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ON THE GROWTH OF SOLUTIONS TO LINEAR COMPLEX DIFFERENTIAL EQUATIONS ON ANNULI

Nan Wu and Zuxing Xuan

Abstract. In this note, we consider the growth of solutions to second order and higher order linear complex differential equations on annuli instead of the complex plane. We establish several theorems that are analogues of the results in the complex plane.

1. Introduction

We assume that the readers are familiar with the fundamental results and standard notations of the Nevanlinna theory in the unit disk $\Delta = \{z : |z| < 1\}$ and in the complex plane $\mathbb{C}$ (see [1,3,11]), such as $T(r, f), N(r, f), m(r, f)$. The growth of solutions to linear complex differential equations in the complex plane $\mathbb{C}$ is an important subject in the value distribution theory (see [8]). According to a result of Gol'dberg ([2]), if f is any entire function with zeros of multiplicity at most $n - 1$, then there exists a linear differential equation of order n with entire coefficients to which f is a solution. Any growth of solutions is possible if one does not care about the growth of the coefficients. Heitiokangas studied the growth of solutions to linear complex differential equations in the unit disk Δ ([4]). The author once studied the growth order of solutions to linear differential equations in an angular domain $\Omega = \{z : \alpha < \arg z < \beta\}(0 < \beta - \alpha < 2\pi)$ of the complex plane ([9]) and in a sector $\omega = \{z : \alpha < \arg z < \beta, |z| < 1\}(0 < \beta - \alpha < 2\pi)$ of the unit disk ([10]). However, the domains such as $\mathbb{C}, \Delta, \Omega, \omega$ are all simply connected domains. Here we pose the following question.

Question. *How does the solutions to linear complex differential equations grow in doubly connected domains of complex plane* $\mathbb{C}$*?*

By the Doubly Connected Mapping Theorem, each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R,\ 0 \le r < R \le +\infty\}$. We consider only two cases: $r = 0$, $R = +\infty$ simultaneously and $0 < r < R < \infty$. In the latter case the homothety $z \to \frac{z}{\sqrt{rR}}$ reduces the given domain to the

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annulus $\frac{1}{R_0} < |z| < R_0$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \to \frac{1}{z}$.

Recently, Khrystiyanyn and Kondratyuk [5,6] have proposed the Nevanlinna theory for meromorphic functions on annuli. Readers can also refer to an important paper [7]. We will show the basic notions of the Nevanlinna theory on annuli in the next section. It is interesting to consider growth of solutions to linear differential equations on annuli. The main purpose of this paper is to deal with this subject. We shall prove several genera1 theorems on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R < R_0 \leq +\infty$.

2. Basic notions in the Nevanlinna theory on annuli and some lemmas

Let f be a meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. We recall the classical notations of Nevanlinna theory as follows:

$$N(r,f) = \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$
$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad T(r,f) = N(r,f) + m(r,f),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t,f)$ is the counting function of poles of function f in $\{z : |z| \leq t\}$. Here we give the notations of the Nevanlinna theory on annuli. Let

$$N_1(r,f) = \int_{\frac{1}{r}}^1 \frac{n_1(t,f)}{t} dt, \quad N_2(r,f) = \int_1^r \frac{n_2(t,f)}{t} dt,$$
$$m_0(r,f) = m(r,f) + m\left(\frac{1}{r}, f\right) - 2m(1,f),$$
$$N_0(r,f) = N_1(r,f) + N_2(r,f),$$

where $n_1(t,f)$ and $n_2(t,f)$ are the counting functions of poles of function f in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. Set

$$\begin{aligned}\overline{N_0}(r, \frac{1}{f-a}) &= \overline{N_1}(r, \frac{1}{f-a}) + \overline{N_2}(r, \frac{1}{f-a}) \\ &= \int_{\frac{1}{r}}^1 \frac{\overline{n}_1(t, \frac{1}{f-a})}{t} dt + \int_1^r \frac{\overline{n}_2(t, \frac{1}{f-a})}{t} dt,\end{aligned}$$

in which each zero of the function $f - a$ is counted only once. The Nevanlinna characteristic of f on the annulus $\mathbb{A}$ is defined by

$$T_0(r,f) = m_0(r,f) + N_0(r,f).$$

Throughout, we denote by $S(r,*)$ quantities satisfying the following cases:

