Bull. Korean Math. Soc. **56** (2019), No. 2, pp. 505–514
https://doi.org/10.4134/BKMS.b180382
pISSN: 1015-8634 / eISSN: 2234-3016

A NOTE ON MONOFORM MODULES

Alireza Hajikarimi and Ali Reza Naghipour

Abstract. Let R be a commutative ring with identity and M be a unitary R-module. A submodule N of M is called a dense submodule if $\mathrm{Hom}_R(M/N, E_R(M)) = 0$, where $E_R(M)$ is the injective hull of M. The R-module M is said to be monoform if any nonzero submodule of M is a dense submodule. In this paper, among the other results, it is shown that any kind of the following module is monoform.

(1) The prime R-module M such that for any nonzero submodule N of M, $\mathrm{Ann}_R(M/N) \neq \mathrm{Ann}_R(M)$.
(2) Strongly prime R-module.
(3) Faithful multiplication module over an integral domain.

1. Introduction

Throughout this paper, R is a commutative ring with identity and all modules are unitary. The injective hull of M and the set of zero divisors of M are denoted by $E_R(M)$ and $Z_R(M)$, respectively. The *annihilator* of M is denoted $\mathrm{Ann}_R(M)$ and for any $x \in M$ the annihilator of Rx is denoted $\mathrm{Ann}_R(x)$. If N is a submodule of an R-module M, then $(N :_R M)$ denotes the ideal $\mathrm{Ann}_R(M/N)$ of R, that is $(N :_R M) = \{r \in R : rM \subseteq N\}$. The R-module M is called *faithful* if $\mathrm{Ann}_R(M) = 0$. A proper submodule N of an R-module M is called *prime submodule* if for $r \in R$ and $x \in M$, $rx \in N$ implies that $x \in N$ or $rM \subseteq N$. Also, M is called *prime module* if the submodule 0 of M is prime. It is easy to see that M is prime if and only if $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(N)$ for any nonzero submodule N of M. This notion of prime submodules was first introduced in [6] and [8] and systematically studied in [4]. Recall that a submodule N of R-module M is said to be *strongly prime submodule* if for any $x, y \in M$, $(Rx + N :_R M)y = 0$ implies that $x \in N$ or $y \in N$. Furthermore, the R-module M is called *strongly prime* if 0 is a strongly prime submodule. It is easy to see that any strongly prime is prime but the converse is not true in general (see [13] and [15]).

Received April 19, 2018; Revised December 2, 2018; Accepted December 10, 2018.
2010 *Mathematics Subject Classification.* 13C05, 13C11, 13E05.
Key words and phrases. dense submodule, prime module, monoform module, injective hull, multiplication module.

Let M be an R-module. A submodule N of M is said to be *essential* if N has nonzero intersection with any nonzero submodule of M. We write $N \leq_e M$ to indicate that N is an essential submodule of M. The notion of a dense submodule is a refinement of that of an essential submodule. A submodule N of M is called *dense submodule* and is written $N \leq_d M$ if for any $x, y \in M$ with $x \neq 0$ there exists $r \in R$ such that $rx \neq 0$ and $ry \in N$. The notion of dense submodule due to Findlay-Lambek plays an important role in the context of commutative (or noncommutative) algebra (see [7]). For example, the main objects in the maximal ring of quotients are dense submodules (see [10, Chapter 5]).

An R-module M is said to be *monoform* if any nonzero submodule of M is a dense submodule. Monoform modules arise in the study of rings and modules with Krull dimensions (see [8]).

We call an R-module M is *essentially monoform* if for any nonzero essential submodule N of M, $N \leq_d M$.

