in [3, 14, 17, 18]. For the purposes of this paper, it is important to mention that Ljunggren [19] in 1942 proved that a Pell number is a square only if $n = 0, 1$ or $7$ and Pethő [26] in 1992 showed that these are the only perfect powers in the Pell sequence (see also [10]). We state their result which we will be used later.

**Theorem 1.** *The Diophantine equation $P_n = y^m$ in positive integers $n, y, m$ with $m \geq 2$ has only the solutions $(n, y, m) = (1, 1, m)$ and $(7, 13, 2)$.*

Many problems in number theory may be reduced to finding the intersection of two sequences of positive integers and it is for this reason that the problem of determining the intersection of two generalized Fibonacci sequences has attracted attention from several number theorists. For example, a conjecture (proposed by Noe and Post [25]) about coincidences between terms of generalized Fibonacci sequences was proved independently by Bravo–Luca [8] and Marques [21]. A similar work for generalized Lucas sequences was performed in [4].

In 2011, Alekseyev [1] established that $F^{(2)} \cap P = \{0, 1, 2, 5\}$ using properties of Lucas sequences, homogeneous quadratic Diophantine equations and Thue equations. Similar results for many other pairs of Lucas sequences were found in [1].

In this paper, we extend the previous work [1] and search for Pell numbers which are $k$-Fibonacci numbers, i.e., we determine all the solutions of the Diophantine equation

$$F_n^{(k)} = P_m, \tag{2}$$

in nonnegative integers $n, k, m$ with $k \geq 2$.

It is important to notice that Mignotte (see [24]) showed that if $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are two linearly recurrence sequences, then under some weak technical assumptions, the equation $u_n = v_m$ has only finitely many solutions in positive integers $m$ and $n$. What he proved is that the intersection of two sequences is finite unless the roots of their characteristic polynomials are multiplicatively dependent. In our case we already know that the intersection $F^{(k)} \cap P$ is finite and so the question arises is how big this intersection can be. In the Fibonacci case, namely when $k = 2$, several well known divisibility properties were used by Alekseyev in [1] to solve the problem. However, similar divisibility properties for $F^{(k)}$ when $k \geq 3$ are not known and therefore it is necessary to attack the problem differently.

First of all, note that $F_0^{(k)} = P_0 = 0$, $F_1^{(k)} = F_2^{(k)} = P_1 = 1$ and $F_3^{(k)} = P_2 = 2$ are valid for all $k \geq 2$. Thus, the triples

$$(n, k, m) \in \{(0, k, 0), (1, k, 1), (2, k, 1), (3, k, 2)\}$$

are solutions of (2) for all $k \geq 2$. These solutions will be called *trivial solutions*.

Our main result is the following:

**Theorem 2.** *The only nontrivial solutions of the Diophantine equation* (2) *in nonnegative integer* $n, k, m$ *with* $k \geq 2$, *are*

$$(n, k, m) \in \{(5, 2, 3), (7, 4, 5)\}.$$

*Namely,* $F_5 = P_3 = 5$ *and* $F_7^{(4)} = P_5 = 29$.

## 2. Preliminary results

In this section we present some basic properties of the Pell and $k$-Fibonacci sequences and a lower bound for a nonzero linear form in logarithms of algebraic numbers. Additionally, we state a reduction lemma, which is an immediate variation of a result due to Dujella and Pethő from [12], and will be the key tool used in this paper to reduce some upper bounds. All these facts will be used in the proof of Theorem 2.