(i) In the case $R_0 = \infty$,

$$S(r,*) = O(\log(rT_0(r,*)))$$

for $r \in (1, +\infty)$ except for the set $\triangle_r$ such that $\int_{\triangle_r} r^{\lambda-1} dr < +\infty (\lambda \geq 0)$;

(ii) If $R_0 < \infty$, then

$$S(r,*) = O\left(\log\left(\frac{T_0(r,*)}{R_0 - r}\right)\right)$$

for $r \in (1, R_0)$ except for the set $\triangle'_r$ such that $\int_{\triangle'_r} \frac{dr}{(R_0-r)^{\lambda+1}} < +\infty (\lambda \geq 0)$;

Thus for an admissible meromorphic function on the annulus $\mathbb{A}$, $S(r,f) = o(T_0(r,f))$ holds for all $1 \leq r < R_0$ except for the set $\triangle_r$ or the above mentioned set $\triangle'_r$, respectively.

Lemma 2.1 ([5, 7])**.** *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then*

(i) $T_0(r,f) = T_0\left(r, \frac{1}{f}\right)$,

(ii) $$\max\left\{T_0(r, f_1 \cdot f_2), T_0\left(r, \frac{f_1}{f_2}\right), T_0(r, f_1 + f_2)\right\} \leq T_0(r,f_1) + T_0(r,f_2) + O(1).$$

According to Lemma 2.1, the first fundamental theorem on the annulus $\mathbb{A}$ is immediately obtained.

Lemma 2.2 ([5, 7] The first fundamental theorem)**.** *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then*

$$T_0\left(r, \frac{1}{f-a}\right) = T_0(r,f) + O(1)$$

for every fixed $a \in \mathbb{C}$.

Lemma 2.3 ([6, 7] The lemma of the logarithmic derivative)**.** *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then for every $k \in \mathbb{N}$*

$$m_0\left(r, \frac{f^{(k)}}{f}\right) = S(r,f).$$

By a simple deduction, we can establish the following lemma.

Lemma 2.4 (The revised lemma of the logarithmic derivative)**.** *Let f be a nonconstant meromorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 \leq r < R_0 \leq +\infty$. Then*

$$m_0\left(r, \frac{f^{(k)}}{f}\right)$$

$$
= \begin{cases} O(\log r), & R_0 = +\infty \text{ and } \rho(f) < +\infty, \\ O\left(\log \frac{1}{R_0 - r}\right), & R_0 < +\infty \text{ and } \rho(f) < +\infty, \\ O\left(\log r + \log T_0(r,f)\right), r \notin \Delta_r & R_0 = +\infty \text{ and } \rho(f) = +\infty, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right), r \notin \Delta_r' & R_0 < +\infty \text{ and } \rho(f) = +\infty. \end{cases}
$$

Next we give two lemmas concerning Pólya peaks of real functions.

Lemma 2.5 ([12])**.** *Let $T(r)$ be a real, increasing and non-negative function defined in $(0,\infty)$ with lower order*

$$\mu = \liminf_{r\to\infty} \log T(r)/\log r < \infty$$

and order

$$0 < \lambda = \limsup_{r\to\infty} \log T(r)/\log r \le \infty.$$

Then for any positive number $\mu \le \sigma \le \lambda$ and a set $\Delta_r \subset (0,\infty)$ with $\int_{\Delta_r} r^{-1}dr < \infty$, there exist five sequences $\{r_n\}$, $\{r_n'\}$, $\{r_n''\}$, $\{\varepsilon_n\}$ and $\{\varepsilon_n'\}$ such that

(1) $r_n \notin \Delta_r$, $\lim_{n\to\infty} \frac{r_n}{r_n'} = \infty$, $\lim_{n\to\infty} \frac{r_n''}{r_n} = \infty$, $\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \varepsilon_n' = 0$;
(2) $\liminf_{n\to\infty} \frac{\log T(r_n)}{\log r_n} \ge \sigma$;
(3) $T(t) < (1+\varepsilon_n)(\frac{t}{r_n})^{\sigma} T(r_n), t \in [r_n', r_n'']$;
(4) $T(t) \le KT(r_n)(\frac{t}{r_n})^{\sigma-\varepsilon_n'}, 1 \le t \le r_n''$ *and for a positive constant K.*

By a transformation $t = R_0 - r^{-1}$, we can establish the following lemma.

