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ON STRONGLY GORENSTEIN HEREDITARY RINGS

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Abstract. In this note, we mainly discuss strongly Gorenstein hereditary rings. We prove that for any ring, the class of SG-projective modules and the class of G-projective modules coincide if and only if the class of SG-projective modules is closed under extension. From this we get that a ring is an SG-hereditary ring if and only if every ideal is G-projective and the class of SG-projective modules is closed under extension. We also give some examples of domains whose ideals are SG-projective.

1. Introduction

Throughout this note, all rings are commutative with identity element and all modules are unitary.

Recall that an R-module M is called *Gorenstein projective* (G-projective for short) in [4] if there exists an exact sequence $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ of projective R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective module. The authors in [3] introduced *strongly Gorenstein projective* (SG-projective for short) modules. An R-module M is called SG-projective if there exists an exact sequence of projective modules $\cdots \to P \to P \to P \to P \to \cdots$ such that all these projective modules are the same and all these arrows in this sequence are the same homomorphism with M to be an image of some arrow and $\mathrm{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective module. The authors in [9] introduced strongly Gorenstein hereditary rings. A ring R is called *strongly Gorenstein hereditary* (for short, SG-hereditary) if every submodule of any projective module is SG-projective. An SG-hereditary domain is called an *SG-Dedekind domain.* Naturally, Dedekind domains are SG-Dedekind domains. Examples of SG-hereditary rings are given in [9]. It is easy to see from the definition that every ideal of an SG-hereditary ring is SG-projective. It is natural to ask whether the converse also holds, that is to say, if every ideal of a ring is SG-projective, can we say that the ring is SG-hereditary? Unfortunately, we can not give a positive answer. But we prove that a ring is SG-hereditary if

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and only if every ideal is G-projective and the class of SG-projective modules is closed under extension. Let R be a G-Dedekind domain with quotient field K and T be its integral closure in K and the ideal $(R :_K T)$ be a nonzero prime ideal of R. If M is a G-projective R-module which has no projective direct summand of rank 1, then M is also a T-module. If M is a finitely generated G-projective R-module, then M is isomorphic to a direct sum of some ideals of R. If every projective ideal of R is principal, then any projective R-module is free. From this, we give some examples of G-Dedekind domains whose ideals are SG-projective. That is, if p is a prime number and $R = \mathbb{Z} + p\mathbb{Z}i$, then every ideal of R is SG-projective. We also get that if $R = \mathbb{Z} + 2\mathbb{Z}i$, then every projective R-module is free.

For unexplained concepts and notations, one can refer to [8, 12].

2. A characterization of *SG*-hereditary rings

In this section, we mainly prove the following:

Theorem 2.1. *Let R be a ring. Then R is an SG-hereditary ring if and only if every ideal of R is G-projective and the class of SG-projective modules is closed under extension.*

In order to prove this theorem, we need some lemmas. We begin with the following:

Lemma 2.2. *Let M be a module and P be a projective module. If $M \oplus P$ is SG-projective, then M is SG-projective.*

Proof. This follows easily from [15, Theorem 2.1]. □

A similar result of the following can be seen in [16, Theorem 3.14], but the proof there is not suitable for SG-projective modules. So, we state this here.

Lemma 2.3. *Let R be a ring and let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of R-modules such that C is SG-projective.*

(1) *If A is projective, then B is SG-projective*;
(2) *If B is projective, then A is SG-projective.*

Proof. (1) Since A is projective, the sequence splits. So $B \cong A \oplus C$ is SG-projective.

(2) This result follows directly from [10, Proposition 2.13]. We give its proof for completeness.

