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CONFORMAL TRANSFORMATION OF LOCALLY DUALLY FLAT FINSLER METRICS

Laya Ghasemnezhad and Bahman Rezaei

Abstract. In this paper, we study conformal transformations between special class of Finsler metrics named **C**-reducible metrics. This class includes Randers metrics in the form $F = \alpha + \beta$ and Kropina metric in the form $F = \frac{\alpha^2}{\beta}$. We prove that every conformal transformation between locally dually flat Randers metrics must be homothetic and also every conformal transformation between locally dually flat Kropina metrics must be homothetic.

1. Introduction

Let F be a Finsler metric on an n-dimensional manifold M. For a non-zero vector $y \in T_xM$, F induces an inner product g_y on T_xM by

$$g_y(u,v) := g_{ij}(x,y)u^iv^j = \frac{1}{2}[F^2]_{y^iy^j}u^iv^j.$$

For two arbitrary non-zero vectors $v, y \in T_xM$ the angle $\theta(v,y)$ between y and v is defined by

$$\cos\theta(y,v) := y_iv^i/F(x,y)\sqrt{g_{ij}(x,y)v^iv^j},$$

where $y_i := g_{ij}(x,y)y^j$. It should be remarked that the notion of angle is not symmetric, in that the angle $\theta(y,v)$ between y and v is different from the angle $\theta(v,y)$ between v and y generally. Now assume that F and $\bar{F}$ are two Finsler metrics on an n-dimensional manifold M. If the angle $\theta(y,v)$ with respect to F is equal to the angle $\bar{\theta}(y,v)$ with respect to $\bar{F}$ for any vectors $v, y \in T_xM\backslash\{0\}$ and any $x \in M$ then F is called conformal to $\bar{F}$. The transformation $F \to \bar{F}$ of the metric is called conformal transformation [1, 2].

In conformal geometry, it is one of interesting issues to study the conformal transformation. In [3], S. Basco and X. Cheng obtained the relations between some geometric quantities of two conformally related Finsler metrics and discussed the properties of those conformal transformations which preserve these

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quantities. Later G. Chen, X. Cheng and Y. Zhu proved that if both conformally related (α, β)-metrics F and $\bar{F}$ are Douglas metrics of non-Randers type, then the conformal transformation must be homothety, and also conformal transformation between two Finsler metrics of isotropic S-curvature must be homothety [4]. In this paper we study the conformal transformation between two special class of Finsler metrics of locally dually flat type. Locally dually flat Finsler metrics are studied in information geometry and naturally arise from the investigation of the flat information structure, this notion is introduced in [12]. X. Cheng and Y. Tian in 2011 found some equations that characterize locally dually flat Randers metrics [8]. By assuming conformally related two locally dually flat Randers metrics we get the following theorem:

Theorem 1.1. *Every conformal transformation between locally dually flat Randers metrics must be homothetic.*

Moreover in 2016 G. Chen and L. Liu studied locally dually flat Kropina metrics [5]. By using the results of this research we prove the following theorem.

Theorem 1.2. *Every conformal transformation between locally dually flat Kropina metrics must be homothetic.*

C-reducible Finsler metrics are a special class of Finsler metrics which was introduced by M. Matsumoto. By Theorems 1.1 and 1.2 we can get this corollary.

Corollary 1.3. *Every conformal transformation between locally dually flat **C**-reducible Finsler metrics must be homothetic.*

2. Preliminaries and notations

Let M be an n-dimensional C^∞ manifold. Denote by T_xM as the tangent space at $x \in M$, and by $TM = \cup_{x\in M} T_xM$ as the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM\backslash\{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber $\pi^*_v TM$ at $v \in TM_0$ is just T_xM, where $\pi(v) = x$. Then

$$\pi^*TM = \{(x, y, v) \,|\, y \in T_xM_0, v \in T_xM\}.$$

A *Finsler metric* on a manifold M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on TM_0; (ii) $F(x, \lambda y) = \lambda F(x, y), \lambda > 0$; and (iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $F^2/2$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^i y^j}$$

is positive definite.

Every Finsler metric F induces a spray $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$ by (see [11])

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\{2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\}y^j y^k. \tag{1}$$

In Finsler geometry there is a quite important class of metrics called (α,β)-metric. An (α,β)-metric is a scalar function F on TM defined by $F := \alpha\phi(\frac{\beta}{\alpha})$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. As we know that the geodesic coefficients G^i of F and geodesic coefficients G^i_α of α are related as follows:

$$G^i = G^i_\alpha + \alpha Q s^i{}_\circ + \alpha^{-1}\Theta\{r_{00} - 2\alpha Q s_\circ\}y^i + \Psi\{r_{00} - 2\alpha Q s_\circ\}b^i, \tag{2}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'},$$
$$\Theta = \frac{\phi\phi' - s(\phi\phi'' - \phi'\phi')}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}},$$
$$\Psi = \frac{\phi''}{2\{(\phi - s\phi') + (b^2 - s^2)\phi''\}}.$$

