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THE k-GOLDEN MEAN OF TWO POSITIVE NUMBERS AND ITS APPLICATIONS

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Abstract. In this paper, we define a mean of two positive numbers called the *k-golden mean* and study some properties of it. Especially, we show that the 2-golden mean refines the harmonic and the geometric means. As an application, we define the *k-golden ratio* and give some properties of it as an generalization of the *golden ratio*. Furthermore, we define the matrix k-golden mean of two positive-definite matrices and give some properties of it. This is an improvement of Lim's results [2] for which the matrix golden mean.

1. Introduction

As early as the ancient Greece, the Pythagoreans had realized the notion of a mean of two numbers called *Pythagorean means* - the *arithmetic mean*, the *geometric mean* and the *harmonic mean* - and they had used in music and astronomy ([4]). Also, it is well-known the arithmetic-geometric-harmonic mean inequalities that

$$H(a,b) \leq G(a,b) \leq A(a,b),$$

where H, G and A denote the arithmetic mean, the geometric mean and the harmonic mean, respectively.

The *logarithmic mean* $L(a,b)$ of two positive numbers a and b is another example of the mean which refines the arithmetic and geometric means as

$$H(a,b) \leq G(a,b) \leq L(a,b) \leq A(a,b).$$

In this paper, we define the notion of k-golden mean of two positive numbers as a generalization of the golden mean and give some properties of it. Especially, we show that the 2-golden mean refines the harmonic and geometric means. Also, we define the k-golden ratio using by k-golden mean and we give some generalized properties of the golden ratio.

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On the other hand, Lim generalized the notion of the golden mean of two positive numbers to the golden mean of two positive definite matrices and apply them to some algebraic and differential Riccati equations in [2].

Motivated by [2], we generalize the notion of the k-golden mean of two positive numbers to the k-golden mean of two positive definite matrices and we list some related results. This is an improvement of Lim's results [2] for which the matrix golden mean. Moreover, we give new inequalities related to matrix means.

2. The k-golden mean

A *mean* of two positive numbers can be defined as a positive function $M(a, b)$ of two variables a and b $(0 < a \leq b)$ satisfying the followings:

(M1) $M(a, a) = a$;
(M2) if $a < b$, then $a < M(a, b) < b$;
(M3) for any positive constant r, $M(ra, rb) = rM(a, b)$;
(M4) $M(a, b)$ is increasing with respect to both a and b.

The most basic examples are the harmonic mean, the geometric mean and the arithmetic mean defined by

$$H(a, b) = \frac{2ab}{a+b}, \; G(a, b) = \sqrt{ab} \; \text{ and } \; A(a, b) = \frac{a+b}{2},$$

respectively. Moreover, they have the well-known mean inequalities:

$$H(a, b) \leq G(a, b) \leq A(a, b).$$

In [1], B. C. Carlson studied the logarithmic mean and some inequalities. The logarithmic mean of two positive numbers a and b is defined by

$$L(a, b) = \begin{cases} \dfrac{b-a}{\log b - \log a} & \text{if } \; a < b, \\ a & \text{if } \; a = b \end{cases}$$

and the logarithmic mean L refines the arithmetic and geometric means:

$$G(a, b) \leq L(a, b) \leq A(a, b),$$

with strict inequalities if $a \neq b$.

On the other hand, the three means $H(a, b)$, $G(a, b)$ and $A(a, b)$ are the positive roots of the three polynomials

$$(a + b)X - 2ab = 0, \; X^2 - ab = 0 \; \text{ and } \; 2X - (a + b) = 0,$$

respectively.

Let's consider the positive root $gld_k(a, b)$ of the quadric equation

$$X^2 - aX - \frac{a(b-a)}{k} = 0, \tag{2.1}$$

where a, b and k are three positive numbers with $a \le b$ and $k > 0$. Equivalently, we have

$$gld_k(a,b) = \frac{a + \sqrt{\{a(4b + (k-4)a)\}/k}}{2}. \tag{2.2}$$

For a fixed positive number k, we call the function gld_k in (2.2) the *k-golden function*. Remark that the domain of the function gld_k of two variables is the set $\{(a,b) \in \mathbb{R}^2 \mid 0 < a \le b\}$.

Proposition 2.1. *If $k \ge 1$, the function gld_k given in (2.2) is a mean, i.e., for two positive numbers a and b with $a \le b$, $gld_k(a,b)$ satisfies the mean properties* (M1)-(M4).

