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ALMOST COHEN-MACAULAYNESS OF KOSZUL HOMOLOGY

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian ring, I an ideal of R and M a non-zero finitely generated R-module. We show that if M and $H_0(I, M)$ are aCM R-modules and $I = (x_1, \ldots, x_{n+1})$ such that $x_1, \ldots, x_n$ is an M-regular sequence, then $H_i(I, M)$ is an aCM R-module for all i. Moreover, we prove that if R and $H_i(I, R)$ are aCM for all i, then $R/(0 : I)$ is aCM. In addition, we prove that if R is aCM and $x_1, \ldots, x_n$ is an aCM d-sequence, then $\operatorname{depth} H_i(x_1, \ldots, x_n; R) \geq i - 1$ for all i.

Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity, I an ideal of R and M a non-zero finitely generated R-module. Let $H_i(I, M)$ denote the ith Koszul homology of the ideal I with respect to some fixed system of generators for I.

The R-module M is called almost Cohen-Macaulay (i.e., aCM) if for every $\mathfrak{p} \in \operatorname{Supp}_R(M)$ $\operatorname{grade}(\mathfrak{p}, M) = \operatorname{grade}(\mathfrak{p}R_{\mathfrak{p}}, M_{\mathfrak{p}})$, and R is called an aCM ring if it is an aCM R-module. It is clear that all CM R-modules are aCM. Several fundamental properties and some characterizations of aCM modules have been proved in [9]. In particular, Kang in [9] proved that if $(R, \mathfrak{m})$ is a local ring, then M is an aCM R-module if and only if $\dim M \leqslant 1 + \operatorname{depth} M$. Moreover, several interesting examples have been given in [10]. After that, several authors studied aCM modules (see for example [2], [8], [12], [13] and [14]).

Huneke in [6] and [7] studied the Cohen-Maculayness of Koszul homology of $H_i(I, R)$. The main aim of this paper is to prove the following:

Theorem 0.1. *Let R be a Noetherian ring and I be an ideal of R.*

(i) *If $I = (x_1, \ldots, x_{n+1})$ such that $x_1, \ldots, x_n$ is an M-regular sequence and $H_0(I, M)$ is an aCM R-module, then $H_i(I, M)$ is an aCM R-module for all $i \geq 0$.*

(ii) *If $(R, \mathfrak{m})$ is an aCM local ring and $H_i(I, R)$ is aCM for all i, then $R/(0 : I)$ is aCM.*

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(iii) *If* $(R,\mathfrak{m})$ *is an aCM local ring and* $x_1,\ldots,x_n$ *is an aCM d-sequence, then* $\operatorname{depth} H_i(x_1,\ldots,x_n;R) \geq i-1$ *for all* $i \geq 0$ *whenever* $H_i(x_1,\ldots, x_n;R) \neq 0$.

For basic definitions and unexplained terminologies, we refer the reader to [1] or [15].

1. The results

We begin this section by the following lemma which is a generalization of [7, Remark 1.5].

Lemma 1.1. *Let* M *be a CM* R*-module and let* $I=(x_1,\ldots,x_n)$ *be an ideal of* R *with* $H_i(I,M)\neq 0$. *Then* $\dim H_i(I,M)=\dim M/IM$.

Proof. It is known that $I+\operatorname{Ann}(M)\subseteq \operatorname{Ann}(H_i(I,M))$. Hence $\dim H_i(I,M)\leq \dim M/IM$. For converse, let $\mathfrak{p}$ be a minimal prime ideal of $\operatorname{Ass}(M/IM)$ and set $\operatorname{grade}_{M_\mathfrak{p}}(IR_\mathfrak{p})=k$. Then $H_{n-k}(IR_\mathfrak{p},M_\mathfrak{p})\cong(\underline{y}M:_M I/\underline{y}M)_\mathfrak{p}$ is the last non-vanishing homology module, where $\underline{y}=y_1,\ldots,y_k$ is an $M_\mathfrak{p}$-regular sequence in $IR_\mathfrak{p}$. This module is a submodule of $(M/\underline{y}M)_\mathfrak{p}$, which is equidimensional and so is all of its submodules. Then by rigidity of the Koszul homology we cannot have any intermediate $H_i(I,M)_\mathfrak{p}$ equal to 0 (see [15, Theorem 5.10]). Thus $\operatorname{Ann}(H_i(I,M))\subseteq\sqrt{I+\operatorname{Ann}(M)}$ and so $\dim M/IM\leq\dim H_i(I,M)$. This completes the proof. ☐