### 2.1. The Pell sequence

An explicit Binet formula for $P$ is well known. Namely, for all $m \geq 0$, we have that

$$P_m = \frac{\gamma^m - \overline{\gamma}^m}{\gamma - \overline{\gamma}} = \frac{\gamma^m - \overline{\gamma}^m}{2\sqrt{2}}, \tag{3}$$

where $(\gamma, \overline{\gamma}) := (1 + \sqrt{2}, 1 - \sqrt{2})$ are the roots of the characteristic equation $x^2 - 2x - 1 = 0$. In particular, it easily implies that the inequality

$$\gamma^{m-2} \leq P_m \leq \gamma^{m-1} \quad \text{holds for all} \quad m \geq 1. \tag{4}$$

Additionally, in view of (3), we can write

$$P_m = \frac{\gamma^m}{2\sqrt{2}} + \xi(m) \quad \text{where} \quad |\xi(m)| < 1/5 \quad \text{for all} \quad m \geq 1. \tag{5}$$

### 2.2. The $k$-Fibonacci sequence

The characteristic polynomial of $F^{(k)}$, namely

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1,$$

is irreducible in $\mathbb{Q}[x]$ and has just one zero real outside the unit circle. Throughout this paper, $\alpha := \alpha(k)$ denotes that single real zero. The other roots are strictly inside the unit circle, so $\alpha(k)$ is a Pisot number of degree $k$. Moreover, it is also known that $\alpha(k)$ is located between $2(1 - 2^{-k})$ and 2, see [16, Lemma 2.3] or [27, Lemma 3.6]. To simplify notation, we shall omit the dependence on $k$ of $\alpha$.

We now consider the function

$$f_k(x) = \frac{x - 1}{2 + (k + 1)(x - 2)}$$

for an integer $k \geq 2$ and $x > 2(1 - 2^{-k})$. It is easy to see that the inequalities

$$1/2 < f_k(\alpha) < 3/4 \qquad \text{and} \qquad |f_k(\alpha^{(i)})| < 1, \qquad 2 \leq i \leq k \tag{6}$$

hold, where $\alpha := \alpha^{(1)}, \dots, \alpha^{(k)}$ are all the zeros of $\Psi_k(x)$. Proofs for (6) can be found in [5].

With the above notation, Dresden and Du showed in [11] that

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}){\alpha^{(i)}}^{n-1} \qquad \text{and} \qquad \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}\right| < \frac{1}{2}$$

hold for all $n \geq 1$ and $k \geq 2$. This allows us to write

$$F_n^{(k)} = f_k(\alpha)\alpha^{n-1} + e_k(n) \text{ where } |e_k(n)| < 1/2 \text{ for all } n \geq 1 \text{ and } k \geq 2. \tag{7}$$

In addition to this, Bravo and Luca proved in [7] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \text{ holds for all } n \geq 1 \text{ and } k \geq 2. \tag{8}$$

### 2.3. Linear forms in logarithms

In order to prove our main result, we need to use a Baker type lower bound for a nonzero linear form in logarithms of algebraic numbers, and such a bound, which plays an important role in this paper, was given by Matveev [23]. We begin by recalling some basic notions from algebraic number theory.

Let $\eta$ be an algebraic number of degree $d$ with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient $a_0$ is positive and the $\eta^{(i)}$'s are the conjugates of $\eta$. Then

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right)$$

is called the *logarithmic height* of $\eta$. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$.

The following properties of the logarithmic heigh, which will be used in the next sections without special reference, are also known:

- $h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2$.
- $h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma)$.
- $h(\eta^s) = |s|h(\eta)$.

Matveev [23] proved the following deep theorem.

**Theorem 3** (Matveev's theorem)**.** *Let $\mathbb{K}$ be a number field of degree $D$ over $\mathbb{Q}$, $\gamma_1, \dots, \gamma_t$ be positive real numbers of $\mathbb{K}$, and $b_1, \dots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \qquad \textit{and} \qquad B \geq \max\{|b_1|, \dots, |b_t|\}.$$

*Let $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ be real numbers for $i = 1, \dots, t$. Then, assuming that $\Lambda \neq 0$, we have*