Proof. (M1) and (M3) are trivial. By using that $a \le b$, $a = (4a + (k-4)a)/k$, the geometric-arithmetic mean inequality and $k \ge 1$, we have (M2) as follows:

$$\begin{aligned} a &= \frac{a + \sqrt{a \cdot a}}{2} = \frac{a + \sqrt{a \cdot (4a + (k-4)a)/k}}{2} \\ &\le \frac{a + \sqrt{a \cdot (4b + (k-4)a)/k}}{2} = gld_k(a,b) \\ &\le \frac{a + \big(a + (4b + (k-4)a)/k\big)/2}{2} = \frac{b + (k-1)a}{k} \\ &\le \frac{b + (k-1)b}{k} = b, \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial a}\big(gld_k(a,b)\big) &= \frac{1}{2}\left\{1 + \frac{2(b-a) + (k-2)a}{\sqrt{a \cdot (4b + (k-4)a)/k}}\right\} \\ &\ge \frac{1}{2}\left\{1 - \frac{a}{\sqrt{a \cdot (4b + (k-4)a)/k}}\right\} \quad (\because\ b \ge a, k \ge 1) \\ &\ge \frac{1}{2}\left\{1 - \frac{a}{\sqrt{a \cdot (4a + (k-4)a)/k}}\right\} = 0. \ (\because\ b \ge a > 0). \end{aligned}$$

Thanks to calculus, $gld_k(a,b)$ is an increasing function with respect to a. For fixed $a > 0$, the function $gld_k(a,b)$ is a root function of b and hence it is increasing with respect to b. □

Let a and b be two positive numbers with $a \le b$. When $k \ge 1$, we denote $gld_k(a,b)$ by $Gld_k(a,b)$ and we call the number $Gld_k(a,b)$ the *k-golden mean* of a and b.

Now, we give some properties of the k-golden mean related to the mean inequalities.

Proposition 2.2. *For any numbers a and b with $0 < a < b$, the mean inequality $H(a,b) < Gld_k(a,b)$ holds if and only if $1 \le k \le 2$.*

Proof. By the direct computation, we have

$$\begin{aligned}&\left\{Gld_k(a,b)-\frac{a}{2}\right\}^2-\left\{H(a,b)-\frac{a}{2}\right\}^2\\&=\frac{a^2(b-a)}{k(a+b)^2}\left\{\big(a-(k-1)b\big)^2+k(2-k)b^2\right\}.\end{aligned}$$

Thus, if $1 \le k \le 2$, then the inequality holds. In the case of $k > 2$, the inequality holds if and only if

$$\frac{a}{b} < k-1-\sqrt{k(k-2)}. \tag{2.3}$$

Since $0 < k-1-\sqrt{k(k-2)} < 1$ for $k > 0$, the inequality (2.3) depends on all variables a, b and k, from which, our mean inequality does not hold. □

Proposition 2.3. *For any numbers a and b with $0 < a < b$,*

(i) *the mean inequality $Gld_k(a,b) < G(a,b)$ holds if and only if $k \ge 2$,*
(ii) *the mean inequality $Gld_k(a,b) > G(a,b)$ holds if and only if $k = 1$.*

Proof. Similar to the proof of Proposition 2.2, it follows from

$$\left\{G(a,b)-\frac{a}{2}\right\}^2-\left\{Gld_k(a,b)-\frac{a}{2}\right\}^2=\frac{a}{k}(\sqrt{b}-\sqrt{a})\big((k-1)\sqrt{b}-\sqrt{a}\big).$$ □

Theorem 2.4. *The k-golden mean Gld_k refines the harmonic and geometric means if and only if $k = 2$.*

Proposition 2.5. *Let a and b be two positive numbers with $0 < a < b$. Then, the inequality $Gld_k(a,b) < A(a,b)$ holds if and only if it holds $b > \frac{4-k}{k}a$. Furthermore, if $k \ge 2$, then $Gld_k(a,b) < A(a,b)$.*

Corollary 2.6 (Harmonic-2-golden-geometric-logarithmic-arithmetic mean inequalities)**.** *For any number a and b with $0 < a \le b$, the following means inequalities hold*:

$$H(a,b) \le Gld_2(a,b) \le G(a,b) \le L(a,b) \le A(a,b), \tag{2.4}$$

with strict inequalities if $a < b$.

When k is a special number, the k-golden mean can be expressed by a simple iterative composition of the arithmetic mean and the geometric mean. For a natural number n, let's set $\tilde{A}_n(a,b)$ inductively by

$$\tilde{A}_0(a,b)=b,\ \tilde{A}_n(a,b)=A(a,\tilde{A}_{n-1}(a,b)),$$

where $A(a,b)$ is the arithmetic mean of a and b. By the simple computation, we get that

$$\begin{aligned}&\tilde{A}_1(a,b)=\frac{a+b}{2},\ \tilde{A}_2(a,b)=A(a,\tilde{A}_1(a,b))=\frac{a+\frac{a+b}{2}}{2}=\frac{b+3a}{4},\ \ldots,\\&\tilde{A}_n(a,b)=\frac{b+\sum_{k=0}^{n-1}2^ka}{2^n}=\frac{b+(2^n-1)a}{2^n}.\end{aligned}$$

Therefore, we have the following theorem.

