# COMPACT SUMS OF TOEPLITZ PRODUCTS ON WEIGHTED DIRICHLET SPACE OF THE UNIT BALL 

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#### Abstract

On the weighted Dirichlet space, by the Sobolev's embedding theorem, we characterize the compactness for operators which are finite sums of products of several Toeplitz operators. Moreover, the essential spectrum of Toeplitz operator is characterized.


## 1. Introduction

For any integer $n \geq 1$, let $\mathbb{B}_{n}$ denote the open unit ball in $\mathbb{C}_{n}$. The boundary of $\mathbb{B}_{n}$ is the sphere $\mathbb{S}_{n}$ and the closure of $\mathbb{B}_{n}$ in the Euclidean metric on $\mathbb{C}_{n}$ is denoted by $\overline{\mathbb{B}_{n}}$. Let $\sigma$ denote the rotation-invariant positive Borel measure on $\mathbb{S}_{n}$, which is normalized so that $\sigma\left(\mathbb{S}_{n}\right)=1$. Let $\mu$ be a positive regular Borel measure on the closed interval $[0,1]$ with $\mu([0,1])=1$, and 1 is in the support of $\mu$. Let $\nu$ be the product measure of $\mu$ and $\sigma$. For $f \in L^{1}\left(\mathbb{B}_{n}, d \nu\right)$, we have

$$
\int_{\mathbb{B}_{n}} f(z) d \nu(z)=\int_{0}^{1}\left(\int_{\mathbb{S}_{n}} f(r \xi) d \sigma(\xi)\right) d \mu(r)
$$

The weighted Sobolev space $W^{1,2}\left(\mathbb{B}_{n}, d \nu\right)$ is the completion of the space of all polynomials $f$ in $z$ and $\bar{z}$ with

$$
\|f\|^{2}=\left|\int_{\mathbb{B}_{n}} f(z) d \nu(z)\right|^{2}+\sum_{i=1}^{n} \int_{\mathbb{B}_{n}}\left\{\left|\frac{\partial f}{\partial z_{i}}\right|^{2}+\left|\frac{\partial f}{\partial \bar{z}_{i}}\right|^{2}\right\} d \nu(z)<\infty
$$

where $\partial$ is the weak partial derivative. Then $W^{1,2}\left(\mathbb{B}_{n}, d \nu\right)$ is a Hilbert space with inner product

$$
\langle f, g\rangle=\int_{\mathbb{B}_{n}} f(z) d \nu(z) \int_{\mathbb{B}_{n}} \overline{g(z)} d \nu(z)+\sum_{i=1}^{n}\left[\left\langle\frac{\partial f}{\partial z_{i}}, \frac{\partial g}{\partial z_{i}}\right\rangle_{2}+\left\langle\frac{\partial f}{\partial \bar{z}_{i}}, \frac{\partial g}{\partial \bar{z}_{i}}\right\rangle_{2}\right],
$$

where $\langle\cdot, \cdot\rangle_{2}$ denotes the inner product in the Bergman space $L_{a}^{2}\left(\mathbb{B}_{n}, d \nu\right)$. The weighted Dirichlet space $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$ is the closure of the space of all holomorphic

[^0]polynomials $f$ in $W^{1,2}\left(\mathbb{B}_{n}, d \nu\right)$ with $f(0)=0$, and let $Q$ denote the orthogonal projection from $W^{1,2}\left(\mathbb{B}_{n}, d \nu\right)$ onto $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$.

Let
$W^{1, \infty}\left(\mathbb{B}_{n}\right)=\left\{\varphi \in W^{1,2}\left(\mathbb{B}_{n}, d \nu\right), \varphi, \frac{\partial \varphi}{\partial z_{i}}, \frac{\partial \varphi}{\partial \overline{z_{i}}} \in L^{\infty}\left(\mathbb{B}_{n}, d \nu\right), i=1,2, \ldots, n\right\}$.
For $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$, a new norm on it is defined by

$$
\|f\|_{*}=\max _{z \in \mathbb{B}_{n}} \max \left\{|f(z)|,\left|\frac{\partial f}{\partial z_{i}}\right|,\left|\frac{\partial f}{\partial \bar{z}_{i}}\right|, i=1, \ldots, n\right\} .
$$

Given a function $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$, the Toeplitz operator $T_{f}$ with symbol $f$ is defined by $T_{f} g=Q(f g)$ for functions $g \in \mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$, then $T_{f}$ is bounded and $\left\|T_{f}\right\| \leq \sqrt{n}\|f\|_{*}$. The Hankel operator $H_{f}$ is defined by $H_{f} g=(I-Q)(f g)$ for functions $g \in \mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$.

In 1996, T. Nakazi and M. Yamada [5] introduced four Riesz's functions. With the help of Riesz's functions, T. Nakazi and R. Yoneda [6] characterized completeness of weighted Bergman space $L_{a}^{2}(\mathbb{D}, d \sigma d \theta / 2 \pi)$, where $d \sigma$ is a positive finite Borel measure on $[0,1)$. They proved that when $\phi$ is continuous on the closed disk, then $T_{\phi}$ is compact if and only if $\phi=0$ on $\partial \mathbb{D}$. T. Le $[1,2]$ proved the characterization is also true for the general rotation-invariant positive Borel measure on the open unit ball and the open unit polydisk.

On the Dirichlet space, Y. J. Lee [3] proved that a compact Toeplitz operator with a special symbol on the Dirichlet space must be zero. In [7], Yu proved that this characterization is true for all Toeplitz operators on the Dirichlet space. In [4], by using complete different arguments, the compactness for operators which are finite sums of products of several Toeplitz operators is characterized on the weighted Bergman and Dirichlet spaces.

In this paper, by the Sobolev's embedding theorem, we characterize the compactness for operators which are finite sums of products of several Toeplitz operators on weighted Dirichlet space of the unit ball. Moreover, the essential spectrum of Toeplitz operator is characterized.

## 2. Preliminaries

For $f \in L^{2}\left(\mathbb{B}_{n}, d \nu\right)$, we put

$$
\|f\|_{2}^{2}=\int_{\mathbb{B}_{n}}|f(z)|^{2} d \nu(z)
$$

The Bergman space $L_{a}^{2}\left(\mathbb{B}_{n}, d \nu\right)$ is the closure of the space of all holomorphic polynomials in $L^{2}\left(\overline{\mathbb{B}_{n}}, d \nu\right)$. Let $P$ denote the orthogonal projection from $L^{2}$ onto $L_{a}^{2}$.

Lemma 2.1. Suppose $f$ is in $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$. Then there exists a positive constant $C$ such that $|f(z)| \leq C\|f\|$.