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_t).$$

We now give estimates for the logarithmic heights of some algebraic numbers that will be used in the sequel. Let $\mathbb{K} = \mathbb{Q}(\alpha)$. Knowing that $\mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha))$ and that $|f_k(\alpha^{(i)})| \leq 1$ for all $i = 1, \ldots, k$ and $k \geq 2$, one can prove the following estimates:

$$h(\alpha) < \log 2/k \qquad \text{and} \qquad h(f_k(\alpha)) < 2\log k \quad \text{for all} \quad k \geq 2. \tag{9}$$

See [5] for further details of the proof of (9). Furthermore,

$$h(2\sqrt{2}f_k(\alpha)) \leq (3/2)\log 2 + 2\log k < 3\log k \quad \text{for all} \quad k \geq 3. \tag{10}$$

Finally, it is easy to see that

$$h(\gamma) = \frac{\log \gamma}{2}. \tag{11}$$

### 2.4. Reduction lemma

In the course of our calculations, we get some upper bounds on our variables which are very large, so we need to reduce them. To this end, we use some results of the theory of continued fractions. Specifically, for a nonhomogenenous linear forms in two integer variables, we will use a slight variation of a result due to Dujella and Pethő from [12], which itself is a generalization of a result of Baker–Davenport [2].

**Lemma 2.** *Let $A, B, \hat{\gamma}, \hat{\mu}$ be positive real numbers and $M$ a positive integer. Suppose that $p/q$ is a convergent of the continued fraction expansion of the irrational $\hat{\gamma}$ such that $q > 6M$. Put $\epsilon := ||\hat{\mu}q|| - M||\hat{\gamma}q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then there is no positive integer solution $(u, v, w)$ to the inequality*

$$0 < |u\hat{\gamma} - v + \hat{\mu}| < AB^{-w},$$

*subject to the restrictions that*

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

## 3. Proof of Theorem 2

Assume throughout that equation (2) holds with $n \geq 4$ and $m \geq 3$. Suppose further that $k \geq 3$ since the case $k = 2$ was already treated by Alekseyev in [1]. In the case $4 \leq n \leq k+1$, we obtain from (1) and (2) that $2^{n-2} = P_m$, which is not possible in view of Theorem 1. Thus, we can assume that $n \geq k+2$.

### 3.1. An initial relation

By (2), (4) and (8) we have $\alpha^{n-2} \leq \gamma^{m-1}$ and $\gamma^{m-2} \leq \alpha^{n-1}$. So, we get

$$(n-2)\frac{\log \alpha}{\log \gamma} + 1 \leq m \leq (n-1)\frac{\log \alpha}{\log \gamma} + 2.$$

It is easy to check that $\alpha > 7/4$ for all $k \geq 3$ by using the fact that $2(1-2^{-k}) < \alpha$. From this and taking into account that

$$\frac{\log\alpha}{\log\gamma} > \frac{\log(7/4)}{\log\gamma} = 0.6349\ldots \quad \text{and} \quad \frac{\log\alpha}{\log\gamma} < \frac{\log 2}{\log\gamma} = 0.7864\ldots$$

we obtain that

$$0.6n - 0.27 < m < 0.8n + 1.22, \tag{12}$$

which is an estimate between $n$ and $m$ as we wanted. We shall use this estimate later.

### 3.2. An inequality for $n$ and $m$ in terms of $k$

By using (2), (5), (7) and taking absolute value, we obtain

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{\gamma^m}{2\sqrt{2}} \right| \leq |e_k(n)| + |\xi(m)| < \frac{7}{10}.$$

Dividing the above inequality by $\gamma^m/(2\sqrt{2})$, we get that

$$\left|(2\sqrt{2}f_k(\alpha))\alpha^{n-1}\gamma^{-m} - 1\right| < \frac{2}{\gamma^m}, \tag{13}$$

where we used the fact $14\sqrt{2}/10 < 2$. In order to use the result of Matveev Theorem 3, we take $t := 3$ and

$$\gamma_1 := 2\sqrt{2}f_k(\alpha), \qquad \gamma_2 := \alpha, \qquad \gamma_3 := \gamma.$$