Theorem 2.7. *For a natural number $n \geq 2$, the 2^n-golden mean of two positive numbers a and b $(a \leq b)$ is expressed by*

$$Gld_{2^n}(a,b) = A(a, G(a, \tilde{A}_{n-2}(a,b))).$$

Remark 2.8. It follows from Theorem 2.7 that $Gld_4(a,b) = A(a, G(a,b))$. Indeed, the 4-golden mean of a and b is the arithmetic mean of a and $\sqrt{ab}$, that is, $Glk_{2^2}(a,b) = \frac{a+\sqrt{ab}}{2}$. Also, the 8-golden mean of a and b is given by $Glk_{2^3}(a,b) = A(a, G(a, A(a,b))) = \frac{a+\sqrt{\frac{a}{2}(a+b)}}{2}$.

3. A generalization of the golden ratio

3.1. The golden ratio vs. the k-golden function

The golden ratio $\phi := \frac{1+\sqrt{5}}{2}$ is one of the most famous numbers and there are various methods to explain it. Among them, we give the *extreme and mean ratios* to define it.

Let A, B, C be three points on a real line $\mathbb{R}$ corresponding the positive numbers a, b and c $(a < c < b)$, respectively. Then, the golden ratio ϕ is a special number given by dividing the line segment $\overline{AB}$ into two parts $\overline{AC}$ and $\overline{CB}$ so that the ratio of the length $|\overline{AC}|$ to the length $|\overline{CB}|$ is equal to the ratio of the length $|\overline{AB}|$ to the length $|\overline{AC}|$, i.e., $\frac{b-a}{c-a} = \frac{c-a}{b-c} = \phi$. Moreover, the point C corresponds to the real number $c = a + \phi(b-a) = \phi \cdot b - \frac{1}{\phi} \cdot a$. Similarly, we consider a point $\bar{C}$ on $\mathbb{R}$ corresponding the positive number $\bar{c}$ such that the ratio of $|\overline{O\bar{C}}|$ to $|\overline{AB}|$ is equal to k times the ratio of $|\overline{OA}|$ to $|\overline{O\bar{C}}|$, i.e.,

$$\frac{b-a}{\bar{c}-a} = k \cdot \frac{\bar{c}}{a} =: \phi_k, \tag{3.1}$$

where we denotes O by the origin on $\mathbb{R}$.

The following proposition is a property of the k-golden function related to the *extreme and mean ratios.*

Proposition 3.1. *Let a, b be two positive real numbers with $a < b$. Then, a number $\bar{c}$ is the k-golden mean of a and b if and only if it holds* (3.1).

Proof. It follows from the equation $a + \sqrt{a\{4b + (k-4)a\}/k} = 2\bar{c}$. □

Remark 3.2. The value $gld_k(a,b)$ of the k-golden function give the number $\bar{c}$ satisfying the ratio condition $\frac{b-a}{\bar{c}-a} = k \cdot \frac{\bar{c}}{a}$, while the golden ratio give the number c satisfying the ratio condition $\frac{b-a}{c-a} = \frac{c-a}{b-c}$.

3.2. The k-golden ratio

For two positive numbers k and ℓ, it satisfies that $gld_k(1, 1+\ell) = \frac{1+\sqrt{1+4\ell/k}}{2}$. Here, the value $gld_k(1, 1+\ell)$ is depend on the ratio of ℓ to k. As setting $\ell = 1$, we can give a generalization of golden ratio.

Definition 3.3. For a positive numbers k, the value $gld_k(1,2)$ is called the *k-golden ratio*. Equivalently, the k-golden ratio is a number $\frac{1+\sqrt{1+4/k}}{2}$ which is the positive root of the quadratic equation $X^2 - X - \frac{1}{k} = 0$.

Trivially, the 1-golden ratio equals to golden ratio. This means that the notion of k-golden ratio is a generalization of golden ratio. Now, we denote by $\phi_k := gld_k(1,2)$.