The Randers metric $F = \alpha + \beta$, the Kropina metric $F = \frac{\alpha^2}{\beta}$, the generalized Kropina metric $F = \alpha^{1-m}\beta^m$ and Matsumoto metric $F = \frac{\alpha^2}{\alpha - \beta}$ are (α,β)-metrics with $\phi(s) = 1 + s$, $\phi(s) = \frac{1}{s}$, $\phi(s) = s^m$ and $\phi = \frac{1}{1-s}$, respectively. Denote the Levi-Civita connection of α by ∇ and define $b_{i|j}$ by $(b_{i|j})\theta^j := db_i - b_j\theta_i{}^j$, where $\theta^i := dx^i$ and $\theta_i{}^j := \Gamma^j_{ik}dx^k$. For a generic (α,β)-metric, we use usually the following notations:

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Furthermore, we denote

$$r^i{}_j := a^{ik}r_{kj}, \quad r_{00} := r_{ij}y^i y^j, \quad r_{i0} := r_{ij}y^j, \quad r := r_{ij}b^i b^j,$$
$$s^i{}_j := a^{ik}s_{kj}, \quad s_j := b^i s_{ij}, \quad s_0 := s_i y^i, \quad s_{i0} := s_{ij}y^j, \quad b^2 := b^i b_i.$$

Randers metric is an important class of Finsler metrics which is a special case of (α,β)-metrics with $\phi(s) = 1 + s$. By (2) the spray coefficients G^i of F and geodesic coefficients G^i_α of α are related as follows:

$$G^i = G^i_\alpha + (\frac{e_{00}}{2F} - s_0)y^i + \alpha s^i{}_0, \tag{3}$$

where $e_{ij} = r_{ij} + b_i s_j + b_j s_i$ and $e_{00} = e_{ij}y^i y^j$ [6].

Another (α,β)-metric that we are interested to study in this paper is Kropina metric. Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on a manifold M, then geodesic coefficients $G^i(x,y)$ are [13]:

$$G^i = G^i_\alpha - \frac{\alpha^2}{2\beta}s^i_0 + \frac{1}{2b^2}(\frac{\alpha^2}{\beta}s_0 + r_{00})b^i - \frac{1}{b^2}(s_0 + \frac{\beta}{\alpha^2}r_{00})y^i. \tag{4}$$

A Finsler metric on a manifold is said to be locally dually flat if at any point there is a local coordinate system in which the spray coefficients of F are in the form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j},$$

where $H = H(x,y)$ is a scalar function on the tangent bundle TM. Such a coordinate system is called an adapted coordinate system. It is easy to see that every locally Minkowskian metric is locally dually flat. But the converse is not true [7]. In [12], Z. Shen proved that a Finsler metric $F(x,y)$ on an open subset $U \subseteq \mathbb{R}^n$ is dually flat if and only if the following PDEs hold:

$$[F^2]_{x^k y^l}y^k - 2[F^2]_{x^l} = 0. \tag{5}$$

In this case, $H = -\frac{1}{6}[F^2]_{x^l}y^l$. Locally dually flat Finsler metrics are studied in Finsler information geometry in [12]. In [7], the authors studied and characterized locally dually flat Randers metrics and obtained the following theorem:

Theorem 2.1 (see [7]). *Let $F = \alpha + \beta$ be a Randers metric on an open subset $U \subseteq \mathbb{R}^n$. Then F is dually flat if and only if in an adapted coordinate system, α and β satisfy*

$$\begin{aligned} G^i_\alpha &= (2\theta + \tau\beta)y^i - \alpha^2(\tau b^i - \theta^i),\\ r_{00} &= 2\theta\beta - 5\tau\beta^2 + (3\tau + 2\tau b^2 - 2b_k\theta^k)\alpha^2,\\ s_{i0} &= \beta\theta_i - \theta b_i, \end{aligned} \tag{6}$$

where $\theta = \theta_k(x)y^k$ is a 1-form on U, $\theta^i := a^{ik}\theta_k$, and $\tau = \tau(x)$ is a scalar function.

The same theorem was studied for Kropina metrics in 2016 as follows.

Theorem 2.2 (see [5]). *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on a manifold M. Then F is dually flat if and only if in an adapted coordinate system, β and α satisfy*

$$G^i_\alpha = \frac{1}{b^2}\Big[\alpha^2\xi^i + (\theta b^2 - \xi)y^i\Big], \tag{7}$$

$$r_{00} = \frac{1}{2b^2}\Big[\beta(\theta b^2 + \xi) - 4(\xi_k b^k)\alpha^2\Big], \tag{8}$$

$$s_{k0} = \frac{1}{4b^2}\Big[b^2(\theta b_k - \beta\theta_k) + 7(\beta\xi_k - b_k\xi)\Big], \tag{9}$$

where $\theta = \theta_i(x)y^i$ and $\xi = \xi_i(x)y^i$ are 1-forms on M and $\xi_i := a^{ij}\xi_j$.