Lemma 2.6. *Let $T(r)$ be a real, increasing and non-negative function defined in $(0,R_0)$ with lower order*

$$\mu = \liminf_{r\to R_0-} \frac{\log T(r)}{\log \frac{1}{R_0-r}} < \infty$$

and order

$$0 < \lambda = \limsup_{r\to\infty} \frac{\log T(r)}{\log \frac{1}{R_0-r}} \le \infty.$$

Then for any positive number $\mu \le \beta \le \lambda$ and a set $\Delta_r \subset (0,R_0)$ with $\int_{\Delta_r} \frac{dr}{R_0-r} < \infty$, there exist five sequences $\{r_n\}$, $\{r_n'\}$, $\{r_n''\}$, $\{\varepsilon_n\}$ and $\{\varepsilon_n'\}$ such that

(1) $r_n \notin \Delta_r$, $0 < r_n' < r_n < r_n'' < R_0$, $r_n' \to R_0-$, $\frac{R_0-r_n'}{R_0-r_n} \to \infty$, $\frac{R_0-r_n}{R_0-r_n''} \to \infty$, $\varepsilon_n \to 0$, $\varepsilon_n' \to 0 (n\to\infty)$;
(2) $\liminf_{n\to\infty} \frac{\log T(r_n)}{-\log(R_0-r_n)} \ge \beta$;
(3) $T(t) < (1+\varepsilon_n)(\frac{R_0-r_n}{R_0-t})^{\beta} T(r_n)$, $t \in [r_n', r_n'']$;
(4) $T(t) \le KT(r_n)(\frac{R_0-r_n}{R_0-t})^{\beta-\varepsilon_n'}$, $0 < t \le r_n''$ *and for a positive constant K.*

3. Results

Before stating the results, we give the definition of order of a mermorphic function on annuli.

Definition 3.1. Let $f(z)$ be a nonconstant mermorphic function on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}$, where $1 < R_0 \leq +\infty$. The function f is called a transcendental or admissible meromorphic function on the annulus $\mathbb{A}$ provided that

$$\limsup_{r\to\infty} \frac{T_0(r,f)}{\log r} = +\infty, \quad 1 < r < R_0 = +\infty$$

or

$$\limsup_{r\to R_0-} \frac{T_0(r,f)}{-\log(R_0 - r)} = +\infty, \quad 1 < r < R_0 < \infty,$$

respectively. The order is defined as

$$\rho_{\mathbb{A}}(f) = \limsup_{r\to\infty} \frac{\log T_0(r,f)}{\log r}, \quad 1 < r < R_0 = +\infty$$

or

$$\rho_{\mathbb{A}}(f) = \limsup_{r\to R_0-} \frac{\log T_0(r,f)}{-\log(R_0 - r)}, \quad 1 < r < R_0 < \infty,$$

respectively. The hyper-order is defined as

$$\rho^2_{\mathbb{A}}(f) = \limsup_{r\to\infty} \frac{\log\log T_0(r,f)}{\log r}, \quad 1 < r < R_0 = +\infty$$

or

$$\rho^2_{\mathbb{A}}(f) = \limsup_{r\to R_0-} \frac{\log\log T_0(r,f)}{-\log(R_0 - r)}, \quad 1 < r < R_0 < \infty,$$

respectively.

Now we are in position to state our results, which are analogues of the results in the complex plane.

Theorem 3.1. *Let $A(z)$ be an admissible (or transcendental) analytic coefficient of*

$$f^{(k)} + A(z)f = 0 \tag{3.1}$$

on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}(1 < R_0 \leq +\infty)$. Then all solutions $f \not\equiv 0$ of (3.1) are of infinite order of growth, i.e., $\rho_{\mathbb{A}}(f) = +\infty$.