The plan of this paper is as follows. In Section 2, we study the connection between the monoform modules and the prime modules. We prove that if M is a prime R-module and $\mathrm{Ann}_R(M/N) \neq \mathrm{Ann}_R(M)$ for any nonzero submodule N of M, then M is monoform. Moreover, if R is Noetherian and M is finitely generated, then the converse is true (see Theorem 2.12). We prove in Theorem 2.16 that if M is an R-module and $E_R(M)$ is a prime R-module, then M is essentially monoform. It is shown that any strongly prime module is monoform (see Theorem 2.13). In Section 3, we prove that any faithful multiplication module over an integral domain is monoform (see Corollary 3.4). Finally, we characterize the finitely generated faithful module which has no proper dense submodule (see Theorem 3.6).

For other notations and terminologies not mentioned in this paper, one can refer to [10] and [16].

2. Some characterizations of dense submodules

First, we recall a proposition about dense submodules which is used widely in the sequel.

Proposition 2.1 (See [10, Proposition 8.6])**.** *Let N be a nonzero submodule of an R-module M. Then the following are equivalent*:

(1) $N \leq_d M$.
(2) $\mathrm{Hom}_R(M/N, E_R(M)) = 0$.
(3) *For any submodule P such that $N \leq P \leq M$,* $\mathrm{Hom}_R(P/N, M) = 0$.

Definition 2.2. An R-module M is called *monoform* (following [8]) if any nonzero submodule of M is a dense submodule.

A nonzero module M is called *uniform* if any two nonzero submodules of M intersect nontrivially (equivalently: any nonzero submodule of M is indecomposable, or else: any nonzero submodule of M is essential in M). In view of

[10, Examples 3.51], we have the obvious implications:

$$\text{simple} \Longrightarrow \text{monoform} \Longrightarrow \text{uniform} \Longrightarrow \text{indecomposable}$$

The following example shows that the set of all simple R-module is strictly contained in the set of all monoform R-modules.

Example 2.3. $\mathbb{Z}$ is monoform but it is not simple.

A cyclic $\mathbb{Z}$-module G is uniform if and only if either G is infinite or $|G| = p^n$ for some prime p and positive integer n. The following example shows that the set of all monoform R-module is strictly contained in the set of all uniform R-modules.

Example 2.4. Let G be a cyclic group of order p^2. Then G is a uniform $\mathbb{Z}$-module but it is not a monoform $\mathbb{Z}$-module (see Corollary 3.2).

Remark 2.5. By [10, Proposition 8.7], if M is an R-module and $N \leq K \leq M$, then $N \leq_d M$ if and only if $N \leq_d K$ and $K \leq_d M$. So, if every nonzero cyclic submodules of M is a dense submodule, then M is a monoform module.

The *support* of an R-module M is the set of all prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}} \neq 0$ and it is denoted by $\mathrm{Supp}_R(M)$. Also, let $\mathfrak{p}$ be a prime ideal of R. $\mathfrak{p}$ is said to be an *associated prime ideal* of M if $\mathfrak{p}$ is the annihilator of some $x \neq 0$ of M. The set of associated primes of M is denoted by $\mathrm{Ass}_R(M)$. If $\mathfrak{a}$ is an ideal of R, then $V(\mathfrak{a})$ is the set of all prime ideals of R which contains $\mathfrak{a}$.

Theorem 2.6. *Let M be a nonzero finitely generated module over a Noetherian ring R and let N be submodule of M. Then the following are equivalent*:

(1) $N \leq_d M$.
(2) $\mathrm{Hom}_R(M/N, E_R(M)) = 0$.
(3) $\mathrm{Ass}_R(\mathrm{Hom}_R(M/N, E_R(M))) = \emptyset$.
(4) $\mathrm{Supp}_R(M/N) \cap \mathrm{Ass}_R(M) = \emptyset$.
(5) $\mathrm{Hom}_R(M/N, M) = 0$.
(6) $V(\mathrm{Ann}_R(M/N)) \cap \mathrm{Ass}_R(M) = \emptyset$.
(7) $\mathrm{Ann}_R(M/N) \nsubseteq Z_R(M)$.

Proof. (1)⇔(2) Follows from Proposition 2.1.

(2)⇔(3) Follows from [16, Corollary 9.35].

(3)⇔(4) Follows from [3, Proposition 10, Chapter IV] and [10, Remarks and Example 3.57(2)].