Two Finsler metrics F and $\bar{F}$ on a manifold M are said to be conformally related if there is a scalar function $\sigma(x)$ on M such that $\bar{F} = e^{\sigma(x)}F$. A Finsler metric which is conformally related to a Minkowski metric is called a conformally flat Finsler metric. Let F and $\bar{F}$ be two conformally related Finsler metrics on an n-dimensional manifold M. It follows from (1) that the geodesic coefficients of F and $\bar{F}$ satisfy

$$\bar{G}^i = G^i + \frac{1}{2\bar{F}}\bar{F}_{;k}y^k y^i + \frac{1}{2}\bar{F}\bar{g}^{il}\{\bar{F}_{;k.l}y^k - \bar{F}_{;l}\}.$$

If $\bar{F} = e^{\sigma(x)}F$, then $\bar{F}_{;k} = \sigma_k e^{\sigma(x)}F$, where $\sigma_k := \frac{\partial\sigma}{\partial x^k}$ and $\sigma_0 = \sigma_k y^k$. Consequently

$$\begin{aligned}\bar{G}^i &= G^i + \frac{1}{2}(\sigma_0)y^i + \frac{F}{2}g^{il}\{(\sigma_k y^k)F_{y^l} - \sigma_l F\}\\ &= G^i + (\sigma_0)y^i - \frac{F^2}{2}\sigma^i. \qquad (10)\end{aligned}$$

By a conformal change $\bar{F} = e^{\sigma(x)}F$ various quantities in (α,β)-metrics are changed as follows:

$$\bar{\alpha} = e^{\sigma(x)}\alpha, \quad \bar{\beta} = e^{\sigma(x)}\beta.$$

Let $\bar{\alpha} = \sqrt{\bar{a_{ij}}y^i y^j}$, $\bar{\beta} = \bar{b}_i(x)y^i$. Then $\bar{a}_{ij} = e^{2\sigma(x)}a_{ij}$, $\bar{a}^{ij} = e^{-2\sigma(x)}a^{ij}$, $\bar{b_i} = e^{\sigma(x)}b_i$, $\bar{b^i} = e^{-\sigma(x)}b^i$. Further, we have [6]

$$\tilde{b}_{i||j} = e^{\kappa(x)}\big(b_{i;j} - b_j\kappa_i + f a_{ij}\big), \qquad (11)$$

where $\bar{b}_{i||j}$ denote the covariant derivative of $\bar{b}_i$ with respect to $\bar{\alpha}$ and $f := b^m\sigma_m$. From (11), we get

$$\begin{aligned}\bar{s}_{ij} &= e^{\sigma(x)}\Big[s_{ij} + \frac{1}{2}(b_i\sigma_j - b_j\sigma_i)\Big],\\ \bar{r}_{ij} &= e^{\sigma(x)}\Big[r_{ij} - \frac{1}{2}(b_i\sigma_j + b_j\sigma_i) + f a_{ij}\Big]. \qquad (12)\end{aligned}$$

3. Proof of Theorem 1.1

Now assume that both $\bar{F} = \bar{\alpha} + \bar{\beta}$ and $F = \alpha + \beta$ are Finsler metrics of Randers type. By Theorem 2.1, if $\bar{F}$ and F are locally dually flat metrics, then by plugging $\bar{F} = e^{\sigma(x)}F$ we obtain the following equations

$$G^i_{\bar{\alpha}} = (2\bar{\theta} + e^{\sigma(x)}\bar{\tau}\beta)y^i - e^{2\sigma(x)}\alpha^2(e^{-\sigma}\bar{\tau}b^i - \bar{\theta}^i), \qquad (13)$$

$$\bar{r}_{00} = 2e^{\sigma}\bar{\theta}\beta - 5e^{2\sigma}\bar{\tau}\beta^2 + (3\bar{\tau} + 2\bar{\tau}b^2 - 2e^{\sigma}b_k\bar{\theta}^k)e^{2\sigma}\alpha^2, \qquad (14)$$

$$\bar{s_{i0}} = e^{\sigma}\beta\bar{\theta}_i - e^{\sigma}\bar{\theta}b_i, \qquad (15)$$

where $\bar{\theta} = \bar{\theta}_k(x)y^k$ is a 1-form on U, $\bar{\theta}^i := \bar{a}^{ik}\bar{\theta}_k$, and $\bar{\tau} = \bar{\tau}(x)$ is a scalar function and $G^i_{\bar{\alpha}}$ denote the spray coefficients of $\bar{F}$. We need the following lemma for the proof of Theorem 1.1.

Lemma 3.1. *Let $\bar{F}$ and F be two Randers metrics which they are conformally related i.e., $\bar{F} = e^{\sigma(x)}F$. Then*

$$\bar{s_0} = e^{\sigma(x)}K(x)s_0, \tag{16}$$
$$\bar{e}_{00} = 2e^{\sigma}K(x)\theta\beta - 5e^{2\sigma}Q(x)\tau\beta^2 + 2e^{\sigma}K(x)s_0\beta + (3Q(x)\tau + 2Q(x)\tau b^2 - 2e^{\sigma}b_kK(x)\theta^k)e^{2\sigma}\alpha^2,$$

where $K(x) = \frac{\bar{\theta}}{\theta}$ and $Q(x) = \frac{\bar{\tau}}{\tau}$.