We also take $b_1 := 1$, $b_2 := n-1$ and $b_3 := -m$. We begin by noticing that the three numbers $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\alpha, \sqrt{2})$, so we can take $D := [\mathbb{K} : \mathbb{Q}] \leq 2k$. The left–hand size of (13) is not zero. Indeed, if this were zero, we would then get that

$$f_k(\alpha)\alpha^{n-1} = \frac{\gamma^m}{2\sqrt{2}}. \tag{14}$$

To see that (14) is not possible, we argue as follows. The conjugates of the number on the left-hand side of (14) are

$$f_k(\alpha_i)\alpha_i^{n-1} \quad \text{for} \quad i = 1, \ldots, k, \tag{15}$$

while the conjugates of the number in the right-hand of (14) are

$$\frac{\gamma^m}{2\sqrt{2}} \quad \text{and} \quad \frac{\overline{\gamma}^m}{-2\sqrt{2}}. \tag{16}$$

Let $\mathbb{L} = \mathbb{Q}(\alpha_1, \ldots, \alpha_k, \gamma)$ be the normal closure of $\mathbb{K}$ and let $\sigma_1, \ldots, \sigma_k$ be elements of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ such that $\sigma_i(\alpha) = \alpha_i$. Then $\sigma_1, \ldots, \sigma_k$ map the elements from the list (15) to the same list as well as to the elements from the list (16). Since $k \geq 3$, there exist $i \neq j$ in $\{1, 2, \ldots, k\}$ such that $\sigma_i(\gamma) = \sigma_j(\gamma)$.

Note that, if we put $\sigma_j^{-1}(\alpha_i) = \alpha_s$, then $s \neq 1$ because $\sigma_j(\alpha_1) = \alpha_j \neq \alpha_i$. Applying $\sigma_j^{-1}\sigma_i$ to the relation (14), we get the relation

$$f_k(\alpha_s)\alpha_s^{n-1} = \frac{\gamma^m}{2\sqrt{2}}.$$

Taking absolute value in the above relation we see that this is not possible since its right-hand side exceeds 1 for all $m \geq 3$, while its left-hand side is smaller than 1.

In view of (9), (10) and (11) we can take $A_1 := 6k \log k$, $A_2 := 2 \log 2$ and $A_3 := k \log \gamma$. In addition, since $m < 0.8n + 1.22 < n + 1$ for all $n \geq 4$ (see (12)), we take $B := n + 1$. Then, Matveev's theorem together with a straightforward calculation gives

$$\left|(2\sqrt{2}f_k(\alpha))\alpha^{n-1}\gamma^{-m} - 1\right| > \exp\left(-2.52 \times 10^{13} k^4 \log^2 k \log n\right), \tag{17}$$

where we used that $1 + \log 2k < 3 \log k$ for all $k \geq 3$ and $1 + \log(n+1) < 2 \log n$ for all $n \geq 4$. Comparing (13) and (17), taking logarithms and then performing the respective calculations, we get that

$$\frac{n}{\log n} < 4.78 \times 10^{13} k^4 \log^2 k. \tag{18}$$

In order to get an upper bound for $n$ depending on $k$ we next use the fact that $x/\log x < A$ implies $x < 2A \log A$ whenever $A \geq 3$. Indeed, taking $x := n$ and $A := 4.78 \times 10^{13} k^4 \log^2 k$, and performing the respective calculations, inequality (18) yields $n < 3.2 \times 10^{15} k^4 \log^3 k$. We record what we have proved so far as a lemma.