By using the proof of Lemma 1.1, we conclude that $\operatorname{Ann}(H_i(I,M))\subseteq\sqrt{(I+\operatorname{Ann}(M))}$. Vasconcelos, in [15, page 286], wrote that in general we have $\operatorname{Ann}(H_i(I,R))\subseteq\sqrt{I}$. But the following example says that $\operatorname{Ann}(H_i(I,M))$ is not contained in $\sqrt{\operatorname{Ann}(M/IM)}$ in general.

For the computation of all examples we use Macaulay 2 [3].

Example 1.2. Let $R=k[x,y,z,u]$ be a polynomial ring with k be a field. Let $I=(x,yz,yu)$ be an ideal of R. Then $\operatorname{Ann}(H_1(I,M))=(x,y)$, where $M=R/I$.

The following example says that the assumption of Cohen-Maculayness of M in Lemma 1.1 is essential.

Example 1.3. Let $R=k[x,y,z]$ be a polynomial ring with k is a field. Let $I=(x,y)$ and $M=R/(x)\oplus R/(x,y,z)$. Then $H_2(I,M)\neq 0$, $\dim H_2(I,M)=0$ but we have $\dim M/IM=1$.

Corollary 1.4. *Let* $(R,\mathfrak{m})$ *be a local ring and let* N *be a CM* R*-modules with* $\operatorname{grade}_N\operatorname{Ann}(M)=g$. *Then* $\dim\operatorname{Ext}^g(M,N)=\dim N/\operatorname{Ann}(M)N$. *In particular,* $\dim\operatorname{Ext}^g(M,N)=\dim N-g$.

Proof. Set $\operatorname{Ann}(M)=(x_1,\ldots,x_n)$. By [11, Corollary 2.2]

$$\dim\operatorname{Ext}^g(R/\operatorname{Ann}(M),N)=\dim\operatorname{Ext}^g(M,N)$$

and also by [1, Theorem 1.6.16]

$$\dim \operatorname{Ext}^g(R/\operatorname{Ann}(M), N) = \dim H_{n-g}(\operatorname{Ann}(M), N). \tag{1}$$

Thus by Lemma 1.1 and [1, Theorem 2.1.2] the result follows. □

The following result easily follows by the proof of Lemma 1.1 and [13, Definition 2.1].

Corollary 1.5. *Let $(R, \mathfrak{m})$ be a local ring and I be an ideal of R. If $\dim M \leq 1$, then M and $H_i(I, M)$ are aCM for all i.*

Proposition 1.6. *Let M be an aCM R-module and let $I = (x_1, \ldots, x_n)$ be an ideal of R such that $x = x_1$ is an M-regular element. Then $H_i(I, M)$ is aCM for all i if and only if $H_i(\overline{I}, \overline{M})$ is aCM for all i, where $\overline{I} = I/(x)$ and $\overline{M} = M/xM$.*

Proof. From the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we have a long exact sequence

$$\cdots \to H_i(I, M) \xrightarrow{x} H_i(I, M) \to H_i(\overline{I}, \overline{M}) \to H_{i-1}(I, M) \xrightarrow{x} H_{i-1}(I, M) \cdots .$$