Proof. Since $f$ is in $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$, by the mean value property, we have

$$
f(z)=\int_{\mathbb{S}_{n}} f(z+r \xi) d \sigma(\xi)
$$

where $0<r<1$.
Hence,

$$
|f(z)| \leq \int_{\mathbb{S}_{n}}|f(z+r \xi)| d \sigma(\xi)
$$

and then

$$
\begin{aligned}
\int_{0}^{1-|z|}|f(z)| d \mu(r) & \leq \int_{0}^{1-|z|} \int_{\mathbb{S}_{n}}|f(z+r \xi)| d \sigma(\xi) d \mu(r) \\
& \leq \int_{\mathbb{B}_{n}}|f(w)| d \nu(w) \\
& \leq\left[\int_{\mathbb{B}_{n}}|f(w)|^{2} d \nu(w)\right]^{\frac{1}{2}}
\end{aligned}
$$

Since 0 is in the support of $\mu$, it is clear that $\int_{0}^{1-|z|} d \mu(r) \neq 0$. Let $f=$ $\sum_{m \in \mathbb{Z}^{+n}} a_{m} z^{m}$, a direct computation gives that

$$
\begin{aligned}
& \|f\|_{2}^{2}=\sum_{m \in \mathbb{Z}^{+n}}\left|a_{m}\right|^{2} \alpha_{|m|} \frac{(n-1)!m!}{(n+|m|-1)!} \\
& \|f\|^{2}=\sum_{m \in \mathbb{Z}^{+n}}\left|a_{m}\right|^{2} \alpha_{|m|-1} \frac{(n-1)!m!|m|}{(n+|m|-2)!}
\end{aligned}
$$

Thus, with the fact that $\|f\|_{2} \leq\|f\|$, we can find a positive constant $C$ (depending only on $z$ and $\mu$ ) such that

$$
|f(z)| \leq C\left[\int_{\mathbb{B}_{n}}|f(w)|^{2} d \nu(w)\right]^{\frac{1}{2}}=C\|f\|_{2} \leq C\|f\|
$$

As a consequence of the above Lemma, each point evaluation is verified to be a bounded linear functional on $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$. Hence, for each $z \in \mathbb{B}_{n}$, there exists a unique function $R_{z}(w) \in \mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$ which has the following reproducing property

$$
g(z)=\left\langle g, R_{z}\right\rangle
$$

for every $g \in \mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$.
For multi-indexes $m, k \in \mathbb{N}^{n}$, from a direct calculation, we have

$$
\begin{aligned}
\left\langle z^{m}, z^{k}\right\rangle & =\int_{\overline{\mathbb{B}_{n}}} \sum_{i=1}^{n} \frac{\partial z^{m}}{\partial z_{i}} \frac{\overline{\partial z^{k}}}{\partial z_{i}} d \nu(z) \\
& =\int_{0}^{1} r^{|m|+|k|-2} d \mu(r) \int_{\mathbb{S}_{n}} m_{i} k_{i} \xi^{m-d_{i}} \bar{\xi}^{k-d i} d \sigma(\xi)
\end{aligned}
$$

$$
= \begin{cases}0 & \text { if } \quad m \neq k \\ \frac{|m|(n-1)!m!}{(n-2+|m|)!} \int_{0}^{1} r^{2(|m|-1)} d \mu(r) & \text { if } \quad m=k\end{cases}
$$

where $d_{k}$ is the ordered $n$-tuple that has 1 in the $k$-th spot and 0 everywhere else. For $s \in \mathbb{N}$, let $\alpha_{s}=\int_{0}^{1} r^{2 s} d \mu(r)$. For $m \in \mathbb{N}^{n}$, put

$$
e_{m}(z)=\left[\frac{(n+|m|-2)!}{|m|(n-1)!m!\alpha_{|m|-1}}\right]^{\frac{1}{2}} z^{m}
$$

It follows that the set $\left\{e_{m}: m \in \mathbb{N}^{n}\right\}$ is an orthonormal basis for $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$. Then

$$
R(z, w)=\sum_{m \in \mathbb{N}^{m}} e_{m}(z) \overline{e_{m}(w)}=\sum_{m \in \mathbb{N}^{m}} \frac{(n+|m|-2)!}{|m|(n-1)!m!\alpha_{|m|-1}} z^{m} \bar{w}^{m}
$$

Lemma 2.2. Suppose $f \in W^{1, \infty}\left(B_{n}\right)$. Then $f$ can be extended to be a continuous function on $\overline{B_{n}}$.

Proof. With the fact that

$$
|f(z)-f(w)| \leq 2 n|z-w| \max _{1 \leq j \leq n} \sup _{z \in B_{n}}\left\{\left|\frac{\partial f}{\partial x_{j}}\right|,\left|\frac{\partial f}{\partial y_{j}}\right|\right\} .
$$

Note that $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$, then it follows $\max _{1 \leq j \leq n} \sup _{z \in B_{n}}\left\{\left|\frac{\partial f}{\partial x_{j}}\right|,\left|\frac{\partial f}{\partial y_{j}}\right|\right\}<\infty$. Thus we get $f$ is uniformly continuous on $B_{n}$. Since $f$ is bounded and uniformly continuous, then it possesses a unique, bounded, continuous extension to the closure $\overline{B_{n}}$ of $B_{n}$.

It is well known that the Sobolev's embedding theorem closely connects the Dirichlet space and Bergman space. In the following lemma, we will proof part of Sobolev's embedding theorem which is in the weighted case.
Lemma 2.3. Suppose $f_{k} \xrightarrow{w} 0$ in the weighted Dirichlet space, and $\left\|f_{k}\right\|=1$ for every $k=1,2, \ldots$. Then $\left\|f_{k}\right\|_{2} \rightarrow 0$ as $k \rightarrow \infty$.
Proof. Suppose $f_{k}(z)=\sum_{m \in \mathbb{Z}^{+n}} b_{m}^{(k)} z^{m}$. Then for given $s \in \mathbb{Z}^{+n}$,

$$
\begin{aligned}
\left\langle f_{k}, z^{s}\right\rangle & =\left\langle\sum_{m \in \mathbb{Z}^{+n}} b_{m}^{(k)} z^{m}, z^{s}\right\rangle \\
& =\left\langle b_{s}^{(k)} z^{s}, z^{s}\right\rangle \\
& =b_{s}^{(k)} \int_{0}^{1} r^{2(|s|-1) d \mu(r)} \int_{\mathbb{S}_{n}} \sum_{i=1}^{n} s_{i}^{2}\left|\xi^{s-d i}\right|^{2} d \sigma(\xi) \\
& =b_{s}^{(k)} \alpha_{|s|-1} \frac{|s|(n-1)!s!}{(n+|s|-2)!}
\end{aligned}
$$

Since $f_{k} \xrightarrow{w} 0$ in $\mathcal{D}$, then $\left\langle f_{k}, z^{s}\right\rangle \rightarrow 0$ as $k \rightarrow \infty$. It follows from the fact that 1 is in the support of $\mu, \alpha_{|s|-1}>0$ for every multi-index $s \in \mathbb{Z}^{+n}$. Then $b_{s}^{(k)} \rightarrow 0$ as $k \rightarrow \infty$.