Proposition 3.4. *For a number $k > 0$, it has that $\phi_k > 1$, $\lim_{k\to 0}\phi_k = \infty$ and $\lim_{k\to\infty}\phi_k = 1$. Moreover, if $k > 1$, then $\phi_{\frac{1}{k(k-1)}} = k$.*

Proof. It follows from $\phi_k = \frac{1+\sqrt{1+4/k}}{2}$. □

It is well-known two representations of the golden ratio ϕ that

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\cdots}}} \quad \text{and} \quad \phi = 1 + \frac{1}{1+\frac{1}{1+\cdots}}$$

called respectively the *nested square root* and the *continued fraction* (see [3]). Similarly, we have two representations of k-golden ratio.

Theorem 3.5 (Representations of k-golden ratio)**.** *For a number $k > 0$, it satisfies that*

$$\phi_k = \sqrt{\frac{1}{k}+\sqrt{\frac{1}{k}+\cdots}} \quad \textit{and} \quad \phi_k = 1 + \frac{1}{k}\cdot\frac{1}{1+\frac{1}{k}\cdot\frac{1}{1+\frac{1}{k}\cdots}}.$$

Proof. It is clear from the quadratic equation $(\phi_k)^2 - \phi_k - \frac{1}{k} = 0$. □

In geometry, the golden ratio ϕ appears when a square is inscribed in a semi-circle. The following theorem is a geometrical property of k-golden ratio ϕ_k as a generalization of the golden ratio ϕ.

Theorem 3.6 (Geometry of the k-golden ratio)**.** *Given a semi-ellipse $\bar{S}$ on x-axis given by $kx^2 + y^2 = a$, $y \geq 0$ and a square $\bar{B}$ inscribed in $\bar{S}$ for fixed positive numbers k and a. Let b and ℓ be the length of one side of $\bar{B}$ and the length of the line between two x-intercepts of $\bar{S}$, respectively* (*see Fig. 1*)*. Then, the ratio of b to $\frac{b+\ell}{2}$ equals to k-golden ratio ϕ_k, that is,*

$$\frac{b+\ell}{2b} = \frac{1}{2}\left(1+\frac{\ell}{b}\right) = \phi_k. \tag{3.2}$$

Proof. It follows from that $\ell = 2\sqrt{\frac{a}{k}}$ and $k\left(\frac{b}{2}\right)^2 + b^2 = a$. □

Remark 3.7. Sometime, a rectangle having sides with the ratio $1 : \phi$ is called the *golden rectangle*. Similarly, we call a rectangle having sides with the ratio $1 : \phi_k$ the *k-golden rectangle* (see Fig. 1).

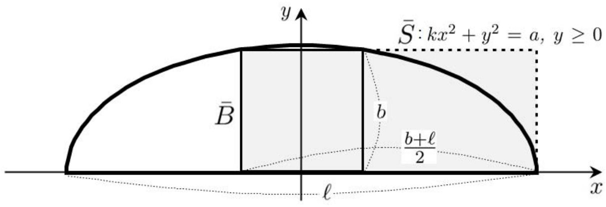
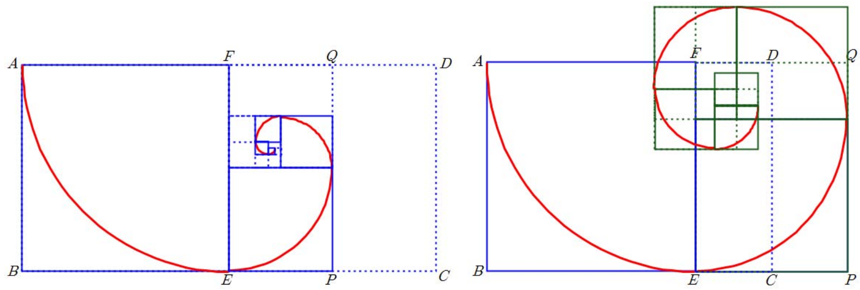
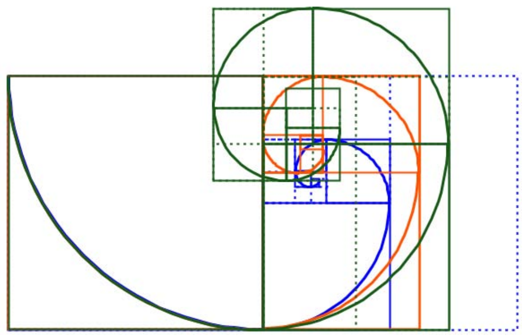


FIGURE 1. A square inscribed in a semi-ellipse and k-golden rectangle



(a) Approximation of $\frac{1}{2}$-golden spiral (b) Approximation of 2-golden spiral



(c) $\frac{1}{2}$-golden(blue) spiral, golden(yellow) sprial and 2-golden(green) spiral