Since $xH_i(I, M) = 0 = xH_{i-1}(I, M)$, we have the exact sequence

$$0 \longrightarrow H_i(I, M) \longrightarrow H_i(\overline{I}, \overline{M}) \longrightarrow H_{i-1}(I, M) \longrightarrow 0 \tag{$*$}$$

for all $i \geq 1$. Set $\dim M/IM = d$. If $H_i(I, M)$ are aCM, then $\dim H_i(I, M) - 1 \leq d - 1 \leq \operatorname{depth} H_i(I, M)$ for all i and so by the exact sequence $(*)$, $\dim H_i(\overline{I}, \overline{M}) - 1 \leq \operatorname{depth} H_i(\overline{I}, \overline{M})$ for all $i \geq 1$. For $i = 0$, $H_i(\overline{I}, \overline{M}) = \overline{M}/\overline{IM} \cong M/IM = H_i(I, M)$ is aCM.

Conversely, suppose $\dim H_i(\overline{I}, \overline{M}) \leq \dim \overline{M}/\overline{IM} = d$. Induct on i to show that $\dim H_i(I, M) - 1 \leq \operatorname{depth} H_i(I, M)$. For $i = 0$, $H_i(\overline{I}, \overline{M}) \cong M/IM = H_i(I, M)$, and by assumption this is aCM. Suppose we have shown $H_{i-1}(I, M)$ is aCM. It follows from $(*)$ that

$$\begin{aligned} \operatorname{depth} H_i(I, M) &\geq \min\{\operatorname{depth} H_i(\overline{I}, \overline{M}), \operatorname{depth} H_{i-1}(I, M) + 1\} \\ &\geq \min\{\operatorname{depth} H_i(\overline{I}, \overline{M}), d\} = \operatorname{depth} H_i(\overline{I}, \overline{M}). \end{aligned}$$

Therefore $\operatorname{depth} H_i(I, M) \geq \dim H_i(I, M) - 1$ and so $H_i(I, M)$ is aCM, as required. □

Theorem 1.7. *Let M be an aCM R-module and $I = (x_1, \ldots, x_{n+1})$ be an ideal of R such that $x_1, \ldots, x_n$ is an M-regular sequence. If $H_0(I, M)$ is an aCM R-module, then $H_i(I, M)$ is an aCM R-module for all $i \geq 0$.*

Proof. Since $\operatorname{grade}(I, M) = n$, by [1, Theorem 1.6.17] we have $H_i(I, M) = 0$ for all $i \geq 2$. By assumption $H_0(I, M) = M/IM$ is aCM. Thus it remains to show that $H_1(I, M)$ is aCM. By [1, Theorem 1.6.16] we have $H_1(I, M) \cong \operatorname{Hom}_R(R/I, M/(x_1, \ldots, x_n)M)$. Consider the following exact sequences

$$0 \to M/((x_1, \ldots, x_n)M :_M x_{n+1}) \to M/(x_1, \ldots, x_n)M \to M/IM \to 0, \tag{$\dagger$}$$

and

(‡) $0 \to H_1(I,M) \to M/(x_1,\dots,x_n)M \to M/((x_1,\dots,x_n)M :_M x_{n+1}) \to 0.$

Since M/IM is aCM, we have $\operatorname{depth} M/IM \geq \dim M/IM - 1 \geq \dim M - \operatorname{grade}_M I - 2$, the second inequality follows by [12, Theorem 2.3]. Since M is aCM, we have

$$\begin{aligned}\operatorname{depth} M/(x_1,\dots,x_n)M &= \operatorname{depth} M - n = \operatorname{depth} M - \operatorname{grade}_M I \\ &\geq \dim M - \operatorname{grade}_M I - 1.\end{aligned}$$

Hence the exact sequence (†) yields that

$$\operatorname{depth} M/((x_1,\dots,x_n)M :_M x_{n+1}) \geq \dim M - \operatorname{grade}_M I - 1.$$

Therefore the exact sequence (‡) yields that

$$\begin{aligned}\operatorname{depth} H_1(I,M) &\geq \dim M - \operatorname{grade}_M I - 1 \\ &\geq \dim M/IM - 1 \geq \dim H_1(I,M) - 1.\end{aligned}$$