Note that $\left\|f_{k}\right\|^{2}=\sum_{j=1}^{n}\left\|\frac{\partial f_{k}}{\partial z_{j}}\right\|_{2}^{2}=1$, then

$$
\sum_{m \in \mathbb{Z}^{+n}-0}\left|b_{m}^{(k)}\right|^{2} \alpha_{|m|-1} \frac{|m|(n-1)!m!}{(n+|m|-2)!}=1(\forall k)
$$

Let $\epsilon>0$ be given. It follows that for sufficient large $|m|$, there is an integer $N_{0}$ independent of $k$ such that for all $m \in \mathbb{Z}^{+n}$ with $|m|>N_{0}$, we have

$$
\begin{aligned}
& \sum_{m \in \mathbb{Z}^{+n}-0}\left|b_{m}^{(k)}\right|^{2} \alpha_{|m|-1} \frac{(n-1)!m!}{(n+|m|-1)!} \\
= & \sum_{m \in \mathbb{Z}^{+n}-0}\left|b_{m}^{(k)}\right|^{2} \alpha_{|m|-1} \frac{|m|(n-1)!m!}{(n+|m|-2)!} \frac{1}{(n+|m|-1)|m|} \\
\leq & \frac{1}{(n+|m|-1)|m|}<\frac{\varepsilon}{2} .
\end{aligned}
$$

Since $b_{m}^{(k)} \rightarrow 0$, there is an integer $K_{0}$ such that for all $k>K_{0}$, we have

$$
\sum_{|m| \leq N_{0}}\left|b_{m}^{(k)}\right|^{2} \alpha_{|m|} \frac{(n-1)!m!}{(n+|m|-1)!}<\frac{\varepsilon}{2}
$$

It follows that

$$
\left\|f_{k}\right\|_{2}^{2}=\sum_{m \in \mathbb{Z}^{+n}}\left|b_{m}^{(k)}\right|^{2} \alpha_{|m|} \frac{(n-1)!m!}{(n+|m|-1)!}<\varepsilon\left(k>K_{0}\right)
$$

Then we get the desired result.

## 3. Compact sums of Toeplitz products

In this section, we characterize the compactness of operators which are finite sums of products of several Toeplitz operators on the weighted Dirichlet space. Our result generalizes the result of [4] to the high dimensional cases with a complete different argument. First, we will characterize the compactness of Toeplitz operators on weighted Dirichlet space of the unit ball. In the following, Toeplitz operator and Hankel operator on the weighted Bergman space over the unit ball are denoted by $\tilde{T}_{f}$ and $\tilde{H}_{f}$.

Theorem 3.1. Suppose that $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$. Then $T_{f}$ is compact if and only if $f(\xi)=0$ for almost every $|\xi|=1$.

Proof. By Lemma 2.2, we have $f$ can be extended to be a continuous function on $\bar{B}_{n}$. On weighted Bergman space of the unit ball, it is shown in [1] that Toeplitz operator $\tilde{T}_{f}$ is compact if and only if $f(\xi)=0$ for almost every $|\xi|=1$.

Suppose that $f(\zeta)=0$ for all $|\zeta|=1$, it is shown in [2] that $\widetilde{T}_{|f|^{2}}$ is a compact Toeplitz operator on $L_{a}^{2}(\mathbb{B}, d \nu)$. It suffices to show that for all $\left\{g_{k}\right\} \subseteq \mathcal{D}$,
$\left\|g_{k}\right\|=1, g_{k} \xrightarrow{w} 0$,

$$
\left\|f g_{k}\right\|^{2}=\left|\int_{\overline{\mathbb{B}_{n}}} f(z) g_{k}(z) d \nu(z)\right|^{2}+\sum_{i=1}^{n}\left\{\left\|\frac{\partial\left(f g_{k}\right)}{\partial z_{i}}\right\|_{2}^{2}+\left\|\frac{\partial\left(f g_{k}\right)}{\partial \overline{z_{i}}}\right\|_{2}^{2}\right\} \rightarrow 0
$$

Since $g_{k} \xrightarrow{w} 0$ on $\mathcal{D}$, then $\frac{\partial g_{k}}{\partial z_{i}} \xrightarrow{w} 0$ on the Bergman space $L_{a}^{2}\left(\overline{\mathbb{B}_{n}}, d \nu\right)$ for every $i=1,2, \ldots, n$.

There exists a constant $M>0$ such that $|f(z)| \leq M$ for all $z \in \overline{\mathbb{B}_{n}}$ since $f$ is continuous on $\overline{\mathbb{B}_{n}}$. According to Lemma 2.3,

$$
\sum_{i=1}^{n}\left\|\frac{\partial\left(f g_{k}\right)}{\partial \overline{z_{i}}}\right\|_{2}^{2}=\sum_{i=1}^{n}\left\langle\frac{\partial f}{\partial \overline{z_{i}}} g_{k}, \frac{\partial f}{\partial \bar{z}_{i}} g_{k}\right\rangle \leq n\|f\|_{*}^{2}\left\|g_{k}\right\|_{2}^{2} \rightarrow 0
$$

and

$$
\left|\int_{\overline{\mathbb{B}_{n}}} f(z) g_{k}(z) d \nu(z)\right|^{2} \leq M^{2}\left\|g_{k}\right\|_{2}^{2} \rightarrow 0
$$

as $k \rightarrow \infty$. Also,

$$
\begin{aligned}
& \left\|\frac{\partial\left(f g_{k}\right)}{\partial z_{i}}\right\|_{2}^{2} \\
= & \left\langle\frac{\partial f}{\partial z_{j}} g_{k}, \frac{\partial f}{\partial z_{j}} g_{k}\right\rangle_{2}+\left\langle\frac{\partial f}{\partial z_{j}} g_{k}, f \frac{\partial g_{k}}{\partial z_{j}}\right\rangle_{2}+\left\langle f \frac{\partial g_{k}}{\partial z_{j}}, \frac{\partial f}{\partial z_{j}} g_{k}\right\rangle_{2}+\left\langle f \frac{\partial g_{k}}{\partial z_{j}}, f \frac{\partial g_{k}}{\partial z_{j}}\right\rangle_{2} \\
= & \left\langle\frac{\partial f}{\partial z_{j}} g_{k}, \frac{\partial f}{\partial z_{j}} g_{k}\right\rangle_{2}+\left\langle\frac{\partial f}{\partial z_{j}} g_{k}, f \frac{\partial g_{k}}{\partial z_{j}}\right\rangle_{2}+\left\langle f \frac{\partial g_{k}}{\partial z_{j}}, \frac{\partial f}{\partial z_{j}} g_{k}\right\rangle_{2}+\left\langle\widetilde{T}_{|f|^{2}} \frac{\partial g_{k}}{\partial z_{j}}, \frac{\partial g_{k}}{\partial z_{j}}\right\rangle_{2} .
\end{aligned}
$$

The proof of $\sum_{i=1}^{n}\left\|\frac{\partial\left(f g_{k}\right)}{\partial z_{i}}\right\|_{2}^{2} \rightarrow 0$ follows from Lemma 2.3 and the fact that $\widetilde{T}_{|f|^{2}}$ is a compact Toeplitz operator on $L_{a}^{2}(\overline{\mathbb{B}}, d \nu)$.