Since C is SG-projective, we have a short exact sequence

$$0 \longrightarrow C \longrightarrow P \longrightarrow C \longrightarrow 0,$$

where P is projective. So we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Now we consider the mapping cone sequence

$$0 \longrightarrow C \longrightarrow P \oplus A \longrightarrow B \longrightarrow 0.$$

Since B is projective, this sequence splits. So $C \oplus B \cong A \oplus P$. Because C and B are SG-projective, $A \oplus P$ is also SG-projective. Therefore, A is SG-projective by Lemma 2.2. □

Let $\mathfrak{X}$ be a class of R-modules. We call $\mathfrak{X}$ *projectively resolving* [5] if $\mathfrak{X}$ contains projective modules, and for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ with $X'' \in \mathfrak{X}$ the conditions $X' \in \mathfrak{X}$ and $X \in \mathfrak{X}$ are equivalent. Denote by SGP the class of SG-projective modules.

Lemma 2.4. *Let R be a ring. If SGP is closed under extension, then SGP is projectively resolving.*

Proof. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence such that C is SG-projective. We must prove that A is SG-projective if and only if B is SG-projective. If A is SG-projective, then B is SG-projective because SGP is closed under extension. If B is SG-projective, then we have a short exact sequence $0 \longrightarrow B \longrightarrow P \longrightarrow B \longrightarrow 0$, where P is projective. So we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & T & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & B & = & B & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Since B and C are SG-projective, it can be seen from the right vertical sequence that T is also SG-projective. Now applying Lemma 2.3 to the middle horizontal sequence, we get that A is SG-projective. □

The following lemma shows some relations between G-projective modules and SG-projective modules.

Lemma 2.5. *The following statements are equivalent for a ring R:*

(1) *Every G-projective module is SG-projective.*
(2) *SGP is closed under extension.*

Proof. (1) $\Rightarrow$ (2) This is obvious since the class of G-projective modules is closed under extension.

(2) $\Rightarrow$ (1) By Lemma 2.4, SGP is projectively resolving. Also notice that the SGP is closed under countable direct sums, and so direct summands by [5, Proposition 1.4]. Since any G-projective module is a direct summand of some SG-projective module by [3, Theorem 2.7], every G-projective module is SG-projective. □

Now we are in the position to prove the main theorem.

Proof of Theorem 2.1. If R is SG-hereditary, then any G-projective module, as a submodule of some projective module, must be SG-projective. So SGP is closed under extension by Lemma 2.5. Also every ideal, as a submodule of R, must be G-projective. For sufficiency part, if every ideal of R is G-projective, then R is a G-hereditary ring by [6, Theorem 1.2]. If SGP is closed under extension, by Lemma 2.5, every G-projective module is SG-projective. Therefore, by [9, Proposition 2.8], R is SG-hereditary. □

3. A special kind of G-Dedekind domains

It is well known that a domain is a Dedekind domain if and only if submodules of any free module are projective. It is proved in [6, Theorem 1.2] that a domain R is a G-Dedekind domain if and only if every ideal of R is G-projective. Since projective modules are SG-projective, Dedekind domains are SG-Dedekind domains. Trivially SG-Dedekind domains are G-Dedekind domains. Unfortunately, we have not found an SG-Dedekind domain which is not a Dedekind domain, but we find some G-Dedekind domains whose ideals are SG-projective. The induction trick of Theorem 3.3 can be found in [2, 6].

Let R be a G-Dedekind domain with quotient field K and T be its integral closure in K. We study such a G-Dedekind domain R that the ideal $(R :_K T)$ is a nonzero prime ideal of R.

In [11, Theorem], E. Matlis proved that every ideal of a domain R can be generated by two elements if and only if R is a Noetherian ring and every finitely generated torsion-free R_M-module is a direct sum of R_M-modules of rank 1. The following lemma is inspired by his work.