These equations can be obtained by using (6), (14) and (15). As we know the spray coefficients of two conformally related Finsler metrics satisfy in (10). Plugging (3) into (10), we get

$$G^i_{\bar{\alpha}} + (\frac{\bar{e}_{00}}{2\bar{F}} - \bar{s}_0)y^i + \bar{\alpha}\bar{s}^i_0 = G^i_{\alpha} + (\frac{e_{00}}{2F} - s_0)y^i + \alpha s^i_0 + \sigma_0 y^i - \frac{F^2}{2}\sigma^i. \tag{17}$$

By replacing quantities that we get in (6), (13) and (16) in (17), one can see

$$\begin{aligned}0 = {} & (2K(x)\theta + e^{\sigma}Q(x)\tau\beta)y^i - e^{2\sigma}\alpha^2(Q(x)\tau e^{-\sigma}b^i - K(x)\theta^i)\\ & + \Big\{\frac{1}{2e^{\sigma}F}(-5e^{2\sigma}Q(x)\tau\beta^2 + e^{\sigma}K(x)\theta\beta + 2e^{\sigma}K(x)s_0\beta) - s_0\\ & + 2(3Q(x)\tau + 2Q(x)\tau b^2 - 2e^{\sigma}b_kK(x)\theta^k)e^{2\sigma}\alpha^2\Big\}y^i - \Big\{ - s_0\\ & + \frac{1}{2F}(2\theta\beta + 2s_0\beta - 5\tau\beta^2 + \alpha^2(3\tau + 2\tau b^2 - 2b_k\theta^k))\Big\}y^i - \alpha^2\theta^i\\ & - (2\theta + \tau\beta)y^i + e^{2\sigma}\alpha(\beta K(x)\theta^i - b^iK(x)\theta) - \alpha(\beta\theta^i - \theta b^i)\\ & + \alpha^2\tau b^i - \sigma_0 y^i + \frac{F^2}{2}\sigma^i.\end{aligned}$$

By multiplying both sides of the equation mentioned above by $2F$ to remove the denominators and sort this equation by α, we get

$$\begin{aligned}0 = {} & \alpha^3\Big\{\frac{1}{2}\sigma^i + K(x)e^{2\sigma}\theta^i + \tau b^i - Q(x)\tau e^{\sigma}b^i - \theta^i\Big\} + \alpha^2\Big\{K(x)e^{2\sigma}\theta^i\beta + \frac{3}{2}\sigma^i\beta\\ & - y^i\tau b^2 + b^iK(x)e^{2\sigma}\theta - y^ie^{2\sigma}b_kK(x)\theta^k + y^ie^{\sigma}Q(x)\tau b^2 + \frac{3}{2}y^ie^{\sigma}Q(x)\tau\\ & + \tau b^i\beta + b^i\theta - Q(x)\tau e^{\sigma}b^i\beta - 2\beta\theta^i + y^ib_k\theta^k + e^{2\sigma}\beta K(x)\theta^i - \frac{3}{2}y^i\tau\Big\}\\ & + \alpha\Big\{2y^iK(x)\theta + e^{2\sigma}\beta^2K(x)\theta^i + y^ie^{\sigma}Q(x)\tau\beta - \beta^2\theta^i - \sigma_0 y^i - y^i\tau\beta\\ & + b^i\theta\beta - e^{2\sigma}b^iK(x)\theta\beta - 2y^i\theta + \frac{3}{2}\sigma^i\beta^2\Big\} - \sigma_0 y^i\beta + K(x)(y^is_0\beta + 3y^i\theta\beta)\\ & - 3y^i\theta\beta + \frac{1}{2}\sigma^i\beta^3 - y^is_0\beta + (\frac{3}{2})y^i\tau\beta^2 - \frac{3}{2}y^ie^{\sigma}Q(x)\tau\beta^2.\end{aligned}$$

We can rewrite this equation as the following

$$0 = \alpha\Big\{\alpha^2\Big\{\frac{1}{2}\sigma^i + K(x)e^{2\sigma}\theta^i + \tau b^i - Q(x)\tau e^{\sigma}b^i - \theta^i\Big\} + \alpha\Big\{K(x)e^{2\sigma}\theta^i\beta$$