Conversely, suppose that $T_{f}$ is compact. For any positive integer $M$, we have

$$
\begin{aligned}
& \sum_{|m|=M}\left\langle T_{f} e_{m}, e_{m}\right\rangle \\
= & \sum_{|m|=M} \frac{(n+|m|-2)!}{|m|(n-1)!m!\alpha_{M-1}} \int_{\overline{\mathbb{B}_{n}}} \sum_{i=1}^{n}\left[\frac{\partial f(z)}{\partial z_{i}} z^{m}+f(z) m_{i} z^{m-d_{i}}\right] m_{i} \bar{z}^{m-d_{i}} d \nu(z) \\
= & \sum_{i=1}^{n} \frac{(n+M-2)!}{(n-1)!M!\alpha_{M-1}} \int_{\overline{\mathbb{B}_{n}}}\left(\frac{\partial f(z)}{\partial z_{i}} z_{i}+f(z) m_{i}\right) \\
& {\left[\sum_{|m|=M, m_{i}>0} \frac{(M-1)!}{m_{1}!\cdots m_{i-1}!\cdots m_{n}!}\left|z_{1}\right|^{2 m_{1}} \cdots\left|z_{i}\right|^{2\left(m_{i}-1\right)} \cdots\left|z_{n}\right|^{2 m_{n}}\right] d \nu(z) } \\
= & \sum_{i=1}^{n} \frac{(n+M-2)!}{(n-1)!M!\alpha_{M-1}} \int_{\overline{\mathbb{B}_{n}}}\left(\left|z_{1}\right|^{2}+\cdots+\left|z_{n}\right|^{2}\right)^{M-1}\left(\frac{\partial f(z)}{\partial z_{i}} z_{i}+f(z) m_{i}\right) d \nu(z) \\
= & \frac{(n+M-2)!}{(n-1)!(M-1)!\alpha_{M-1}} \int_{0}^{1} \int_{\mathbb{S}_{n}} f(r \xi) d \sigma(\xi) r^{2(M-1)} d \mu(r)
\end{aligned}
$$

$$
+\frac{(n+M-2)!}{(n-1)!M!\alpha_{M-1}} \int_{\overline{\mathbb{B}_{n}}} \sum_{i=1}^{n}\left(\frac{\partial f}{\partial z_{i}} z_{i}\right) r^{2(M-1)} d \nu(z) .
$$

Since $f$ is in $\Omega$, there exists a positive integer $C>0$ such that $\left|\frac{\partial f}{\partial z_{i}}\right|<C$ for every $i=1,2, \ldots, n$. Let $\varepsilon>0$ be given. There is an integer $M_{\varepsilon}$ such that for all $m \in \mathbb{N}^{n}$ with $|m|>M_{\varepsilon}$, we have $\left\langle T_{f} e_{m}, e_{m}\right\rangle<\varepsilon$, and $\frac{n C}{|m|}<\varepsilon$. Thus for any $M>M_{\varepsilon}$, then

$$
\begin{aligned}
& \left|\frac{1}{\alpha_{M-1}} \int_{0}^{1} \int_{\mathbb{S}_{n}} f(r \xi) d \sigma(\xi) r^{2(M-1)} d \mu(r)\right| \\
\leq & \frac{(n-1)!(M-1)!}{(n+M-2)!} \sum_{|m|=M}\left|\left\langle T_{f} e_{m}, e_{m}\right\rangle\right|+\frac{1}{M} \int_{\overline{\mathbb{B}_{n}}} \sum_{i=1}^{n}\left|\frac{\partial f}{\partial z_{i}} r^{2(M-1)}\right| d \nu(z) \\
\leq & \frac{(n-1)!(M-1)!}{(n+M-2)!} \frac{(n+M-1)!}{(n-1)!M!} \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\
\leq & \varepsilon+\frac{n-1}{2 M} \varepsilon .
\end{aligned}
$$

Then we have

$$
\lim _{M \rightarrow \infty} \frac{1}{\alpha_{M-1}} \int_{0}^{1} \int_{\mathbb{S}_{n}} f(r \xi) d \sigma(\xi) r^{2(M-1)} d \mu(r)=0 .
$$

For each $0 \leq r \leq 1$, put $\varphi(r)=\int_{\mathbb{S}_{n}} f(r \xi) d \sigma(\xi)$. By Proposition 3.1 in [1], it is shown that

$$
\varphi(1)=\int_{\mathbb{S}_{n}} f(\xi) d \sigma(\xi)=0 .
$$

For all multi-indexes $l_{1}, l_{2} \in \mathbb{N}^{n}$, a direct computation gives that $T_{f z^{l_{1} \bar{z}^{l_{2}}}}=$ $T_{\bar{z}^{l_{2}}} T_{f} T_{z^{l_{1}}}$ which is also compact. Thus we have $\int_{\mathbb{S}_{n}} f(\xi) \xi^{l_{1}} \bar{\xi}^{l_{2}} d \sigma(\xi)=0$. Since this is true for all multi-indexes $l_{1}$ and $l_{2}$, we have $f(\xi)=0$ for $\sigma$-almost all $\xi \in \mathbb{S}_{n}$.

Given a function $f \in L^{\infty}\left(\mathbb{B}_{n}, d \nu(z)\right)$, we define the Hankel operator on weighted Bergman space

$$
\tilde{H}_{f}: L_{a}^{2}(d \nu) \rightarrow\left(L_{a}^{2}(d \nu)\right)^{\perp}=L^{2}\left(\mathbb{B}_{n}, d \nu(z)\right) \ominus L_{a}^{2}(d \nu(z))
$$

by

$$
\tilde{H}_{f} g=(I-P)(f g), \quad g \in L_{a}^{2}(d \nu(z))
$$

For $m \in \mathbb{Z}^{+n}$ and $z \in \mathbb{B}_{n}$, put

$$
\widetilde{e}_{m}(z)=\left(\frac{(n-1+|m|)!}{(n-1)!m!\alpha_{|m|}}\right)^{\frac{1}{2}} z^{m} .
$$

Then the set $\left\{\widetilde{e}_{m}: m \in \mathbb{Z}^{+n}\right\}$ is an orthonormal basis for $L_{a}^{2}\left(\overline{\mathbb{B}_{n}}\right)$ by [2]. Put

$$
a_{m}=\frac{(n-1+|m|)!}{(n-1)!m!\alpha_{|m|}}
$$

then $\widetilde{e}_{m}(z)=a_{m} z^{m}$.
Lemma 3.2. For multi-indexes $\rho, s \in \mathbb{Z}^{+n}$, let $f=z^{\rho} \bar{z}^{s}$ on the closed unit ball. Then $\tilde{H}_{f}$ is compact.
Proof. For multi-indexes $m, k \in \mathbb{N}^{n}$ and $m \neq k$, then

$$
\begin{aligned}
& \left\langle\widetilde{H}_{f}^{*} \widetilde{H}_{f} \widetilde{e}_{m}(z), \widetilde{e}_{k}(z)\right\rangle_{2} \\
= & \left\langle f \widetilde{e}_{m}(z),(I-P)\left(f \widetilde{e}_{k}(z)\right)\right\rangle_{2} \\
= & \left\langle f \widetilde{e}_{m}(z), f \widetilde{e}_{k}(z)\right\rangle_{2}-\left\langle f \widetilde{e}_{m}(z), P\left(f \widetilde{e}_{k}(z),\right)\right\rangle_{2} \\
= & -a_{m} a_{k}\left\langle z^{m+\rho} \bar{z}^{s}, P\left(z^{\rho+k} \bar{z}^{s}\right)\right\rangle_{2} .
\end{aligned}
$$

