BICOMPRESSIBLE SURFACES AND INCOMPRESSIBLE SURFACES

Toshio Saito

Dedicated to Professor Tsuyoshi Kobayashi on the occasion of his 60th birthday

Abstract. We give new evidence that "complicated" Heegaard surfaces behave like incompressible surfaces. More precisely, suppose that a closed connected orientable 3-manifold M contains a closed connected incompressible surface F which separates M into two (connected) components M_1 and M_2. Let S be a Heegaard surface of M. Our result is that if the Hempel distance of S is at least four, then S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$.

1. Introduction

When one studies topological and geometrical properties of 3-manifolds, sub-manifolds of codimension one, which are so-called surfaces, play crucial roles. If surfaces as sub-manifolds are *incompressible* (see below), then such 3-manifolds are extremely well understood. Another good point is that if an irreducible 3-manifold contains two such surfaces with non-empty intersection, then they are isotoped to be in a good position, which means that their intersection is essential in both surfaces.

Though *bicompressible surfaces* (see below) have properties opposite to incompressible surfaces, they are often useful to study 3-manifolds. One of the most important examples of bicompressible surfaces is a *Heegaard surface* of a closed orientable 3-manifold, which divides the 3-manifold into two handlebodies. Unlike incompressible surfaces, it is not always true that two Heegaard surfaces are isotoped to be in a good position. However, it is proved in [7] that if two Heegaard surfaces are "complicated", then they are isotoped to be in a good position. This implies that complicated Heegaard surfaces behave like incompressible surfaces. We here use (Hempel) *distance* in [3], given by a non-negative integer (see below), to measure the complexity of Heegaard surfaces.

Then we can restate that Heegaard surfaces of distance at least two behave like incompressible surfaces.

There is another reason supporting the opinion above. Let F be a closed connected incompressible surface in a closed orientable 3-manifold M that separates M into two components, say M_1 and M_2. Let S be another closed connected separating surface in M which has non-empty intersection with F. It follows by the definition of incompressibility that if S is also incompressible, then S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$. A conclusion similar to the above is obtained by Kobayashi-Qiu [6]: if S is a Heegaard surface of distance at least two, then S is isotoped so that $S \cap M_i$ is incompressible for *some* $i = 1, 2$.

In this paper, we obtain the completely same conclusion by imposing more rigorous requirements: if S is a Heegaard surface of distance at least *four*, then S is isotoped so that $S \cap M_i$ is incompressible for *each* $i = 1, 2$. We give a stronger version of the result in the following section. In the interest of fairness, the author adds the reference that a parallel result is obtained in [4] around the same time.

2. Definitions and results

Throughout this paper, we work in piecewise linear category. Let M be a compact connected orientable 3-manifold. We say that two surfaces properly embedded in M have *non-empty intersection* if they cannot be isotoped to be disjoint.

The key tool in this paper is a well-known ambient isotopy, which is called an *isotopy of type A* by Jaco [5, Chapter II]. Let S and F be surfaces properly embedded in M such that S and F intersect transversely. Suppose that there is a disk δ in M which satisfies the following three conditions: (i) $\delta \cap (S \cup F) = \partial \delta$, (ii) $\partial \delta \cap F$ is a single arc, say α, and (iii) $\partial \delta \cap S$ is a single arc, say β. Then we can consider an ambient isotopy of M so that $\beta \subset S$ goes to α along δ and is further pushed out (cf. Fig. 1). Let S' be the resulting surface obtained from S. Then we say that S' is obtained from S by an *isotopy of type A* along δ.

Suppose that S is a compact orientable surface properly embedded in M. We say that a simple closed curve or a simple arc properly embedded in S is *inessential* in S if it cuts off a disk from S. A simple closed curve or a simple arc is said to be *essential* in S if it is not inessential in S. The surface S is said to be *compressible* in M if there is an embedded disk D in M such that $D \cap S = \partial D$ and ∂D is essential in S. Such a disk D is called a *compressing disk* for S. We say that S is *incompressible* in M if S is not compressible in M. If S cuts off a sub-manifold M' from M and no compressing disks for S are contained in M', then S is said to be *incompressible toward M'*. Suppose that S is connected and separates M into two components V and W. Then S is *bicompressible* if there are compressing disks for S in both V and W. Given a closed bicompressible surface, say S again, let $\mathcal{C}_V$ and $\mathcal{C}_W$ be the sets of
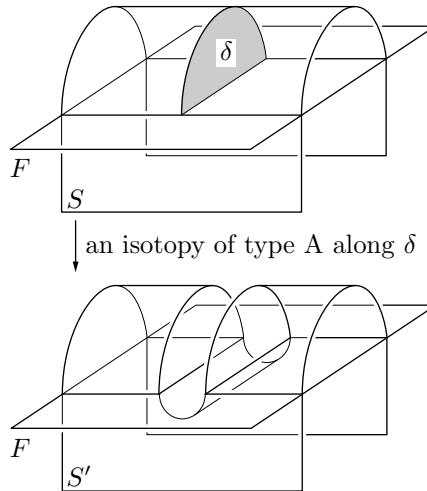
FIGURE 1