Lemma 3.1. *Let M be an R-module. Then M has a projective direct summand of rank 1 if and only if there exists an $f \in M^*$ such that $\mathrm{Im} f$ is projective.*

Proof. Since every projective module of rank 1 is isomorphic to an ideal, if $M = N \oplus P$, where P is a projective submodule of rank 1, then the canonical composition $f : M \to P \to R$ is an element of M^* such that $\mathrm{Im}f$ is projective. Conversely, if there is an $f \in M^*$ such that $\mathrm{Im}f$ is projective, then we have an exact sequence $M \to \mathrm{Im}f \to 0$. Because $\mathrm{Im}f$ is projective, this sequence splits. So, M has a projective direct summand of rank 1. □

Lemma 3.2. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R :_K T)$ be a nonzero prime ideal of R. If M is a G-projective module which has no projective direct summand of rank 1, then $\mathrm{Im}f \subset (R :_K T)$ for any $f \in M^*$.*

Proof. If $\mathrm{Im}f \nsubseteq (R :_K T)$ for some $f \in M^*$, then $\mathrm{Im}f + (R :_K T) = R$ because $(R :_K T)$ is a nonzero prime ideal of R and $\dim(R) \leq 1$ ([14, Corollary 11.7.8]). Therefore $\mathrm{Im}f$ is projective by [7, Lemma 1.8]. This, by Lemma 3.1, means that M has a projective direct summand of rank 1, which contradicts the hypothesis. □

Theorem 3.3. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R :_K T)$ be a nonzero prime ideal of R. If M is a G-projective R-module which has no projective direct summand of rank 1, then M is also a T-module.*

Proof. Let $m \in M$. Since M is a G-projective R-module, we can assume that M is a submodule of some free module F, say $F = \bigoplus_{i \in \Gamma} Rx_i$. So, as an element of F, m has only finitely many coordinates which are not zero. Well-order the set Γ and let nonzero coordinates of m be the beginning ones, say $m \in \bigoplus_{i=1}^n Rx_i$. So every element of Γ is corresponding to an ordinal which is not a limit ordinal. We add those related limit ordinals in Γ to obtain a new set and still denote it by Γ. Then, for some $k \in \Gamma$, if k is corresponding to a limit ordinal, we define $H_k = \bigoplus_{i<k} Rx_i$; if k is corresponding to an ordinal which is not a limit ordinal, we define $H_k = \bigoplus_{i \leqslant k} Rx_i$. Denote $M \cap H_k$ by J_k. Note that $m \in J_n$ and if k is corresponding to a limit ordinal, $J_k = \bigcup_{i<k} J_i$. Now, we prove that for any $f \in J_n^*$ $(\mathrm{Hom}(J_n, R))$, $\mathrm{Im}f$ is not projective. Suppose there is some $f \in J_n^*$ such that $\mathrm{Im}f$ is projective. We consider the following commutative diagram with an exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_n & \longrightarrow & J_{n+1} & \xrightarrow{\alpha} & I_{n+1} & \longrightarrow & 0\ , \\ & & {\scriptstyle f}\Big\downarrow & \swarrow {\scriptstyle f_{n+1}} & & & & & \\ & & \mathrm{Im}f & & & & & & \end{array}$$

where α is the projective map to the $(n+1)$-th coordinate. The existence of f_{n+1} can be verified because I_{n+1}, as an ideal of R, is G-projecitve and $\mathrm{Im}f$ is projective. Since $f : J_n \to \mathrm{Im}f$ is surjective, $f_{n+1} : J_{n+1} \to \mathrm{Im}f$ is also surjective. Next, we prove that there exists a surjective homomorphism

$h : M \to \mathrm{Im} f$. To this end, we show that f can be extended to a homomorphism f_i from J_i to $\mathrm{Im} f$ for every $i \in \Gamma$ such that $f_{i+1}|_{J_i} = f_i$. If there are exceptions, then the set S of those exceptional ordinals admits a minimal element k. If k is a limit ordinal, then $J_k = \bigcup_{i<k} J_i$ and for every $a \in J_k$, there is some $j < k$ such that $a \in J_j$. Therefore we can define $f_k(a) = f_j(a)$. If k is an ordinal which is not a limit ordinal, then $k-1$ exists and we have the following commutative diagram with an exact row