$$+\frac{3}{2}\sigma^i\beta+b^iK(x)e^{2\sigma}\theta-Q(x)\tau e^{\sigma}b^i\beta-y^i\tau b^2+y^ie^{\sigma}Q(x)\tau b^2+\frac{3}{2}y^ie^{\sigma}Q(x)\tau$$
$$+y^ib_k\theta^k+b^i\theta-y^ie^{2\sigma}b_kK(x)\theta^k-2\beta\theta^i+\tau b^i\beta+e^{2\sigma}\beta K(x)\theta^i-\frac{3}{2}y^i\tau\Big\}$$
$$+e^{2\sigma}\beta^2K(x)\theta^i+y^ie^{\sigma}Q(x)\tau\beta-\beta^2\theta^i-y^i(\tau\beta+\sigma_0-2K(x)\theta)+\theta(b^i\beta-2y^i)$$
$$-e^{2\sigma}b^iK(x)\theta\beta+\frac{3}{2}\sigma^i\beta^2\Big\}-\Big\{\sigma_0y^i\beta-y^iK(x)s_0\beta+3y^i\theta\beta+\frac{3}{2}y^ie^{\sigma}Q(x)\tau\beta^2$$
$$-3y^iK(x)\theta\beta+y^is_0\beta-\frac{1}{2}\sigma^i\beta^3-\frac{3}{2}y^i\tau\beta^2\Big\}.$$

From this equation, we know that

$$\begin{aligned}0&=\sigma_0y^i\beta-y^iK(x)s_0\beta+3y^i\theta\beta-3y^iK(x)\theta\beta\\&\quad-\frac{1}{2}\sigma^i\beta^3+y^is_0\beta+\frac{3}{2}y^ie^{\sigma}Q(x)\tau\beta^2-\frac{3}{2}y^i\tau\beta^2.\end{aligned}\tag{18}$$

By contracting (18) with y_i, we have

$$\begin{aligned}0&=\alpha^2\sigma_0\beta+\alpha^2K(x)s_0\beta+3\alpha^2\theta\beta-3\alpha^2K(x)\theta\beta\\&\quad-\frac{1}{2}\sigma_0\beta^3+\alpha^2s_0\beta+\frac{3}{2}\alpha^2e^{\sigma}Q(x)\tau\beta^2-\frac{3}{2}\alpha^2\tau\beta^2.\end{aligned}\tag{19}$$

It follows from (22) that

$$\Big\{K(x)s_0\beta+3\theta\beta-3K(x)\theta\beta+\frac{3}{2}e^{\sigma}Q(x)\tau\beta^2+\sigma_0\beta-\frac{3}{2}\tau\beta^2+s_0\beta\Big\}\alpha^2 \tag{20}$$
$$=\frac{1}{2}\sigma_0\beta^3.$$

This equality holds if and only if $\sigma^iy_i\beta^3=0$ then

$$\sigma^i=0.$$

It means that σ must be a constant function, i.e., conformal transformation must be homothetic.

4. Proof of Theorem 1.2

In this section we try to prove Theorem 1.2. Let $\bar{F}$ be a Kropina metric that satisfies dually flat conditions. Then Theorem 2.2 we have

$$\begin{aligned}G^i_{\bar\alpha}&=\frac{1}{b^2}\Big[\bar\alpha^2\bar\xi^i+(\bar\theta b^2-\bar\xi)y^i\Big],\\ \bar r_{00}&=\frac{1}{2b^2}\Big[\bar\beta(\bar\theta b^2+\bar\xi)-4(\bar\xi_k\bar b^k)\bar\alpha^2\Big],\\ \bar s_{k0}&=\frac{1}{4b^2}\Big[b^2(\overline{\theta}\overline{b}_k-\bar\beta\bar\theta_k)+7(\bar\beta\bar\xi_k-\bar b_k\bar\xi)\Big],\end{aligned}$$

where $\bar\theta=\bar\theta_i(x)y^i$ and $\bar\xi=\bar\xi_i(x)y^i$ are 1-forms on M and $\bar\xi_i:=\bar a^{ij}\bar\xi_j$.

Lemma 4.1. *Let $\bar{F}$ and F be two dually flat Kropina metrics which they are conformally related, i.e., $\bar{F} = e^{\sigma(x)}F$. Then*

$$\bar{r}_{00} = \frac{e^{\sigma}}{2b^2}\Big[\beta(K\theta b^2 + Q\xi) - 4(Q\xi_k b^k)\alpha^2\Big], \tag{21}$$

$$\bar{G}^i_{\bar{\alpha}} = \frac{1}{b^2}\Big[e^{2\sigma}\alpha^2 Q\xi^i + (K\theta b^2 - Q\xi)y^i\Big], \tag{22}$$

$$\bar{s}^i{}_0 = \frac{1}{4b^2}\Big[b^2(Ke^{-\sigma}\theta b^i - e^{\sigma}\beta K\theta^i) + 7(e^{\sigma}\beta Q\xi^i - e^{-\sigma}b^i Q\xi)\Big], \tag{23}$$

$$\bar{s}_0 = \frac{1}{4b^2}\Big[b^2 K(\theta b^2 - \beta\theta_k b^k) + 7Q(\beta\xi_k b^k - b^2\xi)\Big], \tag{24}$$

where $\frac{\bar{\theta}}{\theta} = K(x)$ and $\frac{\bar{\xi}}{\xi} = Q(x)$.

Proof of this lemma by using (7)-(9) is straight.

