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POTENTIALLY EVENTUALLY POSITIVE BROOM SIGN PATTERNS

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Abstract. A sign pattern is a matrix whose entries belong to the set $\{+,-,0\}$. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix or be potentially eventually positive if there exist at least one real matrix A with the same sign pattern as $\mathcal{A}$ and a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. Identifying the necessary and sufficient conditions for an n-by-n sign pattern to be potentially eventually positive, and classifying the n-by-n sign patterns that allow an eventually positive matrix are two open problems. In this article, we focus on the potential eventual positivity of broom sign patterns. We identify all the minimal potentially eventually positive broom sign patterns. Consequently, we classify all the potentially eventually positive broom sign patterns.

1. Introduction

A *sign pattern* is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in the set $\{+,-,0\}$. An n-by-n real matrix A with the same sign pattern as $\mathcal{A}$ is called a *realization* of $\mathcal{A}$. The set of all realizations of sign pattern $\mathcal{A}$ is called the *qualitative class* of $\mathcal{A}$ and is denoted by $Q(\mathcal{A})$. A *subpattern* of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. Furthermore, if $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a *proper subpattern* of $\mathcal{A}$ (equivalently, $\mathcal{A}$ is a *proper superpattern* of $\mathcal{B}$). A sign pattern $\mathcal{A}$ is *reducible* if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A sign pattern is *irreducible* if it is not reducible; see, e.g. [3] for more details.

A sign pattern $\mathcal{A}$ is said to *require* a certain property P with respect to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and *allow* P or be *potentially* P if there is some $A \in Q(\mathcal{A})$ that has the property P.

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Recall that an n-by-n real matrix A is said to be *eventually positive* if there exists a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$; see, e.g., [7]. Eventually positive matrices have applications to dynamical systems in situations where it is of interest to determine whether an initial trajectory reaches positivity at a certain time and remains positive thereafter; see e.g., [8]. An n-by-n sign pattern $\mathcal{A}$ is said to *allow an eventually positive matrix* or be *potentially eventually positive* (*PEP, for short*), if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [2], [5] and the references therein.

PEP sign patterns were studied first in [2], where some sufficient conditions and some necessary conditions for a sign pattern to be PEP were established, and some PEP sign patterns of small orders were classified. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be PEP remains open. Also open is the classification of the general PEP sign patterns. Recently, there are a few literatures on the potential eventual positivity of sign pattern matrices with certain underlying combinatorial structures; see [1, 10–13] for example. More recently, in [14], we identified all the minimal PEP sign patterns $\mathcal{A}_{n,4}$ with the underlying broom graph $G(\mathcal{A}_{n,4})$ consisting of a path P with 4 vertices, together with $(n-4)$ pendent vertices all adjacent to the same end vertex of P, and classified these PEP sign patterns.

In this article, we focus on the eventual positivity of the $(n+m)$-by-$(n+m)$ broom sign patterns $\mathcal{B}_{n,m}$ whose underlying broom graph $G(\mathcal{B}_{n,m})$ consists of a path P with $m+1$ vertices, together with $(n-1)$ pendent vertices all adjacent to the same end vertex of P. As a further investigations of [14], our work extends our preliminary results about the broom sign patterns with parameter $m = 3$, about the star sign patterns with parameter $m = 1$, and about the tridiagonal sign patterns with parameter $n < 3$. Our work is organized as follows. In Section 2, some necessary graph theoretical concepts are introduced, and some sufficient conditions and necessary conditions for an n-by-n sign pattern to be PEP are cited. In Section 3, some necessary conditions for an $(n + m)$-by-$(n + m)$ broom sign patterns to be PEP are established, all the minimal PEP broom sign patterns of order $(n + m)$ are identified as the specific $m + 2$ sign patterns, and consequently all the PEP broom sign patterns are classified.

2. Preliminary results

In this section, we first introduce some graph theoretical concepts as a necessary preparation, and then cite some basic conclusions on PEP sign pattern as four lemmas to state our results clearly.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is *combinatorially symmetric* if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices $1, 2, \ldots, n$ and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the *graph* of the pattern $\mathcal{A}$, see, e.g. [3] and [4] for details. In this article, a combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a *broom sign pattern* if the graph $G(\mathcal{A})$ is a broom graph.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has *signed digraph* $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \dots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). For a digraph $D = (V, E)$, a (directed) *simple cycle of length* k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$ such that the vertices $i_1, \dots, i_k$ are distinct. A digraph D is *primitive* if it is strongly connected and the greatest common divisor of the lengths of its simple cycles is 1. For a nonnegative sign pattern $\mathcal{A}$, if its signed digraph $\Gamma(\mathcal{A})$ is primitive, then $\mathcal{A}$ is said to be *primitive*; see, e.g. [2] for more details.