(4)⇔(5) Follows from [3, Proposition 10, Chapter IV] and [16, Lemma 9.35].

(5)⇔(6) Follows from [3, Proposition 10, Chapter IV] and [16, Lemma 9.20].

(6)⇒(7) Suppose $\mathrm{Ann}_R(M/N) \subseteq Z_R(M)$. By [16, Lemma 9.15], $Z_R(M) = \cup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}$. Also $\mathrm{Ass}_R(M)$ is a finite set, by [11, Theorem 6.5(i)]. So the Prime Avoidance Theorem ([16, Theorem 3.61] implies that $\mathrm{Ann}_R(M/N) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R(M)$. It follows that $\mathfrak{p} \in V(\mathrm{Ann}_R(M/N)) \cap \mathrm{Ass}_R(M)$, a contradiction.

(7)⇒(6) Suppose that $V(\mathrm{Ann}_R(M/N)) \cap \mathrm{Ass}_R(M) \neq \emptyset$ and let

$$\mathfrak{p} \in V(\mathrm{Ann}_R(M/N)) \cap \mathrm{Ass}_R(M).$$

Therefore $\mathrm{Ann}_R(M/N) \subseteq \mathfrak{p} \subseteq Z_R(M)$, a contradiction. □

Corollary 2.7. *Let M be a nonzero module over a Noetherian ring R and $\mathrm{Ass}_R(M) = \mathrm{Supp}_R(M)$. Then M has no proper dense submodule.*

Proof. Let N be a submodule of M. By [16, Exercise 9.19] and Theorem 2.6, $N \leq_d M$ if and only if $\mathrm{Supp}_R(M/N) \cap \mathrm{Supp}_R(M) = \emptyset$ if and only if $\mathrm{Supp}_R(M/N) = \emptyset$ if and only if $M/N = 0$. So, M has no proper dense submodule. □

An example of these modules is weakly Artinian modules. An R-module M is called *weakly Artinian* if $\mathrm{Ass}_R(M)$ consists of finitely many maximal ideals (see [9]).

Following [2], an R-module M is said to be *coretractable* if for any proper submodule N of M, $\mathrm{Hom}_R(M/N, M) \neq 0$.

Proposition 2.8. *Let M be a nonzero finitely generated R-module over a Noetherian ring R. Then M has no proper dense submodule if and only if M is coretractable.*

Proof. It is easy to see that any coretractable module has no proper dense submodule.

Conversely, let M has no proper dense submodule and N be a proper submodule of M. Then $\mathrm{Hom}_R(M/N, E_R(M)) \neq 0$. So, $\mathrm{Supp}_R(M/N) \cap \mathrm{Ass}_R E_R(M) = \mathrm{Supp}_R(M/N) \cap \mathrm{Ass}_R(M) \neq \emptyset$. Therefore $\mathrm{Hom}_R(M/N, M) \neq 0$ and hence M is coretractable. □

Following [12], an R-module M is called *quasi-dedekind* if for any nonzero submodule N of M, $\mathrm{Hom}_R(M/N, M) = 0$. In view of Theorem 2.6, if R is a Noetherian ring and $M \neq 0$ is a finitely generated R-module, then M is monoform if and only if M is quasi-dedekind.

Theorem 2.9. *Let M be a nonzero R-module. Then the following statements are hold*:

(1) *If R is a Noetherian ring, then any finitely generated uniform module is primary.*
(2) *M is monoform if and only if it is uniform prime.*

Proof. (1) By [16, Corollary 9.35] and [10, Lemma 3.59] we have $|\mathrm{Ass}_R(M)| = 1$ and so M is primary from [11, Theorem 6.6].

(2) (⇒) Let N be a nonzero submodule of R-module M and $r \in \mathrm{Ann}_R(N)$. Then the map $r : M \longrightarrow M$ by $x \mapsto rx$ is an R-homomorphism and $N \subseteq \mathrm{Ker}\, r$. So, there exists an R-homomorphism $\varphi : M/N \longrightarrow M$ such that $\varphi(x+N) = rx$. From Proposition 2.1, $\varphi = 0$ and this results that $r \in \mathrm{Ann}_R(M)$.