**Lemma 3.** *If $(n, m, k)$ is a nontrivial solution in positive integers of equation* (2)*, then $n \geq k + 2$ and*

$$m \leq n < 3.2 \times 10^{15} k^4 \log^3 k.$$

### 3.3. The case of large $k$

Suppose that $k > 210$. In this case the following inequalities hold:

$$m \leq n < 3.2 \times 10^{15} k^4 \log^3 k < 2^{k/2}.$$

At this point, we require the following estimate derived from Cooper and Howard's formula from Lemma 1:

$$F_n^{(k)} = 2^{n-2}(1 + \zeta) \quad \text{where} \quad |\zeta| < \frac{1}{2^{k/2}}. \tag{19}$$

Indeed, by Lemma 1 we have

$$F_n^{(k)} = 2^{n-2}\left(1 + \sum_{j=1}^{\lfloor (n+k)/(k+1) \rfloor - 1} \frac{C_{n,j}}{2^{(k+1)j}}\right).$$

Taking into account that $C_{n,1} = -(n-k)$ and defining $\zeta$ as the sum for $j \geq 1$ on the right-hand side of the above expression, we get that

$$\begin{aligned}
|\zeta| &\leq \frac{n-k}{2^{k+1}} + \sum_{j=2}^{\lfloor (n+k)/(k+1)\rfloor -1} \frac{|C_{n,j}|}{2^{(k+1)j}} \\
&< \frac{n}{2^{k+1}} + \sum_{j\geq 2} \frac{2n^j}{2^{(k+1)j}(j-2)!} \\
&< \frac{n}{2^{k+1}} + \frac{2n^2}{2^{2k+2}} \sum_{j\geq 2} \frac{\left(n/2^{k+1}\right)^{j-2}}{(j-2)!} \\
&< \frac{n}{2^{k+1}} + \frac{2n^2}{2^{2k+2}} e^{n/2^{k+1}}.
\end{aligned}$$

Further, since $n < 2^{k/2}$, we have $e^{n/2^{k+1}} < e^{1/2} < 2$ and so

$$|\zeta| < \frac{n}{2^{k+1}} + \frac{4n^2}{2^{2k+2}} < \frac{1}{2^{k/2}}.$$

This proves (19). On the other hand, from (2), (5) and (19), we get

$$\left| 2^{n-2} - \frac{\gamma^m}{2\sqrt{2}} \right| < \frac{2^{n-2}}{2^{k/2}} + \frac{1}{5}$$

leading to

$$\left| 1 - 2^{-(n-1)} (\sqrt{2})^{-1} \gamma^m \right| < \frac{2}{2^{k/2}}, \tag{20}$$

where we have used the fact that $1/2^{n-2} \leq 1/2^k$ because $n \geq k+2$. We lower bound the left-hand side of inequality (20) using again Matveev's result. To do this, we take $t := 3$ and

$$\gamma_1 := 2, \qquad \gamma_2 := \sqrt{2}, \qquad \gamma_3 := \gamma.$$

We also take $b_1 := -(n-1)$, $b_2 := -1$ and $b_3 := m$. Note that the three numbers $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to $\mathbb{K} = \mathbb{Q}(\sqrt{2})$, so we can take $D := [\mathbb{K} : \mathbb{Q}] = 2$. The left–hand size of (20) is not zero. Indeed, if this were zero, we would then get that $\gamma^m = 2^{n-1}\sqrt{2}$. Conjugating the above relation in $\mathbb{Q}(\sqrt{2})$ we get $\overline{\gamma}^m = -2^{n-1}\sqrt{2}$ and therefore $\gamma^m + \overline{\gamma}^m = 0$. Hence, $\gamma^m = |\overline{\gamma}|^m < 1$, which is impossible for positive $m$.