The *positive part* of a sign pattern $\mathcal{A}$ is defined to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. In [2], it has been shown that if $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP. Below, we cite some sufficient conditions and some necessary conditions for an n-by-n sign pattern to be PEP in [2] as Lemmas 1, 2, 3 and 4 to proceed.

Lemma 1. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then every superpattern of $\mathcal{A}$ is also PEP.*

Lemma 2. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then the sign pattern $\hat{\mathcal{A}}$ obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$ is also PEP.*

Lemma 3. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix $A \in Q(\mathcal{A})$ such that*

(1) $\rho(A) = 1$.

(2) $A\mathbf{1} = \mathbf{1}$*, where* $\mathbf{1}$ *is the* $n \times 1$ *all ones vector.*

(3) *If* $n \geq 2$*, the sum of all the off-diagonal entries of* A *is positive.*

Following [14], we denote the square sign pattern of order n consisting entirely of positive (respectively, negative) entries by $[+]_n$ (respectively, $[-]_n$).

Lemma 4. *Let $\mathcal{A}$ be the checkerboard block sign pattern*

$$\begin{pmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

3. Potential eventual positivity of broom sign patterns

In this section, we turn to investigate the potential eventual positivity of broom sign patterns.

Let $\mathcal{B}_{n,m}$ be a broom sign pattern of order $n + m$, where $n \geq 1$ and $m \geq 1$. Then the graph $G(\mathcal{B}_{n,m})$ consists of a path P with $m + 1$ vertices, together with $(n - 1)$ pendant vertices all adjacent to the same end vertex of P; see [9]

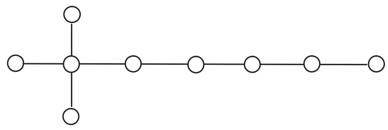

FIGURE 1. The graph of broom sign patten $\mathcal{B}_{4,5}$.

for more details. For example, the graph of broom sign pattern $\mathcal{B}_{4,5}$ is shown in Figure 1.

Note that $\mathcal{B}_{n,m}$ is a tridiagonal sign pattern for $n < 3$, $\mathcal{B}_{n,m}$ is a star sign pattern for $m = 1$, $\mathcal{B}_{n,m}$ is a double star sign pattern for $m = 2$, and $\mathcal{B}_{n,m}$ is a very special broom sign pattern for $m = 3$. All PEP tridiagonal (respectively, star and double star) sign patterns have been classified in [12] (respectively, [11] and [13]). The potential eventual positivity of broom sign patterns with $m = 3$ has been investigated in [14]. To investigate the potential eventual positivity of the more general broom sign patterns, throughout the paper, we assume that $n \geq 3$ and $m \geq 4$. It is clear that sign pattern $\mathcal{A}$ is PEP if and only if $\mathcal{A}^T$ or $\mathcal{P}^T\mathcal{A}\mathcal{P}$ is PEP, for any permutation pattern $\mathcal{P}$. Thus, without loss of generality, let the $(n+m)$-by-$(n+m)$ broom sign pattern $\mathcal{B}_{n,m}$ be of the following form

$$\begin{pmatrix}
? & * & \cdots & * & * & & & \\
* & ? & & & & & & \\
\vdots & & \ddots & & & & & \\
* & & & ? & & & & \\
* & & & & ? & * & & \\
 & & & & * & ? & \ddots & \\
 & & & & & \ddots & \ddots & * \\
 & & & & & & * & ?
\end{pmatrix},$$

where ? denotes an entry from $\{+, -, 0\}$, $*$ denotes a nonzero entry and the unspecified entries are all zeros.

The following proposition is a necessary condition for an $(n+m)$-by-$(n+m)$ broom sign pattern $\mathcal{B}_{n,m}$ to be PEP, which can be obtained by a similar method in [14] and a more complicated discussion. For the reader's conveniences, we give its proof.