By contracting (12) by y^i and y^j we have following equality

$$\bar{r}_{00} = e^{\sigma(x)}(r_{00} + f\alpha^2 - \beta\sigma_0). \tag{25}$$

Input (8) and (21) into (25) we can get

$$\begin{aligned} 0 = {} & \frac{e^{\sigma}}{2b^2}\Big[\beta(K\theta b^2 + Q\xi) - 4(Q\xi_k b^k)\alpha^2\Big] \\ & - e^{\sigma}\Big\{\frac{1}{2b^2}\Big[\beta(\theta b^2 + \xi) - 4(\xi_k b^k)\alpha^2\Big] + f\alpha^2 - \beta\sigma_0\Big\}. \end{aligned}$$

To remove denominators multiply both sides of equation mentioned above by $2b^2$ and sort by α

$$\begin{aligned} 0 = e^{\sigma}\Big\{ & [-4(Q\xi_k b^k) + 4(\xi_k b^k) - 2b^2 f]\alpha^2 \\ & + [(K\theta b^2 + Q\xi) - (\theta b^2 + \xi) + 2b^2\sigma_0]\beta\Big\}. \end{aligned}$$

We know that α^2 and β are relatively prime polynomial then

$$0 = 2(Q\xi_k b^k) - 2(\xi_k b^k) + b^2 f, \tag{26}$$

$$0 = (K\theta b^2 + Q\xi) - (\theta b^2 + \xi) + 2b^2\sigma_0. \tag{27}$$

Moreover the relation between spray geodesic coefficients of two conformally related Finser metrics are as follows:

$$G^i_{\bar{\alpha}} = G^i_{\alpha} + \sigma_0 y^i - \frac{\alpha^2}{2}\sigma^i.$$

By replacing (7) and (22) into this very equation we can get

$$\begin{aligned} 0 = {} & \frac{1}{b^2}\Big[e^{2\sigma}Q\xi^i\alpha^2 + (K\theta b^2 - Q\xi)y^i\Big] \\ & - \frac{1}{b^2}\Big[\alpha^2\xi^i + (\theta b^2 - \xi)y^i\Big] - \sigma_0 y^i + \frac{\alpha^2}{2}\sigma^i. \end{aligned}$$

Multiply both sides of this equation by $2b^2\beta^2 b_i$

$$0 = \Big\{2e^{2\sigma}Q\xi^i b_i - 2\xi^i b_i + b^2 f\Big\}\alpha^2$$

$$+\Big\{2(K\theta b^2 - Q\xi) - 2(\theta b^2 - \xi) - 2b^2\sigma_0\Big\}\beta.$$

This means that

$$0 = 2e^{2\sigma}Q\xi^i b_i - 2\xi^i b_i + b^2 f, \tag{28}$$

$$0 = 2(K\theta b^2 - Q\xi) - 2(\theta b^2 - \xi) - 2b^2\sigma_0. \tag{29}$$

By (28) − (26) we have the following

$$2Q\xi^i b_i(e^{2\sigma} - 1) = 0.$$

Because of $\sigma = \sigma(x)$ is a nonzero function then

$$Q\xi^i b_i = 0. \tag{30}$$

Replace equation (30) into (26)

$$\xi_i b^i = \frac{1}{2}b^2 f. \tag{31}$$

Moreover by computing (27) × 2 − (29) we can get

$$4Q\xi - 4\xi + 6b^2\sigma_0 = 0.$$

Differentiating with respect to y^i and contracting by b^i we have

$$4Q\xi_i b^i - 4\xi_i b^i + 6b^2 f = 0. \tag{32}$$

By using (30) and (31), equation (32) can simplified as follows:

$$4b^2 f = 0.$$

This means that $f = 0$. By (27) × 2 − (29) we can get

$$2b^2(2K\theta - 2\theta - \sigma_0) = 0. \tag{33}$$

Differentiating with respect to y^i and contracting by b^i we have

$$2K\theta_i b^i - 2\theta_i b^i - f = 0.$$

By the above calculation we know that $f = 0$, then

$$\theta_i b^i(K - 1) = 0. \tag{34}$$

Case 1: Let $K = 1$ then by (33) we have

$$\sigma_0 = 0,$$

i.e., the conformal transformation must be homothetic.

Case 2: In the case of $\theta_i b^i = 0$, geodesic coefficients of $\bar{F}$ by (4) is as follows:

$$\bar{G}^i = G^i_{\bar{\alpha}} - \frac{\bar{\alpha}^2}{2\bar{\beta}}\bar{s}^i_0 + \frac{1}{2\bar{b}^2}(\frac{\bar{\alpha}^2}{\bar{\beta}}\bar{s}_0 + \bar{r}_{00})\bar{b}^i - \frac{1}{\bar{b}^2}(\bar{s}_0 + \frac{\bar{\beta}}{\bar{\alpha}^2}\bar{r}_{00})y^i. \tag{35}$$