Theorem 3.2. *Let $A_i(z)(i = 0, 1, \ldots, k-1)$ be the analytic coefficients of*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0 \tag{3.2}$$

on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}(1 < R_0 \leq +\infty)$. If either

(1) $\max_{1\leq j\leq k-1}\{\rho_{\mathbb{A}}(A_j)\} < \rho_{\mathbb{A}}(A_0)$ *or*

(2) $A_j(z)(j = 1, 2, \ldots, k-1)$ *are non-admissible (or non-transcendental) while $A_0(z)$ is admissible (or transcendental) on $\mathbb{A}$,*

then all solutions $f \not\equiv 0$ *of* (3.2) *are of infinite order of growth, i.e.,* $\rho_{\mathbb{A}}(f) = +\infty$.

Theorem 3.3. *Let* $B(z)$ *and* $C(z)$ *be the analytic coefficients of*

$$f'' + B(z)f' + C(z)f = 0 \tag{3.3}$$

on the annulus $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}(1 < R_0 \leq +\infty)$ *satisfying* $\rho_{\mathbb{A}}(C) < \rho_{\mathbb{A}}(B)$. *Then every solution* $f \not\equiv 0$ *of finite order of* (3.3) *satisfies* $\rho_{\mathbb{A}}(f) \geq \rho_{\mathbb{A}}(B)$.

Theorem 3.4. *Let* $A_i(z)(i = 0, 1, \ldots, k-1)$ *be analytic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}(1 < R_0 \leq +\infty)$ *that satisfy* $\max\limits_{1\leq j\leq k-1}\{\rho_{\mathbb{A}}(A_j)\} < \rho_{\mathbb{A}}(A_0)$. *Then all solutions* $f \not\equiv 0$ *of* (3.2) *satisfy* $\rho^2_{\mathbb{A}}(f) \geq \rho_{\mathbb{A}}(A_0)$.

Theorem 3.5. *Let* $A_i(z)(i = 0, 1, \ldots, k-1)$ *be mermorphic functions on the annulus* $\mathbb{A} = \{z : \frac{1}{R_0} < |z| < R_0\}(1 < R_0 \leq +\infty)$ *that satisfy*

$$\max_{0\leq j\leq k-1, j\neq l}\{\rho_{\mathbb{A}}(A_j)\} < \rho_{\mathbb{A}}(A_l)(l = 1, 2, \ldots, k-1).$$

Then all solutions $f \not\equiv 0$ *of* (3.2) *satisfy* $\rho_{\mathbb{A}}(f) \geq \rho_{\mathbb{A}}(A_l)$.

4. Proof of theorems

Proof of Theorem 3.1. Suppose that $f \not\equiv 0$ is a solution to (3.1) of finite order of growth. Then, from Lemma 2.3, we can obtain

$$T_0(r, A) = m_0(r, A) = m_0\left(r, -\frac{f^{(k)}}{f}\right) = \begin{cases} O(\log r), & R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r}\right), & R_0 < +\infty. \end{cases}$$

However, this implies that $A(z)$ is non-admissible (or non-transcendental), which is a contradiction.

Theorem 3.1 follows. $\square$

Proof of Theorem 3.2. Assume that on the contrary to both cases (1) and (2) that $f \not\equiv 0$ is a solution to (3.2) with $\rho(f) < \infty$, writing (3.2) in a form

$$A_0(z) = -A_1(z)\frac{f'(z)}{f(z)} - \cdots - A_{k-1}\frac{f^{(k-1)}(z)}{f(z)} - \frac{f^{(k)}(z)}{f(z)}$$

and make use of Nevanlinna's fundamental theorem and Lemma 2.4, then we obtain

$$T_0(r, A_0) \leq \sum_{j=1}^{k-1} T_0(r, A_j) + \begin{cases} O(\log r), & R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r}\right), & R_0 < +\infty. \end{cases}$$

The desired contradictions to both cases (1) and (2) now follow easily.

Theorem 3.2 follows. $\square$

Proof of Theorem 3.3. Suppose that $f \not\equiv 0$ is a solution to (3.2) with $\rho(f) < +\infty$. It follows from (3.2) that

$$-B = \frac{f''}{f'} + C\frac{f}{f'}.$$

Hence from Nevanlinna's fundamental theorem, i.e., Lemmas 2.1 and 2.2, we have

$$m_0(r,B) = m_0(r,C) + m_0\left(r, \frac{f}{f'}\right) + \begin{cases} O(\log r), & R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r}\right), & R_0 < +\infty. \end{cases} \tag{4.1}$$

It follows that

$$2T_0(r,f) \geq T_0(r,B) - T_0(r,C) - \begin{cases} O(\log r), & R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r}\right), & R_0 < +\infty. \end{cases}$$

Hence the result of the theorem follows from the fact that $\rho_{\mathbb{A}}(C) < \rho_{\mathbb{A}}(B)$.