(⇐) Let N be a nonzero submodule of R-module M. Let $x, y \in M$ with $x \neq 0$. First suppose that $y = 0$. Since M is uniform, there exists $r \in R$ such that $0 \neq rx \in N$ and $0 = ry \in N$. Now let $y \neq 0$. Then there exists $s \in R$ such that $0 \neq sy \in N$ and $sx \neq 0$, since M is prime. So, $N \leq_d M$. $\square$

The following lemma is used at several places in this paper.

Lemma 2.10. *Let N be a (nonzero) submodule of R-module M such that and for any $0 \neq x \in M$, $\operatorname{Ann}_R(M/N) \nsubseteq \operatorname{Ann}_R(x)$. Then $N \leq_d M$.*

Proof. Let $x, y \in M$ with $x \neq 0$. Then $\operatorname{Ann}_R(M/N) \nsubseteq \operatorname{Ann}(x)$. So, there exists $r \in R$ such that $rM \subseteq N$ and $rx \neq 0$. It follows that $ry \in N$. Then $N \leq_d M$. $\square$

Let N be a submodule of an R-module M and $\mathfrak{a}$ an ideal of R. The *residual submodule* of N by $\mathfrak{a}$ is

$$(N :_M \mathfrak{a}) = \{x \in M : x\mathfrak{a} \subseteq N\}.$$

It is a submodule of M containing N (see [16, Definition 6.20]).

Theorem 2.11. *Let M be an R-module. If $\mathfrak{a}$ is an ideal of R such that $(0 :_M \mathfrak{a}) = 0$, then $\mathfrak{a}M$ is a dense submodule of M.*

Proof. Let $\operatorname{Ann}_R(M/\mathfrak{a}M) \subseteq \operatorname{Ann}_R(x)$ for some $0 \neq x \in M$. Then $\mathfrak{a}x = 0$, a contradiction. So, $\mathfrak{a}M$ is a dense submodule of M by the above lemma. $\square$

In the following theorem, we characterize the finitely generated monoform modules over Noetherian rings.

Theorem 2.12. *Let M be a prime R-module and for any nonzero submodule N of M, $\operatorname{Ann}_R(M/N) \neq \operatorname{Ann}_R(M)$. Then M is a monoform R-module. The converse is true if R is a Noetherian ring and M is a finitely generated R-module.*

Proof. Let N be a nonzero submodule of M and $0 \neq x \in M$. Then $\operatorname{Ann}_R(M/N) \nsubseteq \operatorname{Ann}_R M = \operatorname{Ann}_R(x)$. Therefore $N \leq_d M$ by Lemma 2.10, and hence M is a monoform R-module.

Conversely, let R be a Noetherian ring and M a finitely generated monoform R-module. By Theorem 2.9(2), M is prime. Suppose on the contrary that there is a submodule N of M such that $\operatorname{Ann}_R(M/N) = \operatorname{Ann}_R(M)$. Then $\operatorname{Ann}_R(M/N) = \operatorname{Ann}_R(x)$ for all $0 \neq x \in M$. It follows that $\operatorname{Ann}_R(M/N) \in \operatorname{Supp}_R(M/N) \cap \operatorname{Ass}_R(M) \neq \emptyset$, which is a contradiction to Theorem 2.6. This completes the proof. $\square$

In the following theorem, we prove that any strongly prime R-module is a monoform module.

Theorem 2.13. *Let M be a strongly prime R-module. Then M is a monoform module. The converse is true if every finitely generated monoform R-module over a Noetherian ring R is strongly prime.*

Proof. From Remark 2.5, it is enough to prove that any nonzero cyclic submodule of M is a dense submodule. Let $0 \neq x \in M$ and $\mathrm{Ann}_R(M/Rx) \subseteq \mathrm{Ann}_R(y)$ for some $0 \neq y \in M$. Then, for any $r \in R$ if $rM \subseteq Rx$, then $ry = 0$. It follows that $(Rx :_R M)y = 0$ and hence $x = 0$ or $y = 0$, a contradiction. Now, the assertion follows by Lemma 2.10.