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_{k-1} & \longrightarrow & J_k & \xrightarrow{\beta} & I_k & \longrightarrow & 0\ , \\ & & {\scriptstyle f_{k-1}}\big\downarrow & \swarrow{\scriptstyle f_k} & & & & & \\ & & \mathrm{Im} f & & & & & & \end{array}$$

where β is the projective map to the k-th coordinate. The existence of f_k follows because I_k, as an ideal of R, is G-projecitve and $\mathrm{Im} f$ is projective. Both cases contradict the fact that $k \in S$. Now, for every $b \in M = \bigcup_{i\in\Gamma} J_i$, $b \in J_t$ for some $t \in \Gamma$ and we can define $h(b) = f_t(b)$. The existence of h means, by Lemma 3.1, that M has a projective direct summand of rank 1, which contradicts the hypothesis. Therefore, J_n is also a G-projective module which has no projective direct summand of rank 1. For any $q \in T$, we define $qm \in J_n^{**}$ such that $qm(g) = qg(m)$ for every $g \in J_n^*$. Since $g(m) \in (R :_K T)$ by Lemma 3.2 and $q \in T$, this is well defined. Because J_n is a finitely generated G-projective module, J_n is reflexive, i.e., $J_n \cong J_n^{**}$. So $qm \in J_n \subset M$. Thus we have proved that $Tm \subset M$. The arbitrariness of $m \in M$ tells us that M is also a T-module. □

A domain R is called a *Warfield domain* if for any R-submodule A of the quotient field of R, every A-torsion-free $S := \mathrm{End}_R(A)$-module X of finite rank is A-reflexive, that is, the natural homomorphism $X \to \mathrm{Hom}_S(\mathrm{Hom}_S(X, A), A)$ is an isomorphism. In [13], it is proved that a Noetherian domain R is a Warfield domain if and only if each ideal of R is 2-generated. It is proved in [7, Theorem 2.20] that every ideal of a Noetherian Warfield domain is 2-SG-projective. Therefore Noetherian Warfield domains are G-Dedekind domains.

Example 3.4. Let p be a prime number and $R = \mathbb{Z} + p\mathbb{Z}i$. Then every ideal of R is SG-projective.

Proof. Clearly R is a subring of $\mathbb{Z}[i]$. Since $\mathbb{Z}[i]$ is a free $\mathbb{Z}$-module of rank 2, every ideal of R is a submodule of $\mathbb{Z}[i]$. Because $\mathbb{Z}$ is a principal ideal domain, every submodule of a free $\mathbb{Z}$-module of rank n is also free and its rank is at most n. This means that every ideal of R can be generated by two elements as a $\mathbb{Z}$-module, and hence as an ideal. Therefore, R is also a Noetherian Warfield domain, and hence a G-Dedekind domain.

Let $J = (p, pi)$. First we prove that $J^{-1} = \mathbb{Z}[i]$ and J is strongly Gorenstein projective. It is easy to see that the quotient field of R is $\mathbb{Q}[i]$. That $\mathbb{Z}[i]$ is included in J^{-1} is obvious since J is a common ideal of $\mathbb{Z}[i]$ and R. For the reverse inclusion, let $\frac{b}{a}+\frac{d}{c}i$ be an element in J^{-1}, where a,b,c,d are integers and $\gcd(a,b)=1, \gcd(c,d)=1$. So, $(\frac{pb}{a}+\frac{pd}{c}i)\in R$ and $(\frac{pb}{a}i-\frac{pd}{c})\in R$. This means that $a \mid b$ and $c \mid d$. Thus, $\frac{b}{a}+\frac{d}{c}i\in \mathbb{Z}[i]$. Therefore, $J^{-1}=\mathbb{Z}[i]$. To see that J is SG-projective, just notice that $J^{-1}=\mathbb{Z}[i]=R+Ri\cong pR+piR=J$ and an application of [7, Corollary 2.14] will give the result.