In this application of Matveev's result, we can take $A_1 := 2\log 2$, $A_2 := \log 2$ and $A_3 := \log \gamma$. By recalling that $m < n+1$, we can also take $B := n+1$. We thus get that

$$\left| 1 - 2^{-(n-1)} \left(\sqrt{2}\right)^{-1} \gamma^m \right| > \exp\left(-1.65 \times 10^{12} \log n\right), \tag{21}$$

where we used the fact that $1+\log(n+1) \leq 2\log n$ for all $n \geq 4$. Comparing (20) and (21), taking logarithms and then performing the respective calculations, we get that

$$k < 4.8 \times 10^{12} \log n.$$

But, recall that by Lemma 3 we have $n < 3.2 \times 10^{15} k^4 \log^3 k$. Thus,

$$\begin{aligned} k &< 4.8 \times 10^{12} \log(3.2 \times 10^{15} k^4 \log^3 k) \\ &< 4.8 \times 10^{12}(35.8 + 4\log k + 3\log\log k) \\ &< 5.8 \times 10^{13} \log k, \end{aligned}$$

where we used the fact that $35.8 + 4\log k + 3\log\log k < 12\log k$ for all $k > 210$. With the help of *Mathematica* we find that $k < 2.1 \times 10^{15}$. By Lemma 3 once again, we obtain absolute upper bounds for $n$ and $m$, namely

$$n < 2.8 \times 10^{81} \quad \text{and} \quad m < 2.3 \times 10^{81}. \tag{22}$$

**3.3.1.** *Reducing the upper bound for* $\boldsymbol{k}$. In order to apply Lemma 2 we put

$$z := m\log\gamma - (n-1)\log 2 - \log\sqrt{2}$$

and observe that (20) can be rewritten as

$$|e^z - 1| < \frac{2}{2^{k/2}}. \tag{23}$$

Note that $z \neq 0$; thus, we distinguish the following cases. If $z > 0$, the $e^z - 1 > 0$, so from (23) we obtain

$$0 < z \leq e^z - 1 = |e^z - 1| < \frac{2}{2^{k/2}},$$

where we have used the well known fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$. Suppose now that $z < 0$. Note, since $k > 210$, that $2/2^{k/2} < 1/2$. Hence, from (23) we have $|e^z - 1| < 1/2$ and therefore $e^{|z|} < 2$. Since $z < 0$, we have

$$0 < |z| \leq e^{|z|} - 1 = e^{|z|}|e^z - 1| < \frac{4}{2^{k/2}}.$$

In any case, we have that the inequality

$$0 < |z| < \frac{4}{2^{k/2}} \tag{24}$$

holds for all $k > 210$. Replacing $z$ in the above inequality by its formula and dividing it across by $\log 2$, we conclude that

$$0 < \left| m\left(\frac{\log\gamma}{\log 2}\right) - n + \frac{1}{2} \right| < 6 \cdot 2^{-k/2}. \tag{25}$$

We put

$$\hat{\gamma} := \frac{\log\gamma}{\log 2}, \quad \hat{\mu} := \frac{1}{2}, \quad A := 6 \quad \text{and} \quad B := 2.$$

We also put $M := 2.3 \times 10^{81}$, which is an upper bound on $m$ by (22). Then, the above inequality (25) yields

$$0 < |m\hat{\gamma} - n + \hat{\mu}| < A \cdot B^{-k/2}. \tag{26}$$

Clearly, $\hat{\gamma}$ is an irrational number. It then follows from Lemma 2, applied to inequality (26), that

$$\frac{k}{2} < \frac{\log(Aq/\epsilon)}{\log B},$$

where $q > 6M$ is a denominator of a convergent of the continued fraction of $\hat{\gamma}$ such that $\epsilon = ||\hat{\mu}q|| - M||\hat{\gamma}q|| > 0$. A computer search with *Mathematica* revealed that we can take $q := q_{149}$. Hence, we deduce that $k/2 \leq 280.021\ldots$ and so $k \leq 560$. With this new upper absolute bound for $k$ we get

$$n < 8 \times 10^{28} \quad \text{and} \quad m < 6.4 \times 10^{28}.$$

We now take $M := 6.4 \times 10^{28}$ and repeat the process. Indeed, a second application of Lemma 2 finally tells us that $k \leq 205$, which contradicts our assumption that $k > 210$. This completes the analysis in the case $k > 210$.