Hence $H_1(I,M)$ is aCM. □

Theorem 1.8. *Let $(R,\mathfrak{m})$ be an aCM local ring and I be an ideal of R. If $H_i(I,R)$ is aCM for all i, then $R/(0:I)$ is aCM.*

Proof. We can assume that $(0:I) \neq 0$. If $\dim R/I \leq 1$, then by the proof of Lemma 1.1 $\dim H_i(I,R) \leq 1$ and so $\dim R/(0:I) \leq 1$. Thus $R/(0:I)$ is aCM.

Now, we can assume that $\dim R/I \geq 2$ and so there exists a nonzero divisor z on $H_i(I,R)$ and R for all i. The exact sequence

$$0 \longrightarrow R \xrightarrow{z} R \longrightarrow R/zR \longrightarrow 0$$

gives a long exact sequence

$$H_i(I,R) \xrightarrow{z} H_i(I,R) \longrightarrow H_i(I,R/zR) \longrightarrow H_{i-1}(I,R) \xrightarrow{z} H_{i-1}(I,R).$$

Since z is a nonzero divisor on $H_{i-1}(I,R)$ and $H_i(I,R)$, we obtain the exact sequence

$$0 \longrightarrow H_i(I,R) \xrightarrow{z} H_i(I,R) \longrightarrow H_i(I,R/zR) \longrightarrow 0,$$

and so $H_i(I,R)/zH_i(I,R) \cong H_i(I/zR, R/zR)$. Thus it follows that $H_i(I/zR, R/zR)$ are aCM. Set $\overline{I} = I/zI$ and $\overline{R} = R/zR$. We induct on $\dim R/I$ to prove $R/(0:I)$ is aCM. Since $\dim R/I \geq 2$ we choose z as above, $\overline{R}/(0:\overline{I}) \cong R/(z:I)$ is aCM. Let n be the number of generated of I. Since z is not a zero divisor on $H_n(I,R)$ we have $(z:I)/(z) = H_n(\overline{I},\overline{R}) = H_n(I,R)/zH_n(I,R) = (0:I)/z(0:I)$. It follows that $(z:I) = ((0:I),z)$. Since z is not a zero divisor on R, z is not a zero divisor on $R/(0:I)$. As $R/(z:I) = R/((0:I),z)$ is aCM, we conclude that $R/(0:I)$ is aCM, as required. □

Huneke in [4] and [5] defined that a sequence $x_1,\dots,x_n$ in R is a d-sequence which satisfies in the following two conditions:

(i) $x_i \notin (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ for $1 \leq i \leq n$ and

(ii) for all $k \geq i+1$ and all $i \geq 0$, $((x_1,\dots,x_i) : x_{i+1}x_k) = ((x_1,\dots,x_i) : x_k)$.

In the following definition we generalize [7, Definition; page 297].

Definition 1.9. A d-sequence $x_1,\dots,x_n$ is called aCM if the rings $R/((x_1,\dots,x_i) :_R I)$ and $R/(((x_1,\dots,x_i) :_R I) + I)$ are aCM for all $0 \leq i \leq n-1$, where $I = (x_1,\dots,x_n)$.

In the sequel we recall the following example from [7, Example 2.1].

Example 1.10. Let $X = (x_{ij})$ be an r by s matrix ($r \leq s$) of indeterminates over a field k and let I be the ideal in $k[x_{ij}]_{(x_{ij})}$ generated by the t by t minors of X. Set $R = k[x_{ij}]_{(x_{ij})}/I$. Then the images of any row or column of X in R form a CM d-sequence.