Proposition 1. *If an $(n+m)$-by-$(n+m)$ broom sign pattern $\mathcal{B}_{n,m}$ is PEP, then $\mathcal{B}_{n,m}$ is symmetric.*

Proof. Since broom sign pattern $\mathcal{B}_{n,m}$ is PEP, let $B = [b_{ij}] \in Q(\mathcal{B}_{n,m})$ be the eventually positive matrix realization. By Lemma 3, let $\sum_{k=1}^{n+1} b_{1k} = 1$, $b_{ii} = 1 - b_{i1}$, $i = 2, 3, \ldots, n$, $b_{n+1,n+1} = 1 - b_{n+1,1} - b_{n+1,n+2}$, $b_{j,j} = 1 - b_{j,j-1} - b_{j,j+1}$, $j = n+2, n+3, \ldots, n+m-1$, and $b_{n+m,n+m} = 1 - b_{n+m,n+m-1}$. To complete the proof, it suffices to show that $b_{1i}b_{i1} > 0$, $i = 2, 3, \ldots, n+1$, and $b_{j,j+1}b_{j+1,j} > 0$, $j = n+1, n+2, \ldots, n+m-1$. Suppose the corresponding positive left eigenvector of A is $w = (w_1, w_2, \ldots, w_{n+m})^T$. Then by $w^T A = w^T$, we have the following equalities:

$$w_1 b_{1i} + w_i(1 - b_{i1}) = w_i, i = 2, 3, \ldots, n, \tag{1}$$

$$w_{n+m-1}b_{n+m-1,n+m} + w_{n+m}(1 - b_{n+m,n+m-1}) = w_{n+m}, \tag{2}$$

$$\begin{aligned} &w_{n+m-2}b_{n+m-2,n+m-1} + w_{n+m-1}(1 - b_{n+m-1,n+m-2} - b_{n+m-1,n+m}) \\ &+ w_{n+m}b_{n+m,n+m-1} = w_{n+m-1}, \end{aligned} \tag{3}$$

$$\vdots$$

$$w_{n+1}b_{n+1,n+2} + w_{n+2}(1 - b_{n+2,n+1} - b_{n+2,n+3}) + w_{n+3}b_{n+3,n+2} = w_{n+2}, \tag{4}$$

and

$$w_1 b_{1,n+1} + w_{n+1}(1 - b_{n+1,1} - b_{n+1,n+2}) + w_{n+2}b_{n+2,n+1} = w_{n+1}. \tag{5}$$

By Equality (1), we have $w_1 b_{1i} = w_i b_{i1}$. Then $b_{i1}b_{1i} > 0$ for $i = 2, 3, \ldots, n$.

By Equality (2), we have

$$w_{n+m-1}b_{n+m-1,n+m} = w_{n+m}b_{n+m,n+m-1}. \tag{6}$$

It follows that $b_{n+m-1,n+m}b_{n+m,n+m-1} > 0$. By Equalities (3) and (6), we have

$$w_{n+m-2,}b_{n+m-2,n+m-1} = w_{n+m-1}b_{n+m-1,n+m-2}. \tag{7}$$

So $b_{n+m-2,n+m-1}b_{n+m-1,n+m-2} > 0$. Continuing on this way, we obtain that

$$w_{n+m-k,}b_{n+m-k,n+m-k+1} = w_{n+m-k+1}b_{n+m-k+1,n+m-k}, \ k = 1, 2, \ldots, m-1. \tag{8}$$

Thus, $b_{n+m-k,n+m-k+1}b_{n+m-k+1,n+m-k} > 0$, $k = 1, 2, \ldots, m-1$.

For $k = m-1$, we obtain from Equality (8) that

$$w_{n+1,}b_{n+1,n+2} = w_{n+2}b_{n+2,n+1}. \tag{9}$$

By Equalities (9) and (5), we have

$$w_1 b_{1,n+1} = w_{n+1}b_{n+1,1}. \tag{10}$$

It follows that $b_{1,n+1}b_{n+1,1} > 0$. □

It is known that if an n-by-n ($n \geq 2$) sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix realization A such that the sum of all nonzero off-diagonal entries of A is positive. Interestingly enough, the following theorem indicates that all nonzero off-diagonal entries of every matrix realization $B \in Q(\mathcal{B}_{n,m})$ are positive, for an $(n+m)$-by-$(n+m)$ PEP broom sign pattern $\mathcal{B}_{n,m}$.

Theorem 1. *If an $(n+m)$-by-$(n+m)$ broom sign pattern $\mathcal{B}_{n,m} = [\beta_{ij}]$ is PEP, then $\beta_{i1} = \beta_{1i} = +$ for $i = 2, 3, \ldots, n+1$, and $\beta_{j,j+1} = \beta_{j+1,j} = +$ for $j = n+1, n+2, \ldots, n+m-1$.*

Proof. By Proposition 1, the PEP broom sign pattern $\mathcal{B}_{n,m}$ is symmetric. Consequently, it suffices to show that $\beta_{1i} = +$ for $i = 2, 3, \ldots, n$, $\beta_{1,n+1} = +$, and $\beta_{j,j+1} = +$ for $j = n+1, n+2, \ldots, n+m-1$. To state clearly, let s be the number of i such that $\beta_{1i} = -$, $2 \le i \le n$, and let t be the number of j such that $\beta_{j,j+1} = -$ for $j = n+1, n+2, \ldots, n+m-1$. Next, we show that $s = 0$ and $t = 0$ to complete the proof. We obtain contradictions by considering the following three cases: (1) $s > 0$ and $t = 0$; (2) $s = 0$ and $t > 0$; (3) $s > 0$ and $t > 0$.