Conversely, let R be a Noetherian ring and $M \neq 0$ be a finitely generated monoform R-module. By Theorem 2.9, M is prime. Let $x, y \in M$ and $(Rx :_R M)(Ry :_R M)M = 0$. Then $(Rx :_R M)(Ry :_R M) \subseteq \mathrm{Ann}_R(M)$. It results that $(Rx :_R M) \subseteq \mathrm{Ann}_R(M)$ or $(Ry :_R M) \subseteq \mathrm{Ann}_R(M)$. From Theorem 2.12, we conclude that $x = 0$ or $y = 0$. Now, [14, Lemma 2.2] completes the proof. □

Proposition 2.14. *Let M be an R-module. Then the following statements are hold*:

(1) *If $\mathfrak{a}$ is a dense ideal of R and M is a free R-module, then $\mathfrak{a}M$ is a dense submodule of M.*
(2) *If M is a free R-module of finite rank and $\mathfrak{a}$ is a nonzero ideal of R such that $\mathfrak{a}M$ is a dense submodule of M, then $\mathfrak{a}$ is a dense ideal of R.*

Proof. (1) Let $0 \neq x \in M$ and $\mathrm{Ann}_R(M/\mathfrak{a}M) \subseteq \mathrm{Ann}_R(x)$. Then $\mathfrak{a}x = 0$. There exist $r_1, \ldots, r_k$ in R and $e_1, \ldots, e_k$ in a base of M such that $x = r_1e_1 + \cdots + r_ke_k$. So, $r_1\mathfrak{a} = \cdots = r_k\mathfrak{a} = 0$. It follows that $r_1 = \cdots = r_k = 0$, which is a contradiction. Therefore $\mathfrak{a}M$ is a dense submodule of M by Lemma 2.10.

(2) Let $M = \oplus_{i=1}^{k} R$ be a free R-module and let $\mathfrak{a}$ be a nonzero ideal of R such that $\mathfrak{a}M$ a dense submodule of M. We have $\mathrm{Hom}_R(M/\mathfrak{a}M, E_R(M)) \cong \mathrm{Hom}_R(R/\mathfrak{a} \otimes_R (\oplus_{i=1}^{k} R), \oplus_{i=1}^{k} E_R(R)) = 0$. Then $\mathrm{Hom}_R(\oplus_{i=1}^{k} R/\mathfrak{a}, \oplus_{i=1}^{k} E_R(R))$ $= 0$. It follows that $\oplus_{i=1}^{k} \mathrm{Hom}_R(R/\mathfrak{a}, \oplus_{i=1}^{k} E_R(R)) = 0$. Hence,

$$\mathrm{Hom}_R(R/\mathfrak{a}, \oplus_{i=1}^{k} E_R(R)) = 0.$$

Now, from the exact sequence $0 \longrightarrow E_R(R) \longrightarrow \oplus_{i=1}^{k} E_R(R)$ we deduce that $\mathrm{Hom}_R(R/\mathfrak{a}, E_R(R)) = 0$. So, Theorem 2.6 completes the proof. □

Corollary 2.15. *The ring R is an integral domain if and only if there exists a free monoform R-module of finite rank.*

Proof. ($\Rightarrow$) By [10, Corollary 8.4(3)], R is a monoform R-module.

($\Leftarrow$) It is enough to show that R is monoform R-module. Let $0 \neq \mathfrak{a}$ be an ideal of R and M a free monoform R-module of finite rank. Then $\mathfrak{a}M$ is a dense submodule of M and so $\mathfrak{a}$ is a dense ideal of R by Proposition 2.14. □

Following [18], an R-module M is called *essentially monoform* if for any essential submodule N of M,

$$\mathrm{Hom}_R(M/N, E_R(M)) = 0.$$

In the following theorem, we prove if M is an R-module and $E_R(M)$ is a prime R-module, then M is essentially monoform.