By replacing (4), (35) into (10) we have

$$0 = G^i_{\bar{\alpha}} - \frac{\bar{\alpha}^2}{2\bar{\beta}}\bar{s}^i_0 + \frac{1}{2\bar{b}^2}(\frac{\bar{\alpha}^2}{\bar{\beta}}\bar{s}_0 + \bar{r}_{00})\bar{b}^i - \frac{1}{\bar{b}^2}(\bar{s}_0 + \frac{\bar{\beta}}{\bar{\alpha}^2}\bar{r}_{00})y^i - G^i_\alpha$$

$$+\frac{\alpha^2}{2\beta}s^i_0-\frac{1}{2b^2}(\frac{\alpha^2}{\beta}s_0+r_{00})b^i+\frac{1}{b^2}(s_0+\frac{\beta}{\alpha^2}r_{00})y^i-(\sigma_k y^k)y^i+\frac{F^2}{2}\sigma^i. \tag{36}$$

By using (9), it is easy to get the following quantities for dually flat Kropina metric F

$$s^i{}_0=\frac{1}{4b^2}\Big[b^2(\theta b^i-\beta\theta^i)+7(\beta\xi^i-b^i\xi)\Big], \tag{37}$$

$$s_0=\frac{1}{4b^2}\Big[b^2(\theta b^2-\beta\theta_k b^k)+7(\beta\xi_k b^k-b^2\xi)\Big]. \tag{38}$$

Replace (7)-(9), (21)-(24) and (37), (38) into (36)

$$\begin{aligned}0=&\frac{1}{b^2}\Big[e^{2\sigma}\alpha^2Q\xi^i+(K\theta b^2-Q\xi)y^i\Big]+\frac{b^i}{4b^4}\Big[\beta(K\theta b^2+Q\xi)-4(Q\xi_k b^k)\alpha^2\Big)b^i\\&-\frac{e^\sigma\alpha^2}{8\beta b^2}\Big[b^2(Ke^{-\sigma}\theta b^i-e^\sigma\beta K\theta^i)+7(e^\sigma\beta Q\xi^i-e^{-\sigma}b^iQ\xi)\Big]+\frac{2\beta y^i}{b^2}Q\xi_k b^k\\&+\frac{\alpha^2b^i}{8b^4\beta}\Big[b^2K(\theta b^2-\beta\theta_k b^k)+7Q(\beta\xi_k b^k-b^2\xi)\Big]-\frac{\beta^2y^i}{2b^2\alpha^2}(K\theta b^2+Q\xi)\Big]\\&-\frac{y^i}{4b^4}\Big[b^2K(\theta b^2-\beta\theta_k b^k)+7Q(\beta\xi_k b^k-b^2\xi)\Big]-\frac{1}{b^2}\Big[\alpha^2\xi^i+(\theta b^2-\xi)y^i\Big]\\&+\frac{\alpha^2}{8\beta b^4}\Big[b^2(\theta b^i-\beta\theta^i)+7(\beta\xi^i-b^i\xi)\Big]+\frac{\beta y^i}{2\alpha^2b^4}\Big[\beta(\theta b^2+\xi)-4(\xi_k b^k)\alpha^2\\&-\frac{\alpha^2b^i}{8\beta b^4}\Big[b^2(\theta b^2-\beta\theta_k b^k)+7(\beta\xi_k b^k-b^2\xi)\Big]-\frac{b^i}{4b^4}\Big[\beta(\theta b^2+\xi)-4(\xi_k b^k)\alpha^2\Big]\\&+\frac{y^i}{4b^4}\Big[b^2(\theta b^2-\beta\theta_k b^k)+7(\beta\xi_k b^k-b^2\xi)\Big]-\sigma_0y^i+\frac{F^2}{2}\sigma^i.\end{aligned}$$

Multiply both sides of this equation by b_i and replace $\xi_ib^i=\theta_ib^i=f=0$ we get

$$\begin{aligned}0=&\frac{1}{b^2}\big[K\theta b^2-Q\xi\big]\beta-\frac{\alpha^2}{8\beta b^2}\big[b^4K\theta-7b^2Q\xi\big]+\frac{\alpha^2}{8b^2\beta}\big[b^4K\theta-7Qb^2\xi)\big]\\&+\frac{1}{4b^2}\big[\beta(K\theta b^2+Q\xi)\big]-\frac{\beta}{4b^4}\big[b^4K\theta-7Qb^2\xi\big]+\frac{\beta^2}{2b^4\alpha^2}\big[\beta(K\theta b^2+Q\xi)\big]\\&-\frac{1}{b^2}\big[(\theta b^2-\xi)\beta\big]+\frac{\alpha^2}{8\beta b^2}\big[b^4\theta-7b^2\xi)\big]-\frac{\alpha^2}{8\beta b^2}\big[b^2\theta b^2-7b^2\xi\big]\\&-\frac{1}{4b^2}\big[\beta(\theta b^2+\xi)\big])b^2+\frac{\beta}{4b^4}\big[b^4\theta b^2-7b^2\xi)\big]+\frac{\beta^2}{2b^4\alpha^2}\big[\beta(\theta b^2+\xi)\big]-\sigma_0\beta.\end{aligned} \tag{39}$$