Case 1. $s > 0$ and $t = 0$.

By a similar discussion as the proof of Case 1 in Theorem 1 in [14], we obtain that the following two checkerboard block sign patterns

$$\begin{pmatrix} [+]_1 & [-] & [+] & [-] \\ [-] & [+]_s & [-] & [+] \\ [+] & [-] & [+]_{n-s-1} & [-] \\ [-] & [+] & [-] & [+]_m \end{pmatrix}, \text{ and } \begin{pmatrix} [+]_1 & [-] & [+] \\ [-] & [+]_s & [-] \\ [+] & [-] & [+]_{n+m-s-1} \end{pmatrix}$$

are also PEP if $\mathcal{B}_{n,m}$ are PEP. It is a contradiction.

Case 2. $s = 0$ and $t > 0$.

Without loss of generality, let $\beta_{j_1,j_1+1} = \beta_{j_2,j_2+1} = \cdots = \beta_{j_t,j_t+1} = -$, where $n+1 \le j_1 < j_2 < \cdots < j_t \le n+m-1$.

Subcase 2.1. $j_1 > n+1$.

Up to equivalence, the symmetric broom sign pattern

$$\mathcal{B}_{n,m} = \left(\begin{array}{c|ccc|cccc|cccc|c|cccc}
? & + & \cdots & + & \pm & & & & & & & & & & & & \\ \hline
+ & ? & & & & & & & & & & & & & & & \\
\vdots & & \ddots & & & & & & & & & & & & & & \\
+ & & & ? & & & & & & & & & & & & & \\ \hline
\pm & & & & ? & + & & & & & & & & & & & \\
 & & & & + & ? & \ddots & & & & & & & & & & \\
 & & & & & \ddots & \ddots & + & & & & & & & & & \\
 & & & & & & + & ? & - & & & & & & & & \\ \hline
 & & & & & & & - & ? & + & & & & & & & \\
 & & & & & & & & + & ? & \ddots & & & & & & \\
 & & & & & & & & & \ddots & \ddots & + & & & & & \\
 & & & & & & & & & & + & ? & - & & & & \\ \hline
 & & & & & & & & & & & - & \ddots & - & & & \\ \hline
 & & & & & & & & & & & & - & ? & + & & \\
 & & & & & & & & & & & & & + & ? & \ddots & \\
 & & & & & & & & & & & & & & \ddots & \ddots & + \\
 & & & & & & & & & & & & & & & + & ?
\end{array}\right).$$

By changing all 0 and $-$ diagonal entries of $\mathcal{B}_{n,m}$ to $+$, we obtain respectively the PEP sign pattern $\widehat{\mathcal{B}}_{n,m}$ which is a proper subpattern of the following checkerboard block sign pattern

$$\begin{pmatrix} [+]_n & [-] & [+] & \cdots & \cdots & \cdots \\ [-] & [+]_{j_1-n} & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+]_{j_2-j_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & [+]_{j_t-j_{(t-1)}} & [-] \\ \vdots & \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix},$$

or

$$\begin{pmatrix} [+]_{j_1} & [-] & \cdots & \cdots & \cdots \\ [-] & [+]_{j_2-j_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & [+]_{j_t-j_{(t-1)}} & [-] \\ \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix}.$$

Thus, these two checkerboard block sign patterns are also PEP by Lemma 1, which contradicts Lemma 4.

Subcase 2.2. $j_1 = n+1$.

Up to equivalence, the symmetric broom sign pattern

$$\mathcal{B}_{n,m} = \left(\begin{array}{cccc|c|cccc|c|cccc} ? & + & \cdots & + & \pm & & & & & & & & & \\ + & ? & & & & & & & & & & & & \\ \vdots & & \ddots & & & & & & & & & & & \\ + & & & ? & & & & & & & & & & \\ \hline \pm & & & & ? & - & & & & & & & & \\ \hline & & & & - & ? & + & & & & & & & \\ & & & & & + & ? & \ddots & & & & & & \\ & & & & & & \ddots & \ddots & + & & & & & \\ & & & & & & & + & ? & - & & & & \\ \hline & & & & & & & & - & \ddots & - & & & \\ \hline & & & & & & & & & - & ? & + & & \\ & & & & & & & & & & + & ? & \ddots & \\ & & & & & & & & & & & \ddots & \ddots & + \\ & & & & & & & & & & & & + & ? \end{array}\right).$$