### 3.4. The case of small $k$

Suppose now that $k \in [3, 210]$. In order to apply Lemma 2 once again, we put

$$z = (n-1)\log\alpha - m\log\gamma + \log(2\sqrt{2}f_k(\alpha)).$$

Therefore, (13) can be rewritten as

$$|e^z - 1| < \frac{2}{\gamma^m}.$$

Note that $z \neq 0$. By the same arguments used for proving (24), we deduce that

$$0 < |z| < \frac{4}{\gamma^m}$$

holds for all $m \geq 2$ no matter whether $z$ is positive or negative. Replacing $z$ in the above inequality by its formula and dividing it across by $\log\gamma$, we conclude that

$$0 < \left|(n-1)\left(\frac{\log\alpha}{\log\gamma}\right) - m + \frac{\log(2\sqrt{2}f_k(\alpha))}{\log\gamma}\right| < 5\cdot\gamma^{-m}. \tag{27}$$

Putting

$$\hat{\gamma} = \hat{\gamma}(k) := \frac{\log\alpha}{\log\gamma}, \quad \hat{\mu} = \hat{\mu}(k) := \frac{\log(2\sqrt{2}f_k(\alpha))}{\log\gamma}, \quad A := 5 \quad \text{and} \quad B := \gamma,$$

the above inequality (27) implies

$$0 < |(n-1)\hat{\gamma} - m + \hat{\mu}| < A\cdot B^{-m}. \tag{28}$$

Let us show that $\hat{\gamma}$ is an irrational number. If it were not, then with $\hat{\gamma} = a/b$ with coprime integers $a$ and $b$, we would get that $\alpha^b = \gamma^a$. As has been said before,, there exists $\sigma \in \mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ such that $\sigma(\gamma) = \gamma$ and $\sigma(\alpha) = \alpha_s$ for some $s \in \{2, \ldots, k\}$. Applying this to the above relation and taking absolute values we get that $\gamma^a = |\alpha_s|^b < 1$, which is a contradiction because $\gamma > 1$.

We put $M_k := \lfloor 3.2 \times 10^{15} k^4 \log^3 k \rfloor$ which is an upper bound on $n-1$ from Lemma 3. Applying Lemma 2 in (28) for all $k \in [3, 210]$, we obtain that

$$m < \frac{\log(Aq/\epsilon)}{\log B},$$

where $q = q(k) > 6M_k$ is a denominator of a convergent of the continued fraction of $\hat{\gamma}$ such that $\epsilon = \epsilon(k) = ||\hat{\mu}q|| - M_k||\hat{\gamma}q|| > 0$. A computer search with *Mathematica* revealed that if $k \in [3, 210]$, then the maximum value of $\log(Aq/\epsilon)/\log B$ is $\leq 85$ and consequently $n \leq 150$. Finally, a brute force search with *Mathematica* in the range

$$3 \leq k \leq 148, \quad 3 \leq m \leq 85 \quad \text{and} \quad k+2 \leq n \leq 150$$

allows us to conclude that the only nontrivial solutions of (2) are those mentioned in Theorem 2 . This completes the analysis in the case $k \in [3, 210]$ and therefore the proof of Theorem 2.

**Acknowledgements**. J. J. Bravo was supported in part by Projects VRI ID 4689 (Universidad del Cauca) and Colciencias 110371250560. C. A. Gómez was supported in part by Project 71079 (Universidad del Valle). J. L. Herrera was supported by Colciencias (Colombia) through the Program Jóvenes investigadores e innovadores Project VRI ID 4402 (Universidad del Cauca).

## References

[1] M. A. Alekseyev, *On the intersections of Fibonacci, Pell, and Lucas numbers*, Integers **11** (2011), no. 3, 239–259.

[2] A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ *and* $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. (2) **20** (1969), 129–137.

[3] M. Bicknell, *A primer on the Pell sequence and related sequences*, Fibonacci Quart. **13** (1975), no. 4, 345–349.