Theorem 2.16. *Let M be an R-module. Then the following statements are hold*:

(1) *M is essentially monoform if and only if for any nonzero homomorphism $f : M \longrightarrow E_R(M)$, $\operatorname{Ker} f \nleq_e M$.*
(2) *If $E_R(M)$ is a prime R-module, then M is essentially monoform.*

Proof. (1) ($\Rightarrow$) Let $f : M \longrightarrow E_R(M)$ be a nonzero homomorphism and $\operatorname{Ker} f \leq_e M$. Then $g : M/\operatorname{Ker} f \longrightarrow E_R(M)$, by $g(m + \operatorname{Ker} f) = f(m)$ is a nonzero homomorphism. So, $\operatorname{Hom}_R(M/\operatorname{Ker} f, E_R(M)) \neq 0$, that is a contradiction.

($\Leftarrow$) Let there exists $N \leq_e M$ such that $\operatorname{Hom}_R(M/N, E_R(M)) \neq 0$. Let $0 \neq h \in \operatorname{Hom}_R(M/N, E_R(M))$ and $\pi : M \longrightarrow M/N$ be the canonical epimorphism. Then $\phi : M \xrightarrow{\pi} M/N \xrightarrow{h} E_R(M)$ is a nonzero homomorphism and since $N \subseteq \operatorname{Ker} \phi$ we conclude that $\operatorname{Ker} \phi \leq_e M$, a contradiction.

(2) Let $E_R(M)$ be a prime R-module and let $f : M \longrightarrow E_R(M)$ be a monomorphism. In view of part (1), it is enough to show that $\operatorname{Ker} f \nleq_e M$. Suppose on the contrary that $\operatorname{Ker} f \leq_e M$. Since f is nonzero, there exists $0 \neq x \in M$ such that $f(x) \neq 0$. So, there exists $r \in R$ such that $0 \neq rx \in \operatorname{Ker} f$. Since $E_R(M)$ is a prime R-module, we have $r \in \operatorname{Ann}_R(f(x)) = \operatorname{Ann}_R(E_R(M))$. Then $rx = 0$, a contradiction. $\square$

3. Dense submodules of multiplication modules

Let M be an R-module. Then M is called a *multiplication module* if for each submodule N of M, $N = \mathfrak{a}M$ for some ideal $\mathfrak{a}$ of R. If M is a multiplication module, for each submodule N of M, $N = (N :_R M)M$. It is easy to see that any cyclic R-module is multiplication. Also, if M is a multiplication module and N a submodule of M, then M/N is a multiplication module (for more information about multiplication modules see for example [5] and [17]).

[10, Corollary 8.4(3)] shows that the ring R is an integral domain if and only if R is monoform. So, it is natural to ask about dense submodules of multiplication modules. In fact, we prove that any faithful multiplication module over an integral domain is monoform (see Corollary 3.4).

Theorem 3.1. *Let M be a multiplication R-module. Then the following are equivalent*:

(1) *M is monoform.*
(2) *M is uniform prime.*
(3) *M is prime.*
(4) *M is strongly prime.*

Proof. (1)$\Rightarrow$(2) Follows from Theorem 2.9(2).
(2)$\Rightarrow$(3) Is clear.
(3)$\Rightarrow$(4) Follows from [14, Proposition 2.4].
(4)$\Rightarrow$(1) Follows from Theorem 2.13. $\square$

Corollary 3.2. *Let G be a cyclic group. Then G as a $\mathbb{Z}$-module is monoform if and only if either G is infinite or $|G| = p$ for some prime p.*

Proof. ($\Leftarrow$) Let $G = \langle a \rangle$ be infinite and H be a nonzero subgroup of G. There exists positive integer m such that $H = \langle ma \rangle$. If $x, y \in G$ and $x \neq 0$, then $x = sa, y = ta$ for some integer t and nonzero integer s. Therefore, $msa \neq 0$ and $mta \in H$. So, $H \leq_d G$. Now, let $|G| = p$ for some prime p. Then G is simple and so is monoform.