Since $\mathcal{B}_{n,m}$ is PEP, by changing all 0 and $-$ diagonal entries of $\mathcal{B}_{n,m}$ to $+$, we obtain respectively the PEP sign pattern $\widehat{\mathcal{B}}_{n,m}$ which is a proper subpattern of the following checkerboard block sign pattern

$$\begin{pmatrix} [+]_n & [-] & [+] & \cdots & \cdots & \cdots \\ [-] & [+]_1 & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+]_{j_2-j_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & [+]_{j_t-j_{(t-1)}} & [-] \\ \vdots & \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix},$$

or

$$\begin{pmatrix} [+]_{n+1} & [-] & \cdots & \cdots & \cdots \\ [-] & [+]_{j_2-j_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & [+]_{j_t-j_{(t-1)}} & [-] \\ \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix}.$$

Thus, these two checkerboard block sign patterns are also PEP by Lemma 1, which contradicts Lemma 4.

Case 3. $s > 0$ and $t > 0$.

Since s is the number of i such that $\beta_{1i} = -$, $2 \leq i \leq n$, without loss of generality, let $\beta_{12} = \beta_{13} = \cdots = \beta_{1,s+1} = -$. Similarly, let $\beta_{j_1,j_1+1} = \beta_{j_2,j_2+1} = \cdots = \beta_{j_t,j_t+1} = -$, where $n+1 \leq j_1 < j_2 < \cdots < j_t \leq n+m-1$. It suffices to consider the following three possibilities, up to equivalence.

Subcase 3.1. $j_1 > n+1$ and

$$\mathcal{B}_{n,m} = \begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} & \mathcal{B}_{14} & & & \\ (\mathcal{B}_{12})^T & \mathcal{B}_{22} & & & & & \\ (\mathcal{B}_{13})^T & & \mathcal{B}_{33} & & & & \\ (\mathcal{B}_{14})^T & & & \mathcal{B}_{44} & \mathcal{B}_{45} & & \\ & & & (\mathcal{B}_{45})^T & \mathcal{B}_{55} & \ddots & \\ & & & & \ddots & \ddots & \\ & & & & & \mathcal{B}_{t+3,t+3} & \mathcal{B}_{t+3,t+4} \\ & & & & & (\mathcal{B}_{t+3,t+4})^T & \mathcal{B}_{t+4,t+4} \end{pmatrix},$$

where $\mathcal{B}_{11} = (?)$, $\mathcal{B}_{22} = diag(?, \ldots, ?)$ of order s, $\mathcal{B}_{33} = diag(?, \ldots, ?)$ of order $n-s-1$, $\mathcal{B}_{44} = \begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order $j_1 - n$, $\mathcal{B}_{ii} = \begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order $j_{i-3}-j_{i-4}$ for $i=5,6,\dots,t+3$, $\mathcal{B}_{t+4,t+4}=\begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order $n+m-j_t$, $\mathcal{B}_{12}=(-\cdots-)_{1,s}$, $\mathcal{B}_{13}=(+,\dots,+)_{1,n-s-1}$, $\mathcal{B}_{14}=(+,0,\dots,0)_{1,j_1-n}$ with 1 positive entry and j_1-n-1 zero entries, $\mathcal{B}_{45}$ is a (j_1-n)-by-(j_2-j_1) matrix whose $(j_1-n,1)$-entry is $-$ and all the other entries are zeros, $\mathcal{B}_{i,i+1}$ is a $(j_{i-3}-j_{i-4})$-by-$(j_{i-2}-j_{i-3})$ matrix whose $(j_{i-3}-j_{i-4},1)$-entry is $-$ and all the other entries are zeros for $i=5,6,\dots,t+2$, and $\mathcal{B}_{t+3,t+4}$ is a (j_t-j_{t-1})-by-$(n+m-j_t)$ matrix whose $(j_t-j_{t-1},1)$-entry is $-$ and all the other entries.

By changing all 0 and $-$ diagonal entries of $\mathcal{B}_{n,m}$ to $+$, we obtain the PEP sign pattern $\widehat{\mathcal{B}}_{n,m}$ which is a proper subpattern of the checkerboard block sign pattern

$$\begin{pmatrix} [+]_1 & [-] & [+] & [-] & \cdots & \cdots & \cdots \\ [-] & [+]_s & [-] & [+] & \cdots & \cdots & \cdots \\ [+] & [-] & [+]_{j_1-s-1} & [-] & \cdots & \cdots & \cdots \\ [-] & [+] & [-] & [+]_{j_2-j_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & [+]_{j_t-j_{t-1}} & [-] \\ \vdots & \vdots & \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix}.$$

By Lemmas 2 and 1, the above checkerboard block sign pattern is PEP, which is a contradiction by Lemma 4.