FIGURE 2. Approximation of k-golden spirals

A golden rectangle has an interesting geometric property that when a square section is removed, the remainder is also a golden rectangle. Naturally, this property is generalized by the k-golden rectangle as follows:

Proposition 3.8. *Let $\square ABCD$ be a k-golden rectangle having sides with the ratio $|\overline{AB}| : |\overline{BC}| = 1 : \phi_k$ and $\square ABEF$ a square section of $\square ABCD$. Then it satisfies the followings*:

(i) *The rectangle* $\square FECD$ *is also* k*-golden rectangle if and only if* $k = 1$. *In that case, the golden rectangle* $\square FECD$ *has sides with the ratio* $|\overline{FD}| : |\overline{FE}| = 1 : \phi$.

(ii) *For a positive number* k, *let* $\square FEPQ$ *be a rectangle which has the length condition* $|\overline{FQ}| = k \cdot |\overline{FC}|$. *Then the rectangle* $\square FEPQ$ *is a* k*-golden rectangle having sides with the ratio* $|\overline{FQ}| : |\overline{FE}| = 1 : \phi_k$.

Remark 3.9. A golden spiral often appears in nature. It is a well-known algorithm that a golden spiral can be approximated by a golden rectangle. Applying Proposition 3.8(2) into the approximation algorithm of a golden spiral, we can give new spirals approximated by the k-golden rectangle (See Fig. 2).

4. Applications to matrix mean

For two positive definite matrices A and B, the *geometric mean* is given by an explicit formula

$$A\sharp B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}. \tag{4.1}$$

The geometric mean has various characterizations and applications in matrix inequalities, geometry and dynamical systems. In [2], Lim introduced a new matrix mean having characteristics similar to the geometric mean in Riccati matrix equations and called it the *matrix golden mean* which is a matrix version of the golden mean of two positive numbers. He considered in [2] the Riccati matrix equations under $0 < A \leq B$,

$$XA^{-1}X \pm X - (B - A) = 0, \tag{4.2}$$

and the associated Riccati matrix differential equations,

$$\dot{X} = XA^{-1}X \pm X - (B - A) = 0 \tag{4.3}$$

and proved that the matrix golden means are attracting fixed points of the Riccati differential equations (see [2]).

In this section, motivated by [2], we generalize the notion of the k-golden mean of two positive numbers to the k-golden mean of two positive definite matrices and we list some related results. This is an improvement of Lim's results [2] for which the matrix golden mean. Moreover, we give a new inequalities related to matrix means.

Lemma 4.1 (Riccati)**.** *Let* A *be a positive definite and* B *a* (*semi-*)*positive definite. Then the geometric mean* $A\sharp B$ *is a unique* (*semi-*)*positive definite solution of the Riccati equation*

$$XA^{-1}X = B. \tag{4.4}$$

Lemma 4.2 ([2])**.** *The geometric mean has the following properties*:

(i) $A\sharp B = B\sharp A$.

(ii) $(A\sharp B)^{-1} = A^{-1}\sharp B^{-1}$.

(iii) $M(A\sharp B)M^T = (MAM^T)\sharp(MBM^T)$ *for any nonsingular matrix* M.

(iv) $2(A^{-1}+B^{-1})^{-1} \le A\sharp B \le \frac{1}{2}(A+B)$ *for positive definite* A, B.

Theorem 4.3. *Let* k, A *and* B *be a positive number, a positive definite matrix and a positive semi-definite matrix, respectively. Then the nonlinear matrix equations*

$$X^2 \pm kX - A^2 = 0,$$
$$BX^{-1}B - kX \pm A = 0,$$
$$XA^{-1}X \pm kX - B = 0$$

have the unique positive definite solutions

$$S^{\pm}_{k,1}(A) = \frac{1}{2}\big(\mp kI + I\sharp(k^2 I + 4A^2)\big),$$
$$S^{\pm}_{k,2}(A,B) = \frac{1}{2}\big(\pm kA + A\sharp(k^2 A + \frac{4}{k}BA^{-1}B)\big),$$
$$S^{\pm}_{k,3}(A,B) = \frac{1}{2}\big(\mp kA + A\sharp(k^2 A + 4B)\big),$$

respectively.

Proof. (i) From the fact $A\sharp(k^2B) = (kA)\sharp(kB) = k(A\sharp B)$ and $k > 0$, it is easy to check that $S^{\pm}_{k,1}(A) = \frac{1}{2}\big(\mp kI + I\sharp(k^2I + 4A^2)\big) = \frac{k}{2}\big(\mp I + I\sharp(I + \frac{4}{k^2}A^2)\big)$ are positive definite. By the direct computation, we know that $S^{\pm}_{k,1}(A)$ are solutions of $X^2 \pm kX - A^2 = 0$.