isotopy classes of the boundary components of compressing disks for S in V and W respectively. If the genus of S is at least two, then the *distance* $d(S)$ is defined to be the minimal non-negative integer n so that there is a sequence of simple closed curves $c_0, \ldots, c_n$ in S such that $[c_0] \in \mathcal{C}_V$, $[c_n] \in \mathcal{C}_W$, and c_{i-1} is isotoped to be disjoint to c_i for any $i = 1, \ldots, n$. If the genus of S is one, then we define $d(S) = -\infty$ for convenience.

Theorem 2.1. *Suppose that a compact connected orientable irreducible 3-manifold M contains a closed connected incompressible surface F which separates M into two components M_1 and M_2. Let S be a closed bicompressible surface in M which has non-empty intersection with F. If $d(S) \geq 3$, then either*

    (1) *$d(S) = 3$ and S is isotoped so that $S \cap F$ is a simple closed curve which is essential in both S and F, and that $S \cap M_i$ is incompressible for some $i = 1, 2$, or*

    (2) *S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$.*

In the theorem above, suppose that M is closed and that S is a Heegaard surface. Then if M is reducible, we see that $d(S) = 0$ by Haken's lemma [2]. It is also well-known that any closed incompressible surface has non-empty intersection with S. Hence we immediately have the following.

Corollary 2.2. *Suppose that a closed connected orientable 3-manifold M contains a closed connected incompressible surface F which separates M into two components M_1 and M_2. Let S be a Heegaard surface in M. If $d(S) \geq 4$, then S is isotoped so that $S \cap M_i$ is incompressible for each $i = 1, 2$.*

3. Proof of Theorem 2.1

The special case of [1, Corollary 3.8] says that if $d(S) \geq 2$, then S and F can be isotoped so that each component of $S \cap F$ is essential in both surfaces. Hence we may assume that $S \cap F$ is a collection of simple closed curves which are essential in both S and F under our assumption that $d(S) \geq 3$. Suppose that the conclusion (2) in Theorem 2.1 does not hold, i.e., $S \cap M_i$ is compressible for some $i = 1, 2$. Let V and W be 3-manifolds obtained by cutting M along S.

Lemma 3.1. *Both $S \cap M_1$ and $S \cap M_2$ are incompressible toward $V \cap M_1$ and $V \cap M_2$, or toward $W \cap M_1$ and $W \cap M_2$, respectively.*

Proof. If $S \cap M_i$ admits compressing disks in both $V \cap M_i$ and $W \cap M_i$, then we have $d(S) \leq 2$. Hence $S \cap M_i$ is incompressible toward $V \cap M_i$ or $W \cap M_i$. This implies that if $S \cap M_1$ or $S \cap M_2$ is incompressible in M_1 or M_2 respectively, then the lemma holds. The remaining case is that $S \cap M_1$ and $S \cap M_2$ are compressible in M_1 and M_2 respectively. If $S \cap M_1$ contains a compressing disk in $V \cap M_1$ ($W \cap M_1$ resp.) and $S \cap M_2$ contains a compressing disk in $W \cap M_2$ ($V \cap M_2$ resp.), then we have $d(S) \leq 1$, a contradiction. Thus we have the desired conclusion. $\square$

At this moment, S and F satisfy the following condition $(*)$ under the assumption that the conclusion (2) in Theorem 2.1 does not hold.

$(*)$ Each component of $S \cap F$ is essential in both surfaces, and S satisfies one of the three conditions (cf. Fig. 2):

   (i) $S \cap M_1$ admits a compressing disk in $V \cap M_1$ and is incompressible toward $W \cap M_1$, and $S \cap M_2$ is incompressible in M_2.

  (ii) $S \cap M_1$ is incompressible in M_1, and $S \cap M_2$ admits a compressing disk in $V \cap M_2$ and is incompressible toward $W \cap M_2$.

 (iii) Both $S \cap M_1$ and $S \cap M_2$ admit compressing disks in $V \cap M_1$ and $V \cap M_2$ respectively, and are incompressible toward $W \cap M_1$ and $W \cap M_2$ respectively.