Secondly, let I be an ideal of R. If I is projective, it is surely SG-projective. Notice that R is a G-Dedekind domain with quotient field $\mathbb{Q}[i]$ and $\mathbb{Z}[i]$ is its integral closure in $\mathbb{Q}[i]$ and the ideal $J=(R:_K \mathbb{Q}[i])$ is a nonzero prime ideal of R. If I is not projective, by Theorem 3.3, I is also an ideal of $\mathbb{Z}[i]$. But $\mathbb{Z}[i]$ is a principal ideal domain. So $I=a\mathbb{Z}[i]$ for some $a\in I$. Any element of I can be written as the form $a(c+di)$, where $c,d\in\mathbb{Z}$. But $d=qp+m$ for some $m,q\in\mathbb{Z}$, where $0\leqslant m<p$. Therefore $a(c+di)=a(c+qpi+mi)=a(c+qpi)+mai$. Notice that $ai\in I$. So $I=(a,ai)\cong(p,pi)=J$. Therefore I is SG-projective from the first part. □

Considering the localization, we also have the following:

Example 3.5. Let p be a prime number, $S=\mathbb{Z}+p\mathbb{Z}i$, $P=(p,pi)$ be the maximal ideal of S, and $R=S_P$. Then every ideal of R is SG-projective.

Corollary 3.6. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R:_K T)$ be a nonzero prime ideal of R. If M is a G-projective R-module which has no projective direct summand, then M is isomorphic to a direct sum of some ideals of R.*

Proof. By Theorem 3.3, M is also a T-module. Since T is a Dedekind domain, M is isomorphic to a direct sum of some ideals of T. Notice that any ideal I of T, as an R-module, is isomorphic to some ideal aI $(a\in(R:_K T))$ of R. Therefore M is also isomorphic to a direct sum of some ideals of R. □

Lemma 3.7. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R:_K T)$ be a nonzero prime ideal of R. If M is a finitely generated G-projective R-module, then M is isomorphic to a direct sum of some ideals of R.*

Proof. Since R is Noetherian, M is a Noetherian R-module. So we have the decomposition: $M=M'\oplus P$, where M' has no projective direct summand of rank 1 and P is a direct sum of some projective modules of rank 1. Since any projective module of rank 1 is isomorphic to an ideal of R, P is already isomorphic to a direct sum of some ideals of R. But, by Corollary 3.6, M' is also isomorphic to a direct sum of some ideals of R. □

H. Bass proved in [1, Corollary 4.5] that if R is connected and Noetherian, then every non-finitely generated projective R-module is free. In particular, any non-finitely generated projective module over any Noetherian domain is free. Inspired by his work, we have the following:

Proposition 3.8. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R :_K T)$ be a nonzero prime ideal of R. If every projective ideal of R is principal, then any projective R-module is free.*

Proof. First we notice that, by [1, Corollary 4.5], any non-finitely generated projective R-module is free. So we only need to prove that any finitely generated projective R-module M is free. By Lemma 3.7, M is isomorphic to a direct sum of some ideals of R. As direct summands of the projective R-module, these ideals are also projective, and hence free by hypothesis. Therefore M is also free. □

Example 3.9. The domain $R = \mathbb{Z} + 2\mathbb{Z}i$ is a domain which has only two types of ideals: principal ideals and those ideals which are isomorphic to $I = (2, 2i)$. So any projective ideals of R is principal. Therefore any projective R-module is free.

Proof. As in Example 3.4, every ideal of R can be generated by two elements. Let J be any ideal of R. We can assume that $J \cong R\alpha + R\beta$, where $\alpha, \beta \in \mathbb{Z}[i]$ and $\gcd(\alpha, \beta) = 1$. Since $\gcd(\alpha, \beta) = 1$, there exist $u, v \in \mathbb{Z}[i]$ such that $u\alpha + v\beta = 1$. So $2iu\alpha + 2iv\beta = 2i$, which means that $2i \in R\alpha + R\beta$. Likewise, we also have $2 \in R\alpha + R\beta$. We assume that $\alpha = a + bi, \beta = c + di$, where $a, b, c, d \in \mathbb{Z}$.