Subcase 3.2. $j_1>n+1$ and

$$\mathcal{B}_{n,m}=\begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} & \mathcal{B}_{14} & & & & \\ (\mathcal{B}_{12})^T & \mathcal{B}_{22} & & & & & & \\ (\mathcal{B}_{13})^T & & \mathcal{B}_{33} & & & & & \\ (\mathcal{B}_{14})^T & & & \mathcal{B}_{44} & \mathcal{B}_{45} & & & \\ & & & (\mathcal{B}_{45})^T & \mathcal{B}_{55} & \ddots & & \\ & & & & \ddots & \ddots & & \\ & & & & & & \mathcal{B}_{t+3,t+3} & \mathcal{B}_{t+3,t+4} \\ & & & & & & (\mathcal{B}_{t+3,t+4})^T & \mathcal{B}_{t+4,t+4} \end{pmatrix},$$

where $\mathcal{B}_{11}=(?)$, $\mathcal{B}_{22}=diag(?,\dots,?)$ of order s, $\mathcal{B}_{33}=diag(?,\dots,?)$ of order $n-s-1$, $\mathcal{B}_{44}=\begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order j_1-n, $\mathcal{B}_{ii}=\begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order

$j_{i-3}-j_{i-4}$ for $i=5,6,\ldots,t+3$, $\mathcal{B}_{t+4,t+4}=\begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order $n+m-j_t$, $\mathcal{B}_{12}=(-\cdots-)_{1,s}$, $\mathcal{B}_{13}=(+,\ldots,+)_{1,n-s-1}$, $\mathcal{B}_{14}=(-,0,\ldots,0)_{1,j_1-n}$ with 1 negative entry and j_1-n-1 zero entries, $\mathcal{B}_{45}$ is a (j_1-n)-by-(j_2-j_1) matrix whose $(j_1-n,1)$-entry is $-$ and all the other entries are zeros, $\mathcal{B}_{i,i+1}$ is a $(j_{i-3}-j_{i-4})$-by-$(j_{i-2}-j_{i-3})$ matrix whose $(j_{i-3}-j_{i-4},1)$-entry is $-$ and all the other entries are zeros for $i=5,6,\ldots,t+2$, and $\mathcal{B}_{t+3,t+4}$ is a (j_t-j_{t-1})-by-$(n+m-j_t)$ matrix whose $(j_t-j_{t-1},1)$-entry is $-$ and all the other entries.

By changing all 0 and $-$ diagonal entries of $\mathcal{B}_{n,m}$ to $+$, we obtain the PEP sign pattern $\widehat{\mathcal{B}}_{n,m}$ which is a proper subpattern of the checkerboard block sign pattern

$$\begin{pmatrix} [+]_1 & [-] & [+] & [-] & [+] & \cdots & \cdots & \cdots \\ [-] & [+]_s & [-] & [+] & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+]_{n-s-1} & [-] & [+] & \cdots & \cdots & \cdots \\ [-] & [+] & [-] & [+]_{j_1-n} & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+] & [-] & [+]_{j_2-j_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [+]_{j_t-j_{t-1}} & [-] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix}.$$

It follows from Lemmas 2 and 1 that the above checkerboard block sign pattern is PEP. Consequently, Lemma 4 is contradicted.

Subcase 3.3. $j_1=n+1$.

Up to equivalence,

$$\mathcal{B}_{n,m}=\begin{pmatrix} \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{B}_{13} & \mathcal{B}_{14} & & & & \\ (\mathcal{B}_{12})^T & \mathcal{B}_{22} & & & & & & \\ (\mathcal{B}_{13})^T & & \mathcal{B}_{33} & & & & & \\ (\mathcal{B}_{14})^T & & & \mathcal{B}_{44} & \mathcal{B}_{45} & & & \\ & & & (\mathcal{B}_{45})^T & \mathcal{B}_{55} & \mathcal{B}_{56} & & \\ & & & & (\mathcal{B}_{56})^T & \mathcal{B}_{66} & \ddots & \\ & & & & & \ddots & \ddots & \ddots \\ & & & & & & \ddots & \mathcal{B}_{t+3,t+3} & \mathcal{B}_{t+3,t+4} \\ & & & & & & & (\mathcal{B}_{t+3,t+4})^T & \mathcal{B}_{t+4,t+4} \end{pmatrix},$$

where $\mathcal{B}_{11}=\mathcal{B}_{44}=(?)$, $\mathcal{B}_{22}=diag(?,\ldots,?)$ of order s, $\mathcal{B}_{33}=diag(?,\ldots,?)$ of order $n-s-1$, $\mathcal{B}_{ii}=\begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order $j_{i-3}-j_{i-4}$ for $i=5,6,\ldots,t+3$,