We now take a compressing disk D for S in W such that $|D \cap F|$ is minimal among all the bicompressible surfaces which are isotopic to S and satisfy the condition $(*)$, where $| \cdot |$ means the number of connected components. Since F is incompressible, it follows from standard arguments that $D \cap F$ consists of arcs properly embedded in D. Let $\delta \subset D$ be an outermost disk component of D cut along $D \cap F$, i.e., δ is a disk component of D cut along $D \cap F$ such that $\partial \delta$ consists of two arcs α and β with $\alpha \in (D \cap F)$ and $\beta \subset \partial D$. Without loss of generality, we may assume that δ is contained in $W \cap M_1$. Then the following is proved in [6, Claim 2 in the proof of Proposition 2.6]:

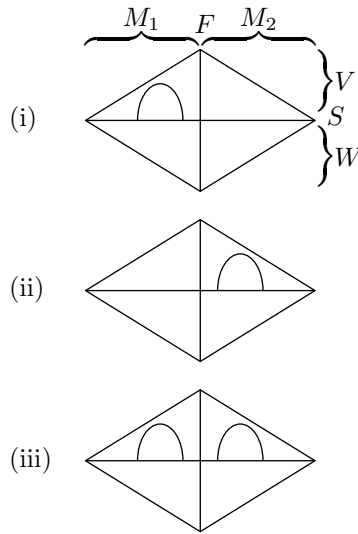**Lemma 3.2** ([6, Claim 2 in the proof of Proposition 2.6])**.** *β is essential in $S \cap M_1$.*

Figure 2

Let S' be a surface obtained from S by an isotopy of type A along δ. Let V' and W' be 3-manifolds obtained from V and W respectively by an influence of the isotopy above. We note that $M = V \cup_S W = V' \cup_{S'} W'$. Set $D' = D \cap W'$. Then D' is a compressing disk for S' in W', and $|D' \cap F| < |D \cap F|$. It follows from Lemma 3.2 that each component of $S' \cap F$ is essential in S'. The following is also obtained as [6, Claim 3 in the proof of Proposition 2.6]:

Lemma 3.3 ([6, Claim 3 in the proof of Proposition 2.6]). *Each component of* $S' \cap F$ *is essential in* F.

Hence the minimality of $|D \cap F|$ shows that (S', V', W') does not satisfy all the cases from (i) to (iii) in the condition ($*$). We here give brief remarks that (a) if $S \cap M_1$ is incompressible in M_1, then so is $S' \cap M_1$, (b) if $S \cap M_1$ is compressible in M_1, then there are both possibilities that $S' \cap M_1$ is compressible or is incompressible, (c) if $S \cap M_2$ is compressible in M_2, then so is $S' \cap M_2$, and (d) if $S \cap M_2$ is incompressible in M_2, then there are both possibilities that $S' \cap M_2$ is compressible or is incompressible. Since $d(S') = d(S) \geq 3$ and hence S' also satisfies Lemma 3.1, we see from the arguments above that S satisfies the case (i) in the condition ($*$) and $S' \cap M_2$ admits a compressing disk in $W' \cap M_2$ and is incompressible toward $V' \cap M_2$ (cf. Fig. 3).

Let c_V be the boundary of a compressing disk for $S \cap M_1$ in $V \cap M_1$ and c_W the boundary of a compressing disk for $S' \cap M_2$ in $W' \cap M_2$. If $|S \cap F| \geq 2$ and $|S' \cap F| \geq 2$, then there is a component of $S \cap F$ or $S' \cap F$ disjoint from $c_V \cup c_W$ under natural identification of S and S'. This implies that $d(S) \leq 2$,
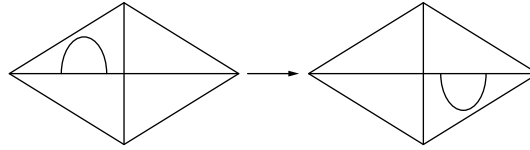
FIGURE 3

a contradiction. Hence we see that either $S \cap F$ or $S' \cap F$ is a simple closed curve which is essential in both surfaces. Let c_1 be a component of $S \cap F$ and c_2 a component of $S' \cap F$. Since we assume that $d(S) \geq 3$, the sequence c_V, c_1, c_2, c_W gives $d(S) = 3$ under natural identification of S and S'. Hence we have the conclusion (1) in Theorem 2.1.

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TOSHIO SAITO
DEPARTMENT OF MATHEMATICS
JOETSU UNIVERSITY OF EDUCATION
1 YAMAYASHIKI, JOETSU, NIIGATA 943-8512, JAPAN
Email address: toshio@juen.ac.jp