The proof is complete. □

Proof of Theorem 3.4. Set $\max\{\rho_{\mathbb{A}}(A_j) : j = 1, 2, \ldots, k-1\} = \rho$, $\rho_{\mathbb{A}}(A_0) = \alpha$. We can rewrite (3.2) as

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}. \tag{4.2}$$

By Lemma 2.1 and Lemma 2.4, the inequality

$$m_0(r,A_0) \leq \sum_{j=1}^{k-1} m_0(r,A_j) + \begin{cases} O(\log r + \log T_0(r,f)), & R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r} + \log T_0(r,f)\right), & R_0 < +\infty, \end{cases} \tag{4.3}$$

holds for all r outside a set $\Delta_r \subset (0, +\infty)$ with a linear measure $\int_{\Delta_r} r^{-1}dr < +\infty$ when $R_0 = +\infty$, and for all r outside a set $\Delta'_r \subset (0, R_0)$ with $\int_{\Delta'_r} \frac{dr}{R_0 - r} < +\infty$ when $R_0 < +\infty$.

Case I. $R_0 = +\infty$.

Since $\rho_{\mathbb{A}}(A_0) = \alpha$, by (2) in Lemma 2.5, there exists a sequence $\{r_n\}$ outside Δ_r such that

$$\liminf_{n\to\infty} \frac{\log m_0(r_n, A_0)}{\log r_n} \geq \alpha.$$

For any given $0 < \varepsilon < (\alpha - \rho)/2$, and for $j = 1, 2, \ldots, k-1$

$$m_0(r_n, A_j) < r_n^{\rho+\varepsilon}, \quad m_0(r_n, A_0) > r_n^{\alpha-\varepsilon} \tag{4.4}$$

hold for sufficiently large r_n. From (4.3) and (4.4), we conclude that for sufficiently r_n,

$$r_n^{\alpha-\varepsilon} < O(r_n^{\rho+\varepsilon}) + O(\log r_n + \log T_0(r_n, f)).$$

Therefore, we have $\rho^2_{\mathbb{A}}(f) \geq \alpha$.

Case II. $R_0 < +\infty$.

Since $\rho_{\mathbb{A}}(A_0)=\alpha$, by (2) in Lemma 2.6, there exists a sequence $\{r_n\}$ outside Δ_r' such that

$$\liminf_{n\to\infty}\frac{\log m_0(r_n,A_0)}{\log\frac{1}{R_0-r_n}}\geq\alpha.$$

For any given $0<\varepsilon<(\alpha-\rho)/2$, and for $j=1,2,\ldots,k-1$

$$m_0(r_n,A_j)<\left(\frac{1}{R_0-r_n}\right)^{\rho+\varepsilon},\quad m_0(r_n,A_0)>\left(\frac{1}{R_0-r_n}\right)^{\alpha-\varepsilon}\tag{4.5}$$

hold for $r_n\to R_0-$. From (4.3) and (4.5), we get for $r_n\to R_0-$

$$\left(\frac{1}{R_0-r_n}\right)^{\alpha-\varepsilon}<O\left(\left(\frac{1}{R_0-r_n}\right)^{\rho+\varepsilon}\right)+O\left(\log\frac{1}{R_0-r_n}+\log T_0(r_n,f)\right).$$

Therefore, we have $\rho_{\mathbb{A}}^2(f)\geq\alpha$.