Case 1: Both b and d are in $2\mathbb{Z}$. Since $\gcd(\alpha, \beta) = 1$, a and c can not be inside $2\mathbb{Z}$ at the same time. Without loss of generality, we assume that $\gcd(a, 2) = 1$. Notice that $2, 2i \in R\alpha + R\beta$, and so we have $R\alpha + R\beta = R(a + bi) + R(c + di) = Ra + Rc + 2Ri + 2R = R$. This means that $R\alpha + R\beta$ is principal.

Case 2: One of the imaginary parts of α and β is not inside $2\mathbb{Z}$. Without loss of generality, we assume that $b = 2k + 1$ and $d = 2l$, where k, l are integers. So $R\alpha + R\beta = R(a + i) + Rc + 2iR + 2R$. If $c \in 2\mathbb{Z}$, then a must also be inside $2\mathbb{Z}$ because $\gcd(\alpha, \beta) = 1$. So $R\alpha + R\beta = R(a + i) + Rc + 2iR + 2R = Ri$, which is principal. If c is not inside $2\mathbb{Z}$, then $R\alpha + R\beta = R(a + i) + Rc + 2iR + 2R = Ri + R \cong (2, 2i)$.

Case 3: Neither b nor d is inside $2\mathbb{Z}$. Under this condition, the imaginary part of $\alpha - \beta$ will be inside $2\mathbb{Z}$. Since $R\alpha + R\beta = R\alpha + R(\alpha - \beta)$, this case will go back to Case 2. □

Let R and T be domains as in Theorem 3.3 which put forward a sufficient condition for a G-projective R-module to be a T-module. The following theorem shows that this condition is also necessary.

Theorem 3.10. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R :_K T)$ be a nonzero prime ideal of R. If M is a G-projective R-module, then M is a T-module if and only if M has no projective direct summand.*

Proof. The sufficiency of this theorem is just Theorem 3.3. Now we assume that M is a T-module. Then any direct summand of M must also be a T-module. This will lead to the result that its projective direct summands will be also T-modules. But it is easy to see that R, and hence any free R-module can not be a T-module. So any nonzero projective R-module N can not be a T-module (otherwise the free $R_{\mathfrak{m}}$-module $N_{\mathfrak{m}}$ will be a $T_{\mathfrak{m}}$-module for the maximal ideal $\mathfrak{m} = (R :_K T)$ of R). This contradiction shows that M has no projective direct summand. □

Let R and T be domains as before. The following theorem gives a sufficient and necessary condition for a finitely generated G-projective R-module to be projective.

Theorem 3.11. *Let R be a G-Dedekind domain with quotient field K, T be its integral closure in K, and the ideal $(R :_K T)$ be a nonzero prime ideal of R. If M is a finite generated G-projective R-module, then M is projective if and only if any direct summand of M is not a T-module.*

Proof. Let N be any direct summand of M. If N is a T-module, then by Theorem 3.10, N is not projective, and hence M is not projective either. Conversely, assume that any nonzero direct summand of M is not a T-module. Then M is not a T-module and by Theorem 3.10, M has a projective direct summand P_1, say $M = P_1 \oplus M_1$. But any nonzero direct summand of M_1 is not a T-module either. So M_1 has a projective direct summand P_2, say $M_1 = P_2 \oplus M_2$. Inductively, we get that $M = P_1 \oplus P_2 \oplus \cdots \oplus P_k \oplus M_k$, where M_k is not a T-modules. Since M is Noetherian, some M_k must be zero. Therefore $M = P_1 \oplus P_2 \oplus \cdots \oplus P_k$ is projective. □

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