($\Rightarrow$) Let G be a finite monoform $\mathbb{Z}$-module. Then $G \cong \mathbb{Z}_n \cong \mathbb{Z}/n\mathbb{Z}$ (as $\mathbb{Z}$-modules) for some n. Now, by Theorem 3.1, n is prime. □

Theorem 3.3. *Let M be a faithful multiplication R-module and $\mathfrak{a}$ be an ideal of R. Then $\mathfrak{a}$ is a dense ideal of R if and only if $\mathfrak{a}M$ is a dense submodule of M.*

Proof. Let $\mathfrak{a}$ be a dense ideal of R. By [10, Examples 8.3(4)], $(0 :_R \mathfrak{a}) = 0$. So, in view of [1, Lemma 2.1(1)], we have $(0 :_M \mathfrak{a}) = (0 :_R \mathfrak{a})M = 0$. Now Theorem 2.11 results that $\mathfrak{a}M$ is a dense submodule of M.

Conversely, assume that $\mathfrak{a}M$ is a dense submodule of M. In view of [10, Examples 8.3(4)], it is enough to show that $\mathrm{Ann}_R(\mathfrak{a}) = 0$. Suppose on the contrary that $0 \neq r \in \mathrm{Ann}_R(\mathfrak{a})$. Since M is faithful, we have $rM \neq 0$. Let $x \in M$ and $rx \neq 0$. Since $\mathfrak{a}M$ is dense, there exists $t \in R$ such that $t(rx) \neq 0$ and $tx \in \mathfrak{a}M$. We have $tx = t_1x_1 + \cdots + t_kx_k$ for some $t_1, \ldots, t_k$ in $\mathfrak{a}$ and $x_1, \ldots, x_k$ in M. It follows that $t(rx) = 0$, a contradiction. □

Corollary 3.4. *Let M be a faithful multiplication R-module. Then the following are equivalent*:

(1) *R is an integral domain.*
(2) *R is monoform.*
(3) *M is monoform.*
(4) *M is prime.*
(5) *M is uniform.*
(6) *M is strongly prime.*

Also, in the case R is Noetherian, the above conditions are equivalent to

(7) *M is prime and $(N :_R M) \neq 0$ for any nonzero submodule N of M.*

Proof. By [5, Theorem 3.4] any faithful multiplication R-module over an integral domain is finitely generated. So, the result follows by [10, Corollary 4.8(3)] and Theorems 2.12, 3.1 and 3.3. □

Let M be an R-module and $x \in R$. For convenience, we simply denote $(0 :_R Rx)$ and $(0 :_M Rx)$ by $(0 :_R x)$ and $(0 :_M x)$, respectively.

Lemma 3.5. *Let M be a finitely generated faithful multiplication R-module, let $\mathfrak{a}$ be an ideal of R, and let $x \in R$. Then the following statements are hold*:

(1) *$\mathfrak{a} = (0 :_R x)$ if and only if $\mathfrak{a}M = (0 :_M x)$.*
(2) *$(0 :_R \mathfrak{a}) = 0$ if and only if $(0 :_M \mathfrak{a}) = 0$.*

(3) $(0:_R (0:_R \mathfrak{a})) = \mathfrak{a}$ *if and only if* $(0:_M (0:_R \mathfrak{a})) = \mathfrak{a}M$.

Proof. In view of [1, Lemma 2.1(1)], we have $(0:_M \mathfrak{a}) = (0:_R \mathfrak{a})M$, so the proof follows by [5, Theorem 3.1]. □

The ring R is called *Kasch ring* if every simple R-module can be embedded in R. Kasch rings are named in honour of Friedrich Kasch. See [10] for more information about Kasch ring. Now we are in position to give a characterization of the finitely generated faithful module which has no proper sense submodule.