Theorem 1.11. *Let $(R,\mathfrak{m})$ be an aCM local ring and $x_1,\dots,x_n$ be an aCM d-sequence. Then* $\operatorname{depth} H_i(x_1,\dots,x_n;R) \geq i-1$ *for all $i \geq 0$ whenever $H_i(x_1,\dots,x_n;R) \neq 0$.*

Proof. Set $I = (x_1,\dots,x_n)$. We proceed by induction on n. Clearly, if $n = 1$, then by [1, Exercise 1.6.31], $H_i(I;R) = 0$ for all $i > 1$ and hence we have nothing to prove. Let $n > 1$ and the assertion holds for all d-sequence of length less than n. We consider two cases.

Case 1: Let $k := \operatorname{grade} I > 0$. By [1, Exercise 1.6.31], $H_i(I;R) = 0$ for all $i > n-k$, and $H_i(I;R) \neq 0$ for all $0 \leq i \leq n-k$. Clearly, if $n = k$ we have nothing to prove. Let $n > k$. Since by [7, Remark 2.6], $x_1,\dots,x_k$ is an R-regular sequence then from [1, Theorem 1.6.16] it follows that $H_{n-k}(I;R) \cong \frac{((x_1,\dots,x_k):_R I)}{(x_1,\dots,x_k)}$. Hence, the exact sequence

$$0 \longrightarrow \frac{((x_1,\dots,x_k) :_R I)}{(x_1,\dots,x_k)} \longrightarrow \frac{R}{(x_1,\dots,x_k)} \longrightarrow \frac{R}{((x_1,\dots,x_k) :_R I)} \longrightarrow 0$$

yields that $\operatorname{depth} H_{n-k}(I;R) \geq \dim R - k - 1$, because by this exact sequence, $\operatorname{depth} \frac{R}{(x_1,\dots,x_k)} = \operatorname{depth} R - k \geq \dim R - k - 1$ and by Definition 1.9 we have $\operatorname{depth} \frac{R}{((x_1,\dots,x_k):_R I)} \geq \dim \frac{R}{((x_1,\dots,x_k):_R I)} - 1$. From [12, Theorem 2.3] it follows that $\operatorname{depth} \frac{R}{((x_1,\dots,x_k):_R I)} \geq \dim R - \operatorname{grade}((x_1,\dots,x_k) :_R I) - 2$. Note that $\operatorname{grade}((x_1,\dots,x_k) :_R I) = k$. Indeed, since $(x_1,\dots,x_k) \subseteq ((x_1,\dots,x_k) :_R I)$ then $\operatorname{grade}((x_1,\dots,x_k) :_R I) \geq k$. Let $\operatorname{grade}((x_1,\dots,x_k) :_R I) > k$. Thus, there exists $\alpha \in ((x_1,\dots,x_k) :_R I)$ such that $\alpha \notin Z_R(R/(x_1,\dots,x_k))$. Now since $\alpha I \subseteq (x_1,\dots,x_k)$ then there exists $x_{k+1} \in I \setminus (x_1,\dots,x_k)$ such that $\alpha x_{k+1} \subseteq (x_1,\dots,x_k)$. Since $\alpha \notin Z_R(R/(x_1,\dots,x_k))$ then $x_{k+1} \in (x_1,\dots,x_k)$. But this is a contradiction with definition of a d-sequence. Hence

$$\operatorname{depth} H_{n-k}(I;R) \geq n-k-1.$$

Now, it remains to show that $\operatorname{depth} H_i(I;R) \geq i-1$ for all $0 \leq i < n-k$. Consider the exact sequence

$$(\sharp) \qquad 0 \longrightarrow H_{n-k}(I;R) \longrightarrow H_{n-k}(\overline{I},\overline{R}) \longrightarrow H_{n-k-1}(I;R) \longrightarrow 0,$$