Multiply both sides of equation mentioned above by $8b^4\alpha^2\beta$ to remove denominators and sort it by α. So we can obtain

$$\begin{aligned}0=\alpha^2\Big\{&8K\theta b^4\beta^2-8b^2Q\xi\beta^2+2\beta^2\big[K\theta b^4+Q\xi b^2\big]-2\beta^2\big[b^4K\theta-7Qb^2\xi\big]\\&-8b^2\beta^2\big[\theta b^2-\xi\big]-2\beta^2\big[\theta b^4+\xi b^2\big]+2\beta^2\big[b^4\theta-7b^2\xi)\big]-8b^4\beta^2\sigma_0\Big\}\\&-4\beta^4\big[K\theta b^2+Q\xi\big]+4\beta^4\big[\theta b^2+\xi\big].\end{aligned}$$

Then we get

$$0 = 4\beta^2\Big\{2\alpha^2[K\theta b^4 + Q\xi b^2 - \theta b^4 - \xi b^2 - b^4\sigma_0] \\ + \beta^2[-K\theta b^2 - Q\xi + \theta b^2 + \xi]\Big\}.$$

Since $\beta \neq 0$ then

$$0 = 2\alpha^2[K\theta b^4 + Q\xi b^2 - \theta b^4 - \xi b^2 - b^4\sigma_0] + \beta^2[-K\theta b^2 - Q\xi + \theta b^2 + \xi].$$

α^2 and β^2 are relatively prime, this means that the coefficients of β^2 and α^2 must be equal to zero

(40) $$0 = -K\theta b^2 - Q\xi + \theta b^2 + \xi,$$

(41) $$0 = b^2[K\theta b^2 + Q\xi - \theta b^2 - \xi] - b^4\sigma_0.$$

Replace (40) into (41) we get

$$\sigma_0 = 0.$$

This completes the proof of theorem.

C-reducible Finsler metrics are a class of Finsler spaces which their Cartan torsion has special form. These spaces first were introduced by M. Matsumoto [9]. In [10] Matsumoto and Hojo proved that a Finsler space is **C**-reducible if and only if the space is either a Randers or a Kropina space. By Theorems 1.1 and 1.2 we can obtain Corollary 1.3.

At the end, we improved the result into general case and get the following theorem.

Theorem 4.2. *Every conformal transformation between locally dually flat Finsler metrics must be homothetic.*

Proof. Let $\bar{F}$ and F be conformally related Finsler metrics such that $\bar{F} = e^\sigma F$ and both of them are locally dually flat, then we have

$$[\bar{F}^2]_{x^k y^l} y^k - 2[\bar{F}^2]_{x^l} = 0, \qquad [F^2]_{x^k y^l} y^k - 2[F^2]_{x^l} = 0.$$

Substituting the relation $\bar{F} = e^\sigma F$ into second equation and by making use the first one, we get following equation

(42) $$4e^{2\sigma}(\sigma_k F F_{y^l} y^k - \sigma_l F^2) = 0.$$

If we put $h^k_l = F^2\delta^k_l - F F_{y^l} y^k$, then $\sigma_k h^k_l = 0$. By fundamental tensor g_{ij} we may also write it as

$$\sigma^j h_{lj} = 0,$$

where $h_{lj} = F^2(g_{lj} - F_{y^l}F_{y^j})$ and $\sigma^j = \sigma_l g^{lj}$. Matrix h_{lj} is symmetric of rank $n-1$, whose kernel is spanned by the vector $(y^1, \ldots, y^n)$. So the vector $(\sigma^1, \ldots, \sigma^n)$ is proportional to $(y^1, \ldots, y^n)$, namely

$$\sigma^k = \lambda . y^k$$

for some function $\lambda = \lambda(x, y)$. Lowering indices yields

$$\sigma_j = \lambda g_{jk} y^k = \lambda . F F_{y^j}.$$

Namely, the vector$(\sigma^1, \ldots, \sigma^n)$ is proportional to $(FF_{y^1}, \ldots, FF_{y^n})$. However, we know that the Legendre transformation, which sends the vector $(y^1, \ldots, y^n)$ to $(FF_{y^1}, \ldots, FF_{y^n})$, is a diffeomorphism between $T_xM \setminus \{0\}$ and $T^*_xM \setminus \{0\}$. For each fixed x, one can choose two values of y such the corresponding vectors $(FF_{y^1}, \ldots, FF_{y^n})$ are linearly independent. Being proportional to these two vectors, $(\sigma^1, \ldots, \sigma^n)$ must be zero. As a result, σ is a constant. □

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Laya Ghasemnezhad
Faculty of Science
Urmia University
Urmia, Iran
Email address: l.ghasemnezhad@urmia.ac.ir

Bahman Rezaei
Faculty of Science
Urmia University
Urmia, Iran
Email address: b.rezae@urmia.ac.ir