Since

$$
\begin{aligned}
P\left(z^{\rho+k} \bar{z}^{s}\right) & =\sum_{q \in \mathbb{Z}+n}\left\langle z^{\rho+k} \bar{z}^{s}, a_{q} z^{q}\right\rangle_{2} \widetilde{e}_{q}(z) \\
& =\left\langle z^{\rho+k} \bar{z}^{s}, a_{\rho+k-s} z^{\rho+k-s}\right\rangle_{2} \widetilde{e}_{\rho+k-s}(z)
\end{aligned}
$$

then

$$
\begin{aligned}
& \left\langle\tilde{H}_{f}^{*} \tilde{H}_{f} \widetilde{e}_{m}(z), \widetilde{e}_{k}(z)\right\rangle_{2} \\
= & -a_{m} a_{k} a_{\rho+k-s}^{2}\left\langle z^{m+\rho} \bar{z}^{s}, z^{\rho+k-s}\right\rangle_{2}\left\langle z^{\rho+k} \bar{z}^{s}, a_{\rho+k-s} z^{\rho+k-s}\right\rangle_{2} \\
= & 0 .
\end{aligned}
$$

Thus $\tilde{H}_{f}{ }^{*} \tilde{H}_{f}$ is a diagonal operator on $L_{a}^{2}\left(\overline{\mathbb{B}_{n}}, d \nu\right)$ with the respect to the standard orthonormal basis $\left\{\widetilde{e}_{m}, m \in \mathbb{Z}^{+n}\right\}$. The eigenvalues of $\tilde{H}_{f}{ }^{*} \tilde{H}_{f}$ are given by

$$
\begin{aligned}
\lambda_{m} & =\left\langle\tilde{H}_{f}{ }^{*} \tilde{H}_{f} \widetilde{e}_{m}(z), \widetilde{e}_{m}(z)\right\rangle_{2} \\
& =\left\|f \widetilde{e}_{m}\right\|_{2}^{2}-\left\|P\left(f \widetilde{e}_{m}\right)\right\|_{2}^{2}
\end{aligned}
$$

for all $m \in \mathbb{Z}^{+n}$. Since

$$
\begin{aligned}
\left\|f \widetilde{e}_{m}\right\|_{2}^{2} & =\frac{(n-1+|m|)!}{(n-1)!m!\alpha_{|m|}} \int_{\overline{\mathbb{B}_{n}}}\left|z^{m+\rho+s}\right|^{2} d \nu(z) \\
& =\frac{(n-1+|m|)!}{(n-1)!m!\alpha_{|m|}} \int_{0}^{1} r^{2|m+\rho+s|} d \mu(r) \int_{\mathbb{S}_{n}}\left|\xi^{m+\rho+s}\right|^{2} d \sigma(\xi) \\
& =\frac{(n-1+|m|)!}{(n-1)!m!\alpha_{|m|}} \alpha_{|m+\rho+s|} \frac{(n-1)!(m+\rho+s)!}{(n+|m|+|\rho|+|s|-1)!} \\
& =\frac{\alpha_{|m+\rho+s|}(m+\rho+s)!(n-1+|m|)!}{\alpha_{|m|} m!(n+|m|+|\rho|+|s|-1)!}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\|P\left(f \widetilde{e}_{m}\right)\right\|_{2}^{2} \\
= & \left|\left\langle f \widetilde{e}_{m}, \widetilde{e}_{m+\rho-s}\right\rangle_{2}\right|^{2}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{(n-1+|m|)!}{(n-1)!m!\alpha_{|m|}} \frac{(n-1+|m|+|\rho|-|s|)!}{(n-1)!(m+\rho-s)!\alpha_{|m|+|\rho|-|s|}}\left(\int_{\overline{\mathbb{B}_{n}}}\left|z^{m+\rho}\right|^{2} d \nu(z)\right)^{2} \\
& =\frac{(m+\rho)!^{2}}{m!(m+\rho-s)!} \frac{(n-1+|m|+|\rho|-|s|)!(n-1+|m|)!\alpha_{|m|+|\rho|}^{2}}{(n-1+|m|+|\rho|)!{ }^{2} \alpha_{|m|+|\rho|-|s|} \alpha_{|m|}}
\end{aligned}
$$

It suffice to prove that $\left\|f \widetilde{e}_{m}\right\|_{2}^{2}-\left\|P\left(f \widetilde{e}_{m}\right)\right\|_{2}^{2} \rightarrow 0$ when $|m| \rightarrow \infty$. By Lemma 3 in [6], we have $\lim _{m \rightarrow \infty} \frac{\alpha_{|m+\rho+s|}}{\alpha_{|m|}}=1$ and $\lim _{m \rightarrow \infty} \frac{\alpha_{|m|+|\rho|}^{2}}{\alpha_{|m|} \alpha_{|m|+|\rho|-|s|}}=1$. The meaning of $m \rightarrow \infty$ is $|m| \rightarrow \infty$. We consider two types of sequences $\{m(k)\}_{k=1}^{\infty}$ in $\mathbb{N}^{n}$ where $m(k)=\left(m_{1}(k), \ldots, m_{n}(k)\right)$ such that $|m(k)| \rightarrow \infty$.

Type 1: some $m_{j}(k)_{k=1}^{\infty}$ is bounded. Without loss of generality, we denote $m=\left(1, m_{2}, \ldots, m_{n}\right)$, which $m_{2}, \ldots, m_{n} \rightarrow \infty$. It follows that

$$
\begin{aligned}
\lim _{m \rightarrow \infty}\left\|f \widetilde{e}_{m}\right\|_{2}^{2} & =\lim _{m \rightarrow \infty} \frac{\alpha_{|m+\rho+s|}(m+\rho+s)!(n-1+|m|)!}{\alpha_{|m|} m!(n+|m|+|\rho|+|s|-1)!} \\
& =\lim _{m \rightarrow \infty} \frac{\left(m_{1}+1\right) \cdots\left(m_{n}+\rho_{n}+s_{n}\right)}{(n+|m|)(n+|m|+|\rho|+|s|-1)} \\
& \leq \lim _{m \rightarrow \infty} \frac{m_{1}+1}{n+|m|}=\lim _{m \rightarrow \infty} \frac{2}{n+|m|}=0 .
\end{aligned}
$$