Suppose that X is a positive definite solution of $X^2+kX-A^2 = 0$. Using the equality $X^2+kX-A^2 = (X+\frac{k}{2}I)^2-(A^2+\frac{k^2}{4}I)$, we have $X+\frac{k}{2} = (A^2+\frac{k^2}{4}I)^{\frac{1}{2}}$, from which, $X = S^{+}_{k,1}(A)$. Similarly, we have a unique positive definite solution $X = S^{-}_{k,1}(A)$ of $X^2 - kX - A^2 = 0$.

(ii) Consider the matrix equations $kX = \pm A + BX^{-1}B$. Setting $Y = A^{-\frac{1}{2}}XA^{-\frac{1}{2}}$ and $D = A^{-\frac{1}{2}}DA^{-\frac{1}{2}}$, we have respectively

$$kY = \pm I + DY^{-1}D. \tag{4.5}$$

Since $A\sharp B$ is a unique positive definite solution of $XA^{-1}X = B$, if Y is a positive definite solution of (4.5), then Y satisfies

$$D = Y\sharp(kY \mp I) = Y^{\frac{1}{2}}\big(Y^{-\frac{1}{2}}(kY \mp I)Y^{-\frac{1}{2}}\big)^{\frac{1}{2}}Y^{\frac{1}{2}} = (kY^2 \mp Y)^{\frac{1}{2}},$$

equivalently,

$$Y^2 \mp \frac{1}{k}Y - (\frac{1}{\sqrt{k}}D)^2 = 0. \tag{4.6}$$

Conversely, if Y is a positive definite solution of (4.6), then Y and D commute and hence Y satisfies (4.5). Therefore (4.5) and (4.6) are equivalent, respectively. Solving Y we then have $Y = S^{\mp}_{k,1}(\frac{1}{\sqrt{k}}D)$ by (i). Thus, we have

$$\begin{aligned} S^{\pm}_{k,2} &= A^{\frac{1}{2}} Y A^{\frac{1}{2}} = A^{\frac{1}{2}} S^{\mp}_{k,1}(\frac{1}{\sqrt{k}} A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \\ &= A^{\frac{1}{2}} \Big(\frac{1}{2} \big(\pm kI + I \sharp (k^2 I + \frac{4}{k} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^2)\big) \Big) A^{\frac{1}{2}} \\ &= \frac{1}{2} \big(\pm kA + A \sharp (k^2 A + \frac{4}{k} B A^{-1} B) \big), \end{aligned}$$

where the last equality follows from (iii) in Lemma 4.2.

(iii) By setting in (ii), we have $Y^2 \pm kY - D = 0$, from which, $Y = S^{\pm}_{k,1}(D^{\frac{1}{2}})$. Thus, we have

$$\begin{aligned} S^{\pm}_{k,3}(A,B) &= A^{\frac{1}{2}} Y A^{\frac{1}{2}} = A^{\frac{1}{2}} S^{\pm}_{k,1}(D^{\frac{1}{2}}) A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}} \Big(\frac{1}{2} \big(\pm kI + I \sharp (k^2 I + 4D) \big) \Big) A^{\frac{1}{2}} \\ &= \frac{1}{2} \big(\mp kA + A \sharp (k^2 A + 4B) \big). \end{aligned}$$

Our assertions are completed. □

Corollary 4.4. *Let k be a fixed number with $k \geq 1$. Suppose that A and B are two positive definite matrices with the order relation $A \leq B$.*

(i) *The Riccati matrix equation*

$$XA^{-1}X - X - \frac{1}{k}(B - A) = 0$$

has a unique positive definite solution

$$X = A \natural_k B := \frac{1}{2} \left(A + A \sharp \frac{1}{k} \big(4B + (k-4)A \big) \right).$$

(ii) *The Riccati matrix equation*

$$XA^{-1}X + X - \frac{1}{k}(B - A) = 0$$

has a unique positive definite solution

$$X = A \bar{\natural}_k B := \frac{1}{2} \left(-A + A \sharp \frac{1}{k} \big(4B + (k-4)A \big) \right).$$

Corollary 4.5. *Let a and b be two positive real number with $a \leq b$. Then, for a fixed $k \geq 1$, it satisfies that*

$$(aA) \natural_k (bA) = Gld_k(a,b) A,$$

where $Gld_k(a,b)$ denotes the k-golden mean of a and b.