where "$^{-}$" denotes the canonical homomorphism from R to $R/(x_1)$ and $H_{n-k}(\overline{I}, \overline{R})$ is the Koszul homology of the elements $\overline{0}, \overline{x_2}, \ldots, \overline{x_n}$. Note that by induction hypothesis, for all i we have $\operatorname{depth} H_i(\overline{I}, \overline{R}) \geq i-1$ as $H_i(\overline{I}, \overline{R}) \cong H_i(\overline{x_2}, \ldots, \overline{x_n}; \overline{R}) \oplus H_{i-1}(\overline{x_2}, \ldots, \overline{x_n}; \overline{R})$ (see [7, Remark 1.4]). So, the exact sequence ($\sharp$) yields that $\operatorname{depth} H_{n-k-1}(I; R) \geq n-k-2$. Hence, the exact sequence

$$0 \longrightarrow H_{n-k-1}(I; R) \longrightarrow H_{n-k-1}(\overline{I}, \overline{R}) \longrightarrow H_{n-k-2}(I; R) \longrightarrow 0$$

yields that $\operatorname{depth} H_{n-k-2}(I; R) \geq n-k-3$. Proceeding in this manner we get $\operatorname{depth} H_i(I; R) \geq i-1$ for all $0 \leq i < n-k$, as required.

Case 2: Let $\operatorname{grade} I = 0$. By [7, Lemma 1.1], for all $i \geq 0$ we have the exact sequence

$$(\natural) \qquad 0 \longrightarrow \oplus(0 :_R I) \longrightarrow H_i(I; R) \longrightarrow H_i(\overline{I}, \overline{R}) \longrightarrow 0,$$

where "$^{-}$" denotes the homomorphism from R to $R/(0 : I) = \overline{R}$. By this exact sequence, $\operatorname{depth}(0 :_R I) \geq i-1$, because by Definition 1.9, $R/(0 :_R I)$ is aCM, and hence by [12, Theorem 2.3] we have $\operatorname{depth} R/(0 :_R I) \geq \dim R/(0 :_R I) - 1 \geq \dim R - \operatorname{grade}(0 :_R I) - 2$. Obversely $\operatorname{grade}(0 :_R I) = 0$ and so from the exact sequence $0 \longrightarrow (0 :_R I) \longrightarrow R \longrightarrow R/(0 :_R I) \longrightarrow 0$ we have $\operatorname{depth}(0 :_R I) \geq \dim R - 1$. Consequently, from [7, Remark 2.4] it follows that $\operatorname{depth}(0 :_R I) \geq \dim R - 1 \geq n-1 \geq i-1$. Since $\operatorname{grade} \overline{I} \geq 1$, by using Case 1 and induction on n we have $\operatorname{depth} H_i(\overline{I}, \overline{R}) \geq i-1$. Now, the exact sequence ($\natural$) yields that $\operatorname{depth} H_i(I; R) \geq i-1$ for all $0 \leq i \leq n$. □

The following examples show that all almost Cohen-Macaulay R-modules are not necessarily Cohen-Macaulay R-module.

Example 1.12. (i) Let k be a field. Set $R := k[\![y]\!]$ and $M := k[\![x, y]\!]/(x^2, xy)$. Then M is a finitely generated R-module as the set $\{\overline{1}, \overline{x}\}$ generates M, where "$^{-}$" denotes the canonical homomorphism $R[\![x]\!] \to M$. So, we have $\dim M = \dim R/\operatorname{Ann}_R M = 1$ as $\operatorname{Ann}_R M = 0$. Clearly, $(y) \subseteq Z_R(M)$, hence $\operatorname{depth} M = 0$. Therefore, by [14, Lemma 1.2], it follows that M is an almost Cohen-Macaulay R-module, however, it is not Cohen-Macaulay R-module.

(ii) Let k be a field. Set $R := k[\![x, y, z]\!]$, and $M := \mathbb{K}[\![x, y, z]\!]/(x, y) \cap (x, y, z)^3$. Clearly, $\dim R = 3$, $\dim M = 1$ and $\operatorname{depth} M = 0$. Thus, by [14, Lemma 1.2], M is an almost Cohen-Macaulay R-module but it is not Cohen-Macaulay.

(iii) All finitely generated R-modules with $\dim M \leq 1$ are almost Cohen-Macaulay.

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