Also we have $\lim _{m \rightarrow \infty}\left\|P\left(f \widetilde{e}_{m}\right)\right\|_{2}^{2}=0$ in the same way.
Type 2: all $m_{j}(k) \rightarrow \infty$. Stirling's approximation of the Gamma functions gives

$$
\lim _{m \rightarrow \infty} \frac{(m+\rho)!^{2}}{(m+\rho+s)!(m+\rho-s)!}=1
$$

and

$$
\lim _{m \rightarrow \infty} \frac{(n-1+|m|+|\rho|)!^{2}}{(n-1+|m|+|\rho|+|s|)!(n-1+|m|+|\rho|-|s|)!}=1
$$

The above two limits imply $\lim _{k \rightarrow \infty} \frac{\left\|P\left(f \tilde{e}_{m(k)}\right)\right\|_{2}^{2}}{\left\|f \tilde{e}_{m(k)}\right\|_{2}^{2}}=1$. Since $\left\|f \widetilde{e}_{m(k)}\right\|_{2}^{2}$ is bounded, we conclude that

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \lambda_{m(k)} & =\lim _{k \rightarrow \infty}\left(\left\|f \widetilde{e}_{m(k)}\right\|_{2}^{2}-\left\|P\left(f \widetilde{e}_{m(k)}\right)\right\|_{2}^{2}\right) \\
& =\lim _{k \rightarrow \infty}\left\|f \widetilde{e}_{m(k)}\right\|_{2}^{2}\left(\frac{\left\|P\left(f \widetilde{e}_{m(k)}\right)\right\|_{2}^{2}}{\left\|f \widetilde{e}_{m(k)}\right\|_{2}^{2}}-1\right)=0
\end{aligned}
$$

Of course these two types still do not cover all possibilities. However, any sequence $\{m(k)\}_{k=1}^{\infty}$ with $|m(k)| \rightarrow \infty$ contains a subsequence of either Type 1 or Type 2. Consequently, any sequence $\left\{\lambda_{m(k)}\right\}_{k=1}^{\infty}$ contains a subsequence that converge to zero. It is an elementary fact that this implies $\lim _{|m| \rightarrow \infty} \lambda_{m}=0$ as desired. Therefore, $\tilde{H}_{f}{ }^{*} \tilde{H}_{f}$ is a compact operator, which implies that $\tilde{H}_{f}$ is also a compact operator.

Thus, for any polynomial $p(z, \bar{z}), \tilde{H}_{p}$ is compact. Since any function $f$ in $C\left(\overline{\mathbb{B}_{n}}\right)$ can be uniformly approximated by polynomials, we conclude that $\tilde{H}_{f}$ is also a compact operator. With this, we get the following theorem.
Theorem 3.3. Suppose $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$. Then $H_{f}$ is compact on the weighted Dirichlet space $\mathcal{D}(d \nu)$.
Proof. For any $h \in \mathcal{D}$ and $g \in \mathcal{D}^{\perp}$,

$$
\begin{aligned}
\left\langle H_{f} h, g\right\rangle= & \langle f h, g\rangle \\
= & \int_{\overline{\mathbb{B}_{n}}} f(z) h(z) d \nu(z) \int_{\overline{\mathbb{B}_{n}}} \overline{g(z)} d \nu(z) \\
& +\sum_{i=1}^{n}\left[\left\langle\frac{\partial f}{\partial z_{i}} h, \frac{\partial g}{\partial z_{i}}\right\rangle_{2}+\left\langle f \frac{\partial h}{\partial z_{i}}, \frac{\partial g}{\partial z_{i}}\right\rangle_{2}+\left\langle\frac{\partial f}{\partial \overline{z_{i}}} h, \frac{\partial g}{\partial \overline{z_{i}}}\right\rangle_{2}\right] \\
= & \int_{\overline{\mathbb{B}_{n}}} f(z) h(z) d \nu(z) \int_{\overline{\mathbb{B}_{n}}} \overline{g(z)} d \nu(z) \\
& +\sum_{i=1}^{n}\left[\left\langle\frac{\partial f}{\partial z_{i}} h, \frac{\partial g}{\partial z_{i}}\right\rangle_{2}+\left\langle\widetilde{H}_{f} \frac{\partial h}{\partial z_{i}}, \frac{\partial g}{\partial z_{i}}\right\rangle_{2}+\left\langle\frac{\partial f}{\partial \bar{z}_{i}} h, \frac{\partial g}{\partial \bar{z}_{i}}\right\rangle_{2}\right] .
\end{aligned}
$$

Suppose $\left\{h_{k}\right\} \xrightarrow{w} 0$ on $\mathcal{D}$, and $\left\|h_{k}\right\|=1$ for any $k=1,2, \ldots$, then $\left\|h_{k}\right\|_{2} \rightarrow 0$ as $k \rightarrow \infty$.

$$
\left\|H_{f} h_{k}\right\| \leq\|f\|_{2}\left\|h_{k}\right\|_{2}+2 \sqrt{n}\|f\|_{*}\left\|h_{k}\right\|_{2}+\left[\sum_{i=1}^{n}\left\|\widetilde{H}_{f} \frac{\partial h_{k}}{\partial z_{i}}\right\|_{2}^{2}\right]^{\frac{1}{2}}
$$

Since $\left\{h_{k}\right\} \xrightarrow{w} 0$ on $\mathcal{D}$, then $\left\{\frac{\partial h_{k}}{\partial z_{i}}\right\} \xrightarrow{w} 0$ on the Bergman space $L_{a}^{2}\left(\overline{\mathbb{B}_{n}}, d \nu\right)$. By Lemma 3.2, we get that $\widetilde{H}_{f}$ is compact on $L_{a}^{2}\left(\overline{\mathbb{B}_{n}}, d \nu\right)$. It is clear that for any $i=1,2, \ldots, n,\left\|\widetilde{H}_{f} \frac{\partial h_{k}}{\partial z_{i}}\right\|_{2} \rightarrow 0$ as $k \rightarrow \infty$.

Then we get $\left\|H_{f} h_{k}\right\| \rightarrow 0$. This completes the proof of the theorem.
Now for $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$, define an operator $W_{f}: \mathcal{D}^{\perp} \rightarrow \mathcal{D}$ by

$$
W_{f} g=Q(f g), \quad g \in \mathcal{D}^{\perp}
$$

It is easy to see that $T_{f g}=T_{f} T_{g}+W_{f} H_{g}$. With above equation, we have the following corollary.
Corollary 3.4. Suppose $f, g \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$. Then $T_{f g}-T_{f} T_{g}$ is compact on $\mathcal{D}$.
More generally, we show that a sum of products of Toeplitz operators can be a compact perturbation of a Toeplitz operator as shown in the following lemma which will be a key ingredient in the proof of the main theorem of this section.
Lemma 3.5. For symbols $u_{i j} \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$, there exists a compact operator $K$ on the weighted Dirichlet space such that

$$
\Sigma_{i=1}^{N} \Pi_{j=1}^{M_{i}} T_{u_{i j}}=T_{\Sigma_{i=1}^{N} \Pi_{j=1}^{M_{i} u_{i j}}}+K .
$$

Proof. Since it is known that $T_{f g}-T_{f} T_{g}$ is compact for $f, g \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$, the proof is similar to Lemma 3 in [4].