For a real number $k \geq 1$ and two positive definite matrices A and B, now, we call the positive definite matrix $G_k(A,B)$ the *k-golden matrix mean.* A suitable choice of positive number k leads to the simple and interesting expressings of the k-golden mean as follows:

Corollary 4.6. *For two positive definite matrices A and B with $A \le B$, the followings are satisfied.*

(i) $A\natural_1 B = A\natural B$, $A\natural_2 B = \frac{1}{2}\big(A + A\sharp(B - \frac{1}{2}A)\big)$.

(ii) $A\natural_4 B = \frac{1}{2}(A + A\sharp B)$, *that is,* $A\natural_4 B$ *is the arithmetic matrix mean of* A *and* $A\sharp B$.

(iii) *For* $m > 0$, $A\natural_{4m} B = \frac{1}{2}\big(A + \frac{1}{\sqrt{m}} A\sharp(B + (m-1)A)\big)$.

We give some properties about the expression of the k-golden mean.

Proposition 4.7. *Let A and B be two arbitrary positive definite matrices with $A \le B$ and k a fixed real number with $k \ge 1$. Then, it satisfies the following properties*:

(i) $M(A\natural_k B)M^T = (MAM^T)\natural_k(MBM^T)$ *and* $M(A\bar{\natural}_k B)M^T = (MAM^T)\bar{\natural}_k(MBM^T)$ *for any nonsingular matrix* M.

(ii) $A\natural_k B = \frac{1}{2}A^{\frac{1}{2}}\Big(I + \frac{1}{\sqrt{k}}\big(4A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (k-4)I\big)^{\frac{1}{2}}\Big)A^{\frac{1}{2}}$.

(iii) *If* $A < B$, *then* $A\natural_k B = \frac{1}{2}\Big(A + \frac{1}{\sqrt{k}}(B-A)\sharp\big(4A + kA(B-A)^{-1}A\big)\Big)$.

(iv) $(A\natural_k B)\sharp(A\bar{\natural}_k B) = \frac{1}{k}A\sharp(B - A)$.

(v) $A\natural_k B = A\sharp\big(\frac{1}{k}(B + (k-1)A) + A\bar{\natural}_k B\big)$ *and* $A\bar{\natural}_k B = A\sharp\big(\frac{1}{k}(B + (k-1)A) - A\natural_k B\big)$.

Proof. (i) It follows from the linearity of the congruence transformations and Lemma 4.2(iii).

(ii) It follows from (i).

(iii) Suppose that $A < B$ and set $C := A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then, it follows that $C - I > 0$. By (ii) and Lemma 4.2(iii), the k-golden mean is calculated by

$$\begin{aligned} A\natural_k B &= \frac{1}{2}A^{\frac{1}{2}}\Big(I + \frac{1}{\sqrt{k}}\big(4A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (k-4)I\big)^{\frac{1}{2}}\Big)A^{\frac{1}{2}} \\ &= \frac{1}{2}A^{\frac{1}{2}}\Big(I + \frac{1}{\sqrt{k}}\big(4C + (k-4)I\big)^{\frac{1}{2}}\Big)A^{\frac{1}{2}} \\ &= \frac{1}{2}A^{\frac{1}{2}}\Big(I + \frac{1}{\sqrt{k}}\big(4(C-I) + kI\big)^{\frac{1}{2}}\Big)A^{\frac{1}{2}} \\ &= \frac{1}{2}A^{\frac{1}{2}}\Big(I + \frac{1}{\sqrt{k}}(C-I)\sharp\big(4I + k(C-I)^{-1}\big)\Big)A^{\frac{1}{2}} \\ &= \frac{1}{2}\Big(A + \frac{1}{\sqrt{k}}(B-A)\sharp\big(4A + kA(B-A)^{-1}A\big)\Big). \end{aligned}$$

(iv) It follows from

$$\begin{aligned} (I\natural_k C)(I\bar{\natural}_k C) &= \frac{1}{4}\big(I + \frac{1}{\sqrt{k}}(4C + (k-4)I)^{\frac{1}{2}}\big)\big(-I + \frac{1}{\sqrt{k}}(4C + (k-4)I)^{\frac{1}{2}}\big) \\ &= \frac{1}{k}(C - I), \end{aligned}$$

the invariance property (i) and the commutativity of $(I\natural_k C)(I\bar{\natural}_k C)$.

