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RINGS IN WHICH SUMS OF *d*-IDEALS ARE *d*-IDEALS

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Abstract. An ideal of a commutative ring is called a d-ideal if it contains the annihilator of the annihilator of each of its elements. Denote by $\mathrm{DId}(A)$ the lattice of d-ideals of a ring A. We prove that, as in the case of f-rings, $\mathrm{DId}(A)$ is an algebraic frame. Call a ring homomorphism "compatible" if it maps equally annihilated elements in its domain to equally annihilated elements in the codomain. Denote by $\mathbf{SdRng_c}$ the category whose objects are rings in which the sum of two d-ideals is a d-ideal, and whose morphisms are compatible ring homomorphisms. We show that $\mathrm{DId}\colon \mathbf{SdRng_c} \to \mathbf{CohFrm}$ is a functor ($\mathbf{CohFrm}$ is the category of coherent frames with coherent maps), and we construct a natural transformation $\mathrm{RId} \longrightarrow \mathrm{DId}$, in a most natural way, where RId is the functor that sends a ring to its frame of radical ideals. We prove that a ring A is a Baer ring if and only if it belongs to the category $\mathbf{SdRng_c}$ and $\mathrm{DId}(A)$ is isomorphic to the frame of ideals of the Boolean algebra of idempotents of A. We end by showing that the category $\mathbf{SdRng_c}$ has finite products.

Introduction

Throughout the paper, by "ring" we mean a commutative ring with identity. All our rings are *reduced*, which is to say they have no nonzero nilpotent elements. The ideals that will be the subject of study in this paper have appeared in various guises with different names. They were first studied by Speed [21] in 1972 in the context of Baer rings. He called them "Baer ideals". He proved that the lattice of Baer ideals of any Baer ring is complete and relatively pseudocomplemented. Thus, one of our theorems significantly improves this result of Speed. Baer ideals were also put to good use in 1972 by Evans [9] when characterizing Baer rings that are finite direct sums of integral domains.

In 1984, Jayaram [15] studied these ideals in general reduced rings, and not just the Baer ones. He used them to characterize quasiregular and von

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Neumann regular rings. Contessa [7] called these ideals "B-ideals" when they were not restricted to Baer rings. She obtained a fascinating characterization [7, Proposition 2.2(2)] in terms of ring homomorphisms that will play a crucial role in this paper.

The name "d-ideal" in rings was introduced by Mason [19] in 1988 to describe exactly what Contessa had called B-ideals. It should be mentioned though that Mason's rings in that paper are not assumed to be commutative. His choice of name was motivated by the notion of d-ideal in Riesz spaces. In Riesz spaces, d-ideals have been studied by several authors, mainly Luxemburg [17] and Huijsmans and de Pagter [13].

This now brings us to 2003; the time that Martínez and Zenk [18] abstracted the notion of d-ideal from Riesz spaces to algebraic frames. They introduced what they termed the d-nucleus on any algebraic frame with the finite intersection property (abbreviated FIP) on compact elements. They called the fixed elements of the d-nucleus on such a frame L the "d-elements" of L. A typical algebraic frame with FIP is the frame $\mathrm{RId}(A)$ of radical ideals of A. So, what are the d-elements of $\mathrm{RId}(A)$?

In [8], the authors study d-elements of $\mathrm{RId}(A)$ for A a reduced f-ring, and prove that they are exactly the d-ideals of A. This then proves that the lattice $\mathrm{DId}(A)$, for A a reduced f-ring is a frame. However, at the bottom of page 2 of [8] the authors start by saying "let A be a reduced ring", with the prefix "f-" omitted, which then gives the impression that $\mathrm{DId}(A)$ is the frame of d-ideals for any reduced ring A.

We point out at the beginning of Section 2 how the omission of the prefix impacts the intended result. We then show that the d-elements of $\mathrm{RId}(A)$, for any reduced ring A, is actually the lattice of what Artico, Marconi, and Moresco [2] call ζ-ideals (Proposition 2.1). We then proceed to show that although $\mathrm{DId}(A)$ is not necessarily the set of d-elements of $\mathrm{RId}(A)$, it *is* a coherent frame (Theorem 2.2), and its d-elements are also exactly the ζ-ideals of A (Proposition 2.5).

Section 3 is mainly about characterizations of Baer rings in terms of frames of d-ideals. It starts with an observation that in any Baer ring the sum of two d-ideals is a d-ideal (Lemma 3.1), which then yields that A is a Baer ring if and only if the sum of two d-ideals of A is a Baer ring and $\mathrm{DId}(A)$ is a regular frame (Proposition 3.3). This, in turn, enables us to show that A is a Baer ring if and only if the sum of two d-ideals in A is a d-ideal and $\mathrm{DId}(A)$ is isomorphic to the frame of ideals of the Boolean algebra of idempotents of A (Theorem 3.5).