In Theorem 3.1, it is shown that $T_{f}$ is compact on weighted Dirichlet space if and only if $f=0$ a.e on $\mathbb{S}_{n}$. Using this fact together with Lemma 3.5, we prove the following theorem which generalizes the result of [4] to the high dimensional case.
Theorem 3.6. Suppose $u_{i j} \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$. Then the following statements are equivalent:
(1) $\Sigma_{i=1}^{N} \Pi_{j=1}^{M_{i}} T_{u_{i j}}$ is compact on $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$.
(2) $\Sigma_{i=1}^{N} \Pi_{j=1}^{M_{i}} u_{i j}=0$ on $\mathbb{S}_{n}$.

Proof. The compactness of $\Sigma_{i=1}^{N} \Pi_{j=1}^{M_{i}} T_{u_{i j}}$ and $T_{\Sigma_{i=1}^{N} \Pi_{j=1}^{M_{i} u_{i j}}}$ are the same by Lemma 3.5. Thus the result follows from Theorem 3.1. The proof is complete.

## 4. Essential spectrum

In this section, the essential spectrum of Toeplitz operator on weighted Dirichlet space over the ball is characterized.
Lemma 4.1. Suppose $f \in W^{1, \infty}\left(\mathbb{B}_{n}\right)$. Then $T_{f}^{*}-T_{\bar{f}}$ is compact.
Proof. For any $g, h \in \mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$, it follows

$$
\left\langle\left(T_{f}^{*}-T_{\bar{f}}\right) g, h\right\rangle=\sum_{i=1}^{n}\left[\left\langle\frac{\partial g}{\partial z_{i}}, \frac{\partial f}{\partial z_{i}} h\right\rangle_{2}-\left\langle g, \frac{\partial f}{\partial \bar{z}_{i}} \frac{\partial h}{\partial z_{i}}\right\rangle_{2}\right] .
$$

Hence

$$
\left|\left\langle\left(T_{f}^{*}-T_{\bar{f}}\right) g, h\right\rangle\right| \leq \sqrt{n}\|f\|_{*}\|h\|_{2}\|g\|+\sqrt{n}\|f\|_{*}\|g\|_{2}\|h\| .
$$

For any $\left\{h_{k}\right\} \subset \mathcal{D}\left(\mathbb{B}_{n}, d \nu\right),\left\|h_{k}\right\|=1, h_{k} \xrightarrow{w} 0$, we have
$\left\|\left(T_{f}^{*}-T_{\bar{f}}\right) h_{k}\right\|^{2} \leq \sqrt{n}\|f\|_{*}\left\|h_{k}\right\|_{2}\left\|\left(T_{f}^{*}-T_{\bar{f}}\right) h_{k}\right\|+\sqrt{n}\|f\|_{*}\left\|\left(T_{f}^{*}-T_{\bar{f}}\right) h_{k}\right\|_{2}\left\|h_{k}\right\|$.
Since $\left(T_{f}^{*}-T_{\bar{f}}\right) h_{k} \xrightarrow{w} 0$, it follows that

$$
\left\|\left(T_{f}^{*}-T_{\bar{f}}\right) h_{k}\right\|_{2} \rightarrow 0,
$$

which implies $\left\|\left(T_{f}^{*}-T_{\bar{f}}\right) h_{k}\right\| \rightarrow 0$. This completes the proof.
Lemma 4.2. Suppose $|\xi|=1$, and let $f_{k}(z)=\left(\frac{1+\langle z, \xi\rangle}{2}\right)^{k}-\left(\frac{1}{2}\right)^{k}$. Then $\frac{f_{k}}{\left\|f_{k}\right\|} \xrightarrow{w}$ 0 on $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$, where $\langle z, \xi\rangle=\sum_{i=1}^{n} z_{i} \overline{\xi_{i}}$.

Proof. It is sufficient to prove $\frac{1}{\left\|f_{k}\right\|} \frac{\partial f_{k}}{\partial z_{j}} \xrightarrow{w} 0$ as $k \rightarrow \infty$, in weighted Bergman space for each $j=1, \ldots, n$. If not, there exists a $z_{0} \in \mathbb{B}_{n}$ such that $\frac{1}{\left\|f_{k}\right\|} \frac{\partial f_{k}}{\partial z_{j}}\left(z_{0}\right)$ does not converge to 0 as $k \rightarrow \infty$. Hence there exists a $c>0$ such that

$$
k\left|\frac{1+\left\langle z_{0}, \xi\right\rangle}{2}\right|^{k-1} \geq c\left\|f_{k}\right\|
$$

Also we can find a $\rho>1$, define $S_{\rho}(\xi)=\left\{z \in \mathbb{B}_{n},\left|\frac{1+\langle z, \xi\rangle}{2}\right|>\rho\left|\frac{1+\left\langle z_{0}, \xi\right\rangle}{2}\right|\right\}$. $\forall z \in S_{\rho}(\xi)$, it follows that

$$
k\left|\frac{1+\langle z, \xi\rangle}{2}\right|^{k-1} \geq k\left|\frac{1+\left\langle z_{0}, \xi\right\rangle}{2}\right|^{k-1} \rho^{k-1} \geq c \rho^{k-1}\left\|f_{k}\right\|
$$

Thus

$$
\begin{aligned}
\left\|f_{k}\right\|^{2} & =\int_{\mathbb{B}_{n}} k^{2}\left|\frac{1+\langle z, \xi\rangle}{2}\right|^{2 k-2} \frac{1}{4} d \nu(z) \\
& \geq \int_{S_{\rho}(\xi)} \frac{1}{4} c^{2} \rho^{2 k-2}\left\|f_{k}\right\|^{2} d \nu(z) \geq \nu\left(S_{\rho}(\xi)\right) \frac{1}{4} c^{2} \rho^{2 k-2}\left\|f_{k}\right\|^{2}
\end{aligned}
$$

which is contradiction.
Theorem 4.3. Suppose $g \in \Omega$. Then $\sigma_{e}\left(T_{g}\right)=g\left(\mathbb{S}_{n}\right)$.
Proof. Suppose $0 \in g\left(\mathbb{S}_{n}\right)$, then there exists $\xi \in \mathbb{S}_{n}$ such that $g(\xi)=0$. Let $f_{k}(z)$ be defined as in Lemma 4.6, then $\frac{f_{k}}{\left\|f_{k}\right\|} \xrightarrow{w} 0$ on $\mathcal{D}$.