$\mathcal{B}_{t+4,t+4} = \begin{pmatrix} ? & + & & \\ + & ? & \ddots & \\ & \ddots & \ddots & + \\ & & + & ? \end{pmatrix}$ of order $n+m-j_t$, $\mathcal{B}_{12} = (-,\ldots,-)_{1,s}$, $\mathcal{B}_{13} = (+,\ldots,+)_{1,n-s-1}$, $\mathcal{B}_{14} = (+)$ or $(-)$, $\mathcal{B}_{45}$ is a 1-by-(j_2-n-1) matrix whose $(1,1)$-entry is $-$ and all other entries are zeros, $\mathcal{B}_{i,i+1}$ is a $(j_{i-3}-j_{i-4})$-by-$(j_{i-2}-j_{i-3})$ matrix whose $(j_{i-3}-j_{i-4},1)$-entry is $-$ and all other entries are zeros for $i=5,6,\ldots,t+2$, and $\mathcal{B}_{t+3,t+4}$ is a (j_t-j_{t-1})-by-$(n+m-j_t)$ matrix whose $(j_t-j_{t-1},1)$-entry is $-$ and all other entries are zeros. By changing all 0 and $-$ diagonal entries of $\mathcal{B}_{n,m}$ to $+$, we obtain the PEP sign pattern $\widehat{\mathcal{B}}_{n,m}$ which is a proper subpattern of one of the checkerboard block sign patterns

$$\begin{pmatrix} [+]_1 & [-] & [+] & [-] & [+] & \cdots & \cdots & \cdots \\ [-] & [+]_s & [-] & [+] & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+]_{n-s-1} & [-] & [+] & \cdots & \cdots & \cdots \\ [-] & [+] & [-] & [+]_1 & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+] & [-] & [+]_{j_2-n-1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [+]_{j_t-j_{t-1}} & [-] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix}$$

and

$$\begin{pmatrix} [+]_1 & [-] & [+] & [-] & \cdots & \cdots & \cdots \\ [-] & [+]_s & [-] & [+] & \cdots & \cdots & \cdots \\ [+] & [-] & [+]_{n-s} & [-] & \cdots & \cdots & \cdots \\ [+] & [-] & [+] & [+]_{j_2-n-1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & [+]_{j_t-j_{t-1}} & [-] \\ \vdots & \vdots & \vdots & \vdots & \vdots & [-] & [+]_{n+m-j_t} \end{pmatrix}.$$

Since $\mathcal{B}_{n,m}$ is PEP, the above checkerboard block sign patterns are PEP by Lemmas 2 and 1. Consequently, Lemma 4 is contradicted. $\square$

The following proposition indicates that an $(n+m)$-by-$(n+m)$ PEP broom sign pattern has at least one positive diagonal entry, and its proof is similar to the proof of Proposition 2 in [14].

Proposition 2. *If an $(n+m)$-by-$(n+m)$ broom sign pattern $\mathcal{B}_{n,m} = [\beta_{ij}]$ is PEP, then there exists some $i \in \{1,2,\ldots,n+m\}$ such that $\beta_{ii} = +$.*

Now we turn to identify the minimal PEP broom sign patterns. For the sake of convenience, let $\mathcal{B}_{n,m}^{(i)}$ be the broom sign pattern $\mathcal{B}_{n,m} = [\beta_{ij}]$ with all nonzero off-diagonal entries equal to $+$, $\beta_{ii} = +$ and $\beta_{jj} = 0$ for all $j \neq i$,

$i \in \{1, 2, \ldots, n+m\}$. For example,

$$\mathcal{B}_{n,m}^{(1)} = \begin{pmatrix} + & + & \cdots & + & + & & & \\ + & 0 & & & & & & \\ \vdots & & \ddots & & & & & \\ + & & & 0 & & & & \\ + & & & & 0 & + & & \\ & & & & + & 0 & \ddots & \\ & & & & & \ddots & \ddots & + \\ & & & & & & + & 0 \end{pmatrix},$$

and

$$\mathcal{B}_{n,m}^{(n+1)} = \begin{pmatrix} 0 & + & \cdots & + & + & & & \\ + & 0 & & & & & & \\ \vdots & & \ddots & & & & & \\ + & & & 0 & & & & \\ + & & & & + & + & & \\ & & & & + & 0 & \ddots & \\ & & & & & \ddots & \ddots & + \\ & & & & & & + & 0 \end{pmatrix}.$$

Note that sign patterns $\mathcal{B}_{n,m}^{(2)}$, $\mathcal{B}_{n,m}^{(3)}$, $\ldots$, $\mathcal{B}_{n,m}^{(n)}$ are equivalent to each other.