The proof is complete. □

Proof of Theorem 3.5. Set $\max\limits_{0\leq j\leq k-1,j\neq l}\{\rho_{\mathbb{A}}(A_j)\}=\rho$, $\rho_{\mathbb{A}}(A_l)=\alpha$. We can rewrite (3.2) as

$$-A_l=\frac{f^{(k)}}{f^{(l)}}+A_{k-1}\frac{f^{(k-1)}}{f^{(l)}}+\cdots+A_0\frac{f}{f^{(l)}}.\tag{4.6}$$

By Nevanlinna theory, i.e., Lemma 2.1 and the inequality $T_0(r,f^{(k)})\leq(k+1)T_0(r,f)+m_0\left(r,\frac{f^{(k)}}{f}\right)$, the following

$$\begin{aligned}T_0(r,A_l)&\leq\sum_{j=0,\neq l}^{k-1}T_0(r,A_j)+\sum_{j=0,\neq l}^{k}T_0\left(r,\frac{f^{(j)}}{f^{(l)}}\right)\\&\leq\sum_{j=0,\neq l}^{k-1}T_0(r,A_j)+O(T(r,f))\\&\quad+\begin{cases}O(\log r+\log T_0(r,f)), & R_0=+\infty,\\ O\left(\log\frac{1}{R_0-r}+\log T_0(r,f)\right), & R_0<+\infty,\end{cases}\end{aligned}\tag{4.7}$$

holds for all r outside a set $\Delta_r\subset(0,+\infty)$ with a linear measure $\int_{\Delta_r}r^{-1}dr<+\infty$ when $R_0=+\infty$ and for all r outside a set $\Delta_r'\subset(0,R_0)$ with $\int_{\Delta_r'}\frac{dr}{R_0-r}<+\infty$ when $R_0<+\infty$.

Case I. $R_0=+\infty$.

Since $\rho_{\mathbb{A}}(A_l)=\alpha$, by (2) in Lemma 2.5, there exists a sequence $\{r_n\}$ outside Δ_r such that

$$\liminf_{n\to\infty}\frac{\log T_0(r_n,A_l)}{\log r_n}\geq\alpha.$$

For any given $0<\varepsilon<(\alpha-\rho)/2$, and for $j\neq l$

$$T_0(r_n,A_j)<r_n^{\rho+\varepsilon},\quad T_0(r_n,A_l)>r_n^{\alpha-\varepsilon}\tag{4.8}$$

hold for sufficiently large r_n. From (4.7) and (4.8), we get for sufficiently r_n,

$$r_n^{\alpha-\varepsilon} < O(r_n^{\rho+\varepsilon}) + O(T_0(r_n, f)) + O(\log r_n + \log T_0(r_n, f)).$$

Therefore, we have $\rho_{\mathbb{A}}(f) \geq \alpha$.

Case II. $R_0 < +\infty$.

Since $\rho_{\mathbb{A}}(A_l) = \alpha$, by (2) in Lemma 2.6, there exists a sequence $\{r_n\}$ outside Δ_r' such that

$$\liminf_{n\to\infty} \frac{\log T_0(r_n, A_l)}{\log \frac{1}{R_0 - r_n}} \geq \alpha.$$

For any given $0 < \varepsilon < (\alpha - \rho)/2$, and for $j \neq l$

$$T_0(r_n, A_j) < \left(\frac{1}{R_0 - r_n}\right)^{\rho+\varepsilon}, \quad T_0(r_n, A_l) > \left(\frac{1}{R_0 - r_n}\right)^{\alpha-\varepsilon} \tag{4.9}$$

hold for $r_n \to R_0-$. From (4.7) and (4.9), we get for $r_n \to R_0-$

$$\begin{aligned}\left(\frac{1}{R_0 - r_n}\right)^{\alpha-\varepsilon} &< O\left(\left(\frac{1}{R_0 - r_n}\right)^{\rho+\varepsilon}\right) + O(T_0(r_n, f)) \\ &\quad + O\left(\log \frac{1}{R_0 - r_n} + \log T_0(r_n, f)\right).\end{aligned}$$

Therefore, we have $\rho_{\mathbb{A}}(f) \geq \alpha$.

The proof is complete. □

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Nan Wu
Department of Mathematics
School of Science
China University of Mining and Technology (Beijing)
Beijing, 100083, P. R. China
Email address: `wunan2007@163.com`

Zuxing Xuan
High-speed Railway Economic Research Institute
Beijing Key Laboratory of Information Service Engineering
Department of General Education
Beijing Union University
No.97 Bei Si Huan Dong Road
Chaoyang District
Beijing, 100101, P. R. China
Email address: `zuxingxuan@163.com`