$$
\begin{aligned}
\left\|g f_{k}\right\|^{2}= & \left|\int_{\overline{\mathbb{B}}_{n}} g(z) f_{k}(z) d \nu(z)\right|^{2}+\sum_{i=1}^{n}\left[\left\langle\frac{\partial g}{\partial z_{i}} f_{k}, \frac{\partial g}{\partial z_{i}} f_{k}\right\rangle_{2}+\left\langle\frac{\partial g}{\partial z_{i}} f_{k}, g \frac{\partial f_{k}}{\partial z_{i}}\right\rangle_{2}\right. \\
& \left.+\left\langle g \frac{\partial f_{k}}{\partial z_{i}}, \frac{\partial g}{\partial z_{i}} f_{k}\right\rangle_{2}+\left\langle g \frac{\partial f_{k}}{\partial z_{i}}, g \frac{\partial f_{k}}{\partial z_{i}}\right\rangle_{2}+\left\langle f_{k} \frac{\partial g}{\partial \bar{z}_{i}}, f_{k} \frac{\partial g}{\partial \bar{z}_{i}}\right\rangle_{2}\right]
\end{aligned}
$$

Then by Lemma 2.3 and the proof of Theorem 3.1, we have

$$
\lim _{k \rightarrow \infty} \frac{\left|\left\langle\frac{\partial g}{\partial z_{i}} f_{k}, \frac{\partial g}{\partial z_{i}} f_{k}\right\rangle_{2}\right|+\left|\left\langle\frac{\partial g}{\partial z_{i}} f_{k}, g \frac{\partial f_{k}}{\partial z_{i}}\right\rangle_{2}\right|+\left|\left\langle g \frac{\partial f_{k}}{\partial z_{i}}, \frac{\partial g}{\partial z_{i}} f_{k}\right\rangle_{2}\right|+\left|\left\langle f_{k} \frac{\partial g}{\partial z_{i}}, f_{k} \frac{\partial g}{\partial z_{i}}\right\rangle_{2}\right|}{\left\|f_{k}\right\|^{2}}=0
$$

for every $i=1,2, \ldots, n$. And it is clear that

$$
\begin{gathered}
\lim _{k \rightarrow \infty} \frac{\left|\int_{\overline{\mathbb{B}_{n}}} g(z) f_{k}(z) d \nu(z)\right|^{2}}{\left\|f_{k}\right\|^{2}} \leq \lim _{k \rightarrow \infty} \frac{\|g\|_{2}^{2}\left\|f_{k}\right\|_{2}^{2}}{\left\|f_{k}\right\|^{2}}=0 \\
\lim _{k \rightarrow \infty} \sum_{i=1}^{n} \frac{\left\langle g \frac{\partial f_{k}}{\partial z_{i}}, g \frac{\partial f_{k}}{\partial z_{i}}\right\rangle_{2}}{\left\|f_{k}\right\|^{2}}=\lim _{k \rightarrow \infty} \frac{\int_{\overline{\mathbb{B}_{n}}}|g(z)|^{2}\left|\frac{1+\langle z, \xi\rangle}{2}\right|^{2 k-2} d \nu(z)}{\int_{\overline{\mathbb{B}_{n}}}\left|\frac{1+\langle z, \xi\rangle}{2}\right|^{2 k-2} d \nu(z)}=|g(\xi)|=0 .
\end{gathered}
$$

The proof of the above is similar to the proof of Lemma 4.2. Since $\frac{\left\|T_{g} f_{k}\right\|}{\left\|f_{k}\right\|} \leq$ $\frac{\left\|g f_{k}\right\|}{\left\|f_{k}\right\|}$, then $T_{g}$ is not a Fredholm operator. And it is clear that $0 \in \sigma_{e}\left(T_{g}\right)$. Then $g\left(\mathbb{S}_{n}\right) \subseteq \sigma_{e}\left(T_{g}\right)$.

Conversely, suppose that $0 \notin g\left(\mathbb{S}_{n}\right)$, it follows for any $\xi \in \mathbb{S}_{n}$, there exists $\varepsilon_{0}>0$ such that $|g(\xi)|>\varepsilon_{0}>0$. Hence $\tilde{T}_{|g|^{2}}$ is a Fredholm operator on the weighted Bergman space. Since

$$
\left\langle T_{|g|^{2}} f, h\right\rangle=\sum_{i=1}^{n}\left[\left\langle\frac{\partial|g|^{2}}{\partial z_{i}} f, \frac{\partial h}{\partial z_{i}}\right\rangle_{2}+\left\langle\tilde{T}_{|g|^{2}} \frac{\partial f}{\partial z_{i}}, \frac{\partial h}{\partial z_{i}}\right\rangle_{2}\right] .
$$

Let $U_{j}$ be an operator defined by $U_{j} f=\frac{\partial f}{\partial z_{j}}, j=1, \ldots, n$, then $U_{j}$ is a bounded operator from weighted Dirichlet space to weighted Bergman space. Thus we get

$$
T_{|g|^{2}}=\sum_{j=1}^{n}\left(A_{j}+U_{j}^{*} \tilde{T}_{|g|^{2}} U_{j}\right),
$$

where $\left\langle A_{j} f, h\right\rangle=\left\langle\frac{\partial|g|^{2}}{\partial z_{j}} f, \frac{\partial h}{\partial z_{j}}\right\rangle_{2}$. In order to prove that $T_{|g|^{2}}$ is also Fredholm operator, it suffice to prove that $A_{j}$ is compact. Let $f_{\alpha} \xrightarrow{w} 0$ in $\mathcal{D}\left(\mathbb{B}_{n}, d \nu\right)$, it follows that $\left\|f_{\alpha}\right\|_{2} \rightarrow 0$ by Lemma 2.3.

$$
\begin{aligned}
\left\|A_{j} f_{\alpha}\right\| & =\sup _{\|h\|=1}\left|\left\langle A_{j} f_{\alpha}, h\right\rangle\right|=\sup _{\|h\|=1}\left|\left\langle\frac{\partial|g|^{2}}{\partial z_{j}} f_{\alpha}, \frac{\partial h}{\partial z_{j}}\right\rangle_{2}\right| \\
& \leq \sup _{\|h\|=1}\left\|f_{\alpha}\right\|_{2}\left\|\frac{\partial|g|^{2}}{\partial z_{j}}\right\|_{\infty}\left\|\frac{\partial h}{\partial z_{j}}\right\|_{2} \leq\left\|f_{\alpha}\right\|\left\|_{2}\right\| \frac{\partial|g|^{2}}{\partial z_{j}} \|_{\infty},
\end{aligned}
$$

it follows that $\lim _{\alpha \rightarrow \infty}\left\|A_{j} f_{\alpha}\right\|=0$. Hence we get that $A_{j}$ is compact for every $j=1, \ldots, n$. Thus $T_{|g|^{2}}$ is also Fredholm operator. As $T_{\bar{g}} T_{g}-T_{|g|^{2}}, T_{g} T_{\bar{g}}-T_{|g|^{2}}$, $T_{g}^{*}-T_{\bar{g}}$ is compact, hence $T_{g}$ is also Fredholm operator and $0 \notin \sigma_{e}\left(T_{g}\right)$. The proof is complete.

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