Theorem 2. *$\mathcal{B}_{n,m}^{(1)}$, $\mathcal{B}_{n,m}^{(2)}$, $\mathcal{B}_{n,m}^{(n+1)}$, $\mathcal{B}_{n,m}^{(n+2)}$, $\ldots$, $\mathcal{B}_{n,m}^{(n+m)}$ are minimal PEP broom sign patterns.*

Proof. $\mathcal{B}_{n,m}^{(1)}$, $\mathcal{B}_{n,m}^{(2)}$, $\mathcal{B}_{n,m}^{(n+1)}$, $\mathcal{B}_{n,m}^{(n+2)}$, $\ldots$, $\mathcal{B}_{n,m}^{(n+m)}$ are PEP for their positive parts are primitive, respectively. The minimality follows readily from Proposition 2 and the fact that PEP sign patterns must be irreducible. □

The following proposition follows directly from Theorem 2 and Proposition 2.

Proposition 3. *If an $(n+m)$-by-$(n+m)$ broom sign pattern $\mathcal{B}_{n,m}$ is a minimal PEP sign pattern, then there exists exactly one $i \in \{1, 2, \ldots, n+m\}$ such that $\beta_{ii} = +$ and $\beta_{jj} = 0$ for all $j \neq i$.*

Below, all minimal PEP broom sign patterns are identified, which implies that there are only $m+2$ minimal PEP broom sign patterns, up to equivalence.

Theorem 3. *Let $\mathcal{B}_{n,m}$ be an $(n+m)$-by-$(n+m)$ broom sign pattern. Then $\mathcal{B}_{n,m}$ is a minimal PEP sign pattern if and only if $\mathcal{B}_{n,m}$ is equivalent to one of sign patterns $\mathcal{B}_{n,m}^{(1)}$, $\mathcal{B}_{n,m}^{(2)}$, $\mathcal{B}_{n,m}^{(n+1)}$, $\mathcal{B}_{n,m}^{(n+2)}$, $\ldots$, $\mathcal{B}_{n,m}^{(n+m)}$.*

Proof. The sufficiency is clear by Theorem 2. The necessity follows from Theorem 1 and Proposition 3. □

In the following theorem that follows readily from Theorem 3, all PEP broom sign patterns are classified.

Theorem 4. *Let $\mathcal{B}_{n,m}$ be an $(n+m)$-by-$(n+m)$ broom sign pattern. Then $\mathcal{B}_{n,m}$ is PEP if and only if $\mathcal{B}_{n,m}$ is equivalent to a superpattern of one of sign patterns $\mathcal{B}_{n,m}^{(1)}$, $\mathcal{B}_{n,m}^{(2)}$, $\mathcal{B}_{n,m}^{(n+1)}$, $\mathcal{B}_{n,m}^{(n+2)}$, $\ldots$, $\mathcal{B}_{n,m}^{(n+m)}$.*

We conclude this paper by establishing a general result on the $(n+m)$-by-$(n+m)$ PEP broom sign patterns with exactly one nonzero diagonal entry and the parameters $m \geq 1$ and $n \geq 1$.

Proposition 4. *Let $\mathcal{B}_{n,m}$ be an $(n+m)$-by-$(n+m)$ broom sign pattern with exactly one nonzero diagonal entry, $m \geq 1$ and $n \geq 1$. Then the following statements are equivalent*:
(1) *$\mathcal{B}_{n,m}$ is a minimal PEP sign pattern*;
(2) *$\mathcal{B}_{n,m}$ requires eventual positivity*;
(3) *$\mathcal{B}_{n,m}$ is nonnegative and primitive.*

Proof. If $n < 3$, then Proposition 4 follows readily from Corollary 3 in [12]. If $m = 1$, then Proposition 4 follows readily from Theorem 4.4 in [11] and Theorem 2.3 in [6]. If $m = 2$, then Proposition 4 follows readily from Theorem 2 in [13]. If $n \geq 3$ and $m \geq 3$, then Proposition 4 follows readily from Proposition 4 in [14]. If $n \geq 3$ and $m \geq 4$, then Proposition 4 follows readily from Theorem 3 and Theorem 2.3 in [6]. □

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