[4] E. F. Bravo, J. J. Bravo, and F. Luca, *Coincidences in generalized Lucas sequences*, Fibonacci Quart. **52** (2014), no. 4, 296–306.

[5] J. J. Bravo, C. A. Gómez, and F. Luca, *Powers of two as sums of two k-Fibonacci numbers*, Miskolc Math. Notes **17** (2016), no. 1, 85–100.

[6] J. J. Bravo and F. Luca, *Powers of two in generalized Fibonacci sequences*, Rev. Colombiana Mat. **46** (2012), no. 1, 67–79.

[7] ———, *On a conjecture about repdigits in k-generalized Fibonacci sequences*, Publ. Math. Debrecen **82** (2013), no. 3-4, 623–639.

[8] ———, *Coincidences in generalized Fibonacci sequences*, J. Number Theory **133** (2013), no. 6, 2121–2137.

[9] ———, *On the largest prime factor of the k-Fibonacci numbers*, Int. J. Number Theory **9** (2013), no. 5, 1351–1366.

[10] J. H. E. Cohn, *Perfect Pell powers*, Glasgow Math. J. **38** (1996), no. 1, 19–20.

[11] G. P. B. Dresden and Z. Du, *A simplified Binet formula for k-generalized Fibonacci numbers*, J. Integer Seq. **17** (2014), no. 4, Article 14.4.7, 9 pp.

[12] A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford Ser. (2) **49** (1998), no. 195, 291–306.

[13] C. A. Gómez and F. Luca, *On the largest prime factor of the ratio of two generalized Fibonacci numbers*, J. Number Theory **152** (2015), 182–203.

[14] A. F. Horadam, *Applications of modified Pell numbers to representations*, Ulam Quart. **3** (1995), no. 1, 34–53.

[15] F. T. Howard and C. Cooper, *Some identities for r-Fibonacci numbers*, Fibonacci Quart. **49** (2011), no. 3, 231–242.

[16] L. K. Hua and Y. Wang, *Applications of number theory to numerical analysis*, translated from the Chinese, Springer-Verlag, Berlin, 1981.

[17] E. Kilic and D. Tasci, *The linear algebra of the Pell matrix*, Bol. Soc. Mat. Mexicana (3) **11** (2005), no. 2, 163–174.

[18] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2001.

[19] W. Ljunggren, *Zur Theorie der Gleichung* $x^2+1=Dy^4$, Avh. Norske Vid. Akad. Oslo. I. **1942** (1942), no. 5, 27 pp.

[20] F. Luca, *Fibonacci and Lucas numbers with only one distinct digit*, Portugal. Math. **57** (2000), no. 2, 243–254.

[21] D. Marques, *The proof of a conjecture concerning the intersection of k-generalized Fibonacci sequences*, Bull. Braz. Math. Soc. (N.S.) **44** (2013), no. 3, 455–468.

[22] ———, *On k-generalized Fibonacci numbers with only one distinct digit*, Util. Math. **98** (2015), 23–31.

[23] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II*, Izv. Math. **64** (2000), no. 6, 1217–1269; translated from Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), no. 6, 125–180.

[24] M. Mignotte, *Intersection des images de certaines suites récurrentes linéaires*, Theoret. Comput. Sci. **7** (1978), no. 1, 117–122.

[25] T. D. Noe and J. Vos Post, *Primes in Fibonacci n-step and Lucas n-step sequences*, J. Integer Seq. **8** (2005), no. 4, Article 05.4.4, 12 pp.

[26] A. Pethő, *The Pell sequence contains only trivial perfect powers*, in Sets, graphs and numbers (Budapest, 1991), 561–568, Colloq. Math. Soc. János Bolyai, 60, North-Holland, Amsterdam, 1992.

[27] D. A. Wolfram, *Solving generalized Fibonacci recurrences*, Fibonacci Quart. **36** (1998), no. 2, 129–145.

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