(v) By the direct computation, we have

$$\begin{aligned}
&(A\natural_k B)A^{-1}(A\natural_k B)\\
&= \frac{1}{4}\Big(A + \frac{1}{\sqrt{k}}\cdot A\sharp\big(4B+(k-4)A\big)\Big)A^{-1}\Big(A + \frac{1}{\sqrt{k}}\cdot A\sharp\big(4B+(k-4)A\big)\Big)\\
&= \frac{1}{4}\Big(I + \frac{1}{\sqrt{k}}\cdot A\sharp\big(4B+(k-4)A\big)\cdot A^{-1}\Big)\Big(A + \frac{1}{\sqrt{k}}\cdot A\sharp\big(4B+(k-4)A\big)\Big)\\
&= \frac{1}{4}\Big(A + \frac{2}{\sqrt{k}}\cdot A\sharp\big(4B+(k-4)A\big) + \frac{1}{k}\big(4B+(k-4)A\big)\Big)\\
&= \frac{1}{2}\Big(-A + \frac{1}{\sqrt{k}}\cdot A\sharp\big(4B+(k-4)A\big)\Big) + \frac{1}{k}\big(B+(k-1)A\big)\\
&= A\bar{\natural}_k B + \frac{1}{k}\big(B+(k-1)A\big),
\end{aligned}$$

where the third equality follows from the Riccati lemma for the geometric mean. Thus, we have $A\natural_k B = A\sharp\big(A\bar{\natural}_k B + \frac{1}{k}(B+(k-1)A)\big)$ from the Riccati lemma. Similarly, we have the second equation of (v). □

Now, we give some inequalities related to the k-golden mean.

Proposition 4.8. *Let A and B be two arbitrary positive definite matrices with $A \leq B$ and k a fixed real number with $k \geq 1$. Then, it satisfies the following properties*:

(i) *If $1 \leq k_1 < k_2$, then $A\natural_{k_2} B < A\natural_{k_1} B$.*

(ii) *$A \leq A\natural_k B \leq B$ with strict inequalities if $A < B$.*

(iii) *$A\natural_k B = A\sharp B$ if and only if $k = 1$ and $A = B$.*

(iv) (The harmonic-k-golden mean inequalities) If $1 \leq k \leq 2$, then

$$2(A^{-1}+B^{-1})^{-1} \leq A\natural_k B$$

with strict inequalities if $A < B$.

(v) (The k-golden-geometric mean inequalities) *If $k \geq 2$, then*

$$A\natural_k B \leq A\sharp B$$

with strict inequalities if $A < B$.

(vi) ([2]) (1-golden-geometric mean inequalities)

$$A\sharp B \leq A\natural_1 B = A\natural B$$

with strict inequalities if $A < B$.

(vii) (The harmonic-2-golden-geometric-arithmetic inequalities)

$$A \leq 2(A^{-1}+B^{-1})^{-1} \leq A\natural_2 B \leq A\sharp B \leq \frac{1}{2}(A+B) \leq B$$

with strict inequalities if $A < B$.

Proof. (i) If $k_1 \neq k_2$, then $\frac{1}{k_1}\big(4B+(k_1-4)A\big) - \frac{1}{k_2}\big(4B+(k_2-4)A\big) = \frac{4(k_2-k_1)}{k_1k_2}(B-A)$. Hence, it follows from the geometric mean inequality that if $B_1 < B_2$, then $A\sharp B_1 < A\sharp B_2$.

(ii) Since $A = \frac{1}{k}\big(4A + (k-4)A\big) \le \frac{1}{k}\big(4B + (k-4)A\big)$, $A = \frac{1}{2}(A + A) = \frac{1}{2}(A + A\sharp A) \le \frac{1}{2}\big(A + A\sharp\frac{1}{k}(4B + (k-4)A)\big) = A\natural_k B$.

(ii) If $k_1 \neq k_2$, then $\frac{1}{k_1}\big(4B + (k_1-4)A\big) - \frac{1}{k_2}\big(4B + (k_2-4)A\big) = \frac{4(k_2-k_1)}{k_1k_2}(B - A)$. Hence, it follows from the property of the geometric mean that if $A \le B_1 < B_2$, then $A\sharp B_1 < A\sharp B_2$.

(iii) In the case of $k = 1$, $A\natural_1 B = A\sharp B$ if and only if $A = B$ (see [2]). Since $A\natural_k \le A\natural_1 B$, from (ii), our assertion (iii) holds.

(iv) Using Proposition 2.2 and the spectral decomposition of a positive definite matrix D, we have

$$2(I + D^{-1})^{-1} \le \frac{1}{2}\left\{I + I\sharp\frac{1}{\sqrt{k}}\big(4D + (k-4)I\big)\right\},$$

from which, our inequality is reduced by putting $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$.

(v) It follows by the similar method of (iv).

(vi) See [2].

(vii) It follows from (iv) and (v). □

5. Conclusions

In this paper, using the notion of the k-golden mean of two positive numbers, we show that the 2-golden mean refines the harmonic, geometric mean inequalities and we apply in the matrix mean inequalities.

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