Theorem 3.6. *Let M be a finitely generated faithful multiplication R-module. Then the following are equivalent*:

(1) *R is a Kasch ring.*
(2) *Any maximal submodule of M has the form $(0:_M x)$ for some $x \in R$.*
(3) *For any maximal submodule N of M, $(0:_R N) \neq 0$.*
(4) *For any maximal ideal $\mathfrak{m}$ of R, $\mathfrak{m}M = (0:_M (0:_R \mathfrak{m}))$.*
(5) *M has no proper dense submodule.*
(6) *For any submodule $N \subsetneqq M$, $(0:_R N) \neq 0$.*

Proof. We note that the maximal submodules of M is the form $\mathfrak{m}M$ for some maximal ideal $\mathfrak{m}$ and any proper submodule is contained in a maximal submodule. Now, we deduce the result by [10, Corollary 8.28] and Lemma 3.5. □

Acknowledgments. The authors are deeply grateful to the referees for careful reading of the manuscript and helpful comments. The second author would like to thank Monash University for hospitality during his sabbatical leave in 2017–18.

References

[1] M. M. Ali, *Residual submodules of multiplication modules*, Beiträge Algebra Geom. **46** (2005), no. 2, 405–422.

[2] B. Amini, M. Ershad, and H. Sharif, *Coretractable modules*, J. Aust. Math. Soc. **86** (2009), no. 3, 289–304.

[3] N. Bourbaki, *Commutative algebra. Chapters 1–7*, translated from the French, reprint of the 1989 English translation, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.

[4] J. Dauns, *Prime modules*, J. Reine Angew. Math. **298** (1978), 156–181.

[5] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra **16** (1988), no. 4, 755–779.

[6] E. H. Feller and E. W. Swokowski, *Prime modules*, Canad. J. Math. **17** (1965), 1041–1052.

[7] G. D. Findlay and J. Lambek, *A generalized ring of quotients. I, II*, Canad. Math. Bull. **1** (1958), 77–85, 155–167.

[8] R. Gordon and J. C. Robson, *Krull dimension*, American Mathematical Society, Providence, RI, 1973.

[9] A. Hajikarimi, *Local cohomology modules which are supported only at finitely many maximal ideals*, J. Korean Math. Soc. **47** (2010), no. 3, 633–643.

[10] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, **189**, Springer-Verlag, New York, 1999.

[11] H. Matsumura, *Commutative Ring Theory*, translated from the Japanese by M. Reid, second edition, Cambridge Studies in Advanced Mathematics, **8**, Cambridge University Press, Cambridge, 1989.

[12] A. S. Mijbass, *Quasi-Dedekind modules*, Ph.D. Thesis, College of Science University of Baghdad, 1997.

[13] A. R. Naghipour, *Strongly prime submodules*, Comm. Algebra **37** (2009), no. 7, 2193–2199.

[14] ———, *Some results on strongly prime submodules*, J. Algebr. Syst. **1** (2013), no. 2, 79–89.

[15] D. E. Rush, *Strongly prime submodules, G-submodules and Jacobson modules*, Comm. Algebra **40** (2012), no. 4, 1363–1368.

[16] R. Y. Sharp, *Steps in commutative algebra*, second edition, London Mathematical Society Student Texts, **51**, Cambridge University Press, Cambridge, 2000.

[17] S. Singh and Y. Al-Shaniafi, *Multiplication modules*, Comm. Algebra **29** (2001), no. 6, 2597–2609.

[18] J. M. Zelmanowitz, *Representation of rings with faithful polyform modules*, Comm. Algebra **14** (1986), no. 6, 1141–1169.

Alireza Hajikarimi
Mobarakeh Branch
Islamic Azad University
Isfahan, Iran
Email address: alihajikarimi@yahoo.com

Ali Reza Naghipour
Department of Mathematical Sciences
Shahrekord University
P.O. Box 115, Shahrekord, Iran
and
Faculty of Information Technology
Monash University
Australia
Email address: naghipour@sci.sku.ac.ir