In [5, p. 351], Banaschewski remarks that the significance of RId "lies in the fact that it represents the frame of open sets of the prime spectrum without any reference to the latter". This is so because, with the Axiom of Choice, $\mathrm{RId}(A)$ is isomorphic to the frame $\mathfrak{O}(\mathrm{Spec}(A))$. Letting $\mathrm{Min}(A)$ be the space of minimal prime ideals of A with the Zariski topology, we prove (Theorem 3.7) that, for any Baer ring A, the frame $\mathrm{DId}(A)$ is isomorphic to $\mathfrak{O}(\mathrm{Min}(A))$. Thus, in Baer rings, DId plays an analogous role to that of RId for the minimal spectrum because $\mathrm{DId}(A)$ is a frame constructed with no reference made to minimal prime ideals.

In Section 4, we make DId into a functor. We start by showing that the assignment $A \mapsto \mathrm{DId}(A)$ is a functorial from the category **BRng** of Baer rings with all ring homomorphisms to the category **CohFrm** of coherent frames and coherent maps (Theorem 4.3). If we wish to have a natural transformation $\mathrm{RId} \to \mathrm{DId}$, we are forced (as Lemma 4.4 attests) to restrict the ring homomorphisms to those that send elements with the same annihilator to images with the same annihilator. With the morphisms so restricted, we then broaden the class of objects in the domain of DId to be all rings in which the sum of two d-ideals is a d-ideal. The resulting category is denoted $\mathbf{SdRng_c}$. We then have that $\mathrm{DId}\colon \mathbf{SdRng_c} \to \mathbf{CohFrm}$ is a functor, and when the domain of RId is also restricted to $\mathbf{SdRng_c}$, we again have the same natural transformation $\mathrm{RId} \to \mathrm{DId}$ as in the case of Baer rings (Theorem 4.7).

We conclude by showing that the category $\mathbf{SdRng_c}$ has finite products. They are constructed exactly as in the category of all rings with ring homomorphisms (Theorem 5.3).

1. Preliminaries

1.1. Rings

Let us reiterate that all our rings are commutative, reduced, and have the identity. We write $\mathrm{Ann}(S)$ for the annihilator of $S \subseteq A$, and abbreviate $\mathrm{Ann}(\{a\})$ as $\mathrm{Ann}(a)$. The annihilator of the annihilator (or, colloquially speaking, the double-annihilator) of an element a will be written as $\mathrm{Ann}^2(a)$. We shall frequently use the identity

$$\mathrm{Ann}^2(xy) = \mathrm{Ann}^2(x) \cap \mathrm{Ann}^2(y)$$

which was proved by Henriksen and Jerison [12, Lemma 3.1].

The ideal generated by a set S will be written as $\langle S\rangle$. If $S = \{a_1, \dots, a_n\}$ we shall write $\langle a_1, \dots, a_n\rangle$, so that $\langle a\rangle$ is the principal ideal generated by a. Following Kist [16], we say a ring A is a *Baer ring* if for every $a \in A$ there is an idempotent $e \in A$ such that $\mathrm{Ann}(a) = \langle e\rangle$. This is equivalent to saying for every $a \in A$ there is an idempotent $e \in A$ such that $\mathrm{Ann}^2(a) = \langle e\rangle$. Observe that every Baer ring is reduced. Indeed, if $a^2 = 0$ in a Baer ring, then $a \in \mathrm{Ann}(a) = \langle e\rangle$ for some idempotent e. But then if $a \in \langle e\rangle$, then $a = re$ for some r, and so $a = (re)e = ae = 0$ because $\mathrm{Ann}(a) = \langle e\rangle$.

We should point out that the moniker "Baer ring" is in certain articles used to describe rings in which every annihilator ideal is generated by an idempotent. We shall write **BRng** for the class of Baer rings.

As already recalled in the Abstract, an ideal I of a ring A is a *d-ideal* if, for every $a \in I$, $\mathrm{Ann}^2(a) \subseteq I$. In [19], Mason gives several characterizations. One we shall frequently use is that I is a d-ideal if and only if whenever $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ and $a \in I$, then also $b \in I$. We shall denote by **SdRng** the class of rings in which the sum of two d-ideals is a d-ideal. If A is an object in **SdRng**, we shall also say A is an *sd-ring*.

The *radical* of an ideal I of A is the ideal

$$\sqrt{I} = \{x \in A \mid x^n \in I \text{ for some positive integer } n\}.$$

An ideal is called a *radical ideal* if it coincides with its radical. The smallest radical ideal containing an element a is denoted by $[a]$. That is, $[a] = \sqrt{\langle a \rangle}$.

1.2. Algebraic frames

Our reference for frames and their homomorphisms is [20]. Let L be a frame. An element $a \in L$ is *compact* if, for any $X \subseteq L$, $a \leq \bigvee X$ implies that there is a finite $Y \subseteq X$ with $a \leq \bigvee Y$. We denote by $\mathfrak{k}(L)$ the set of all compact elements of L. If every element of L is the join of compact elements below it, then L is said to be *algebraic*. If $a \wedge b \in \mathfrak{k}(L)$ for every $a, b \in \mathfrak{k}(L)$, then L is said to have the *finite intersection property*, throughout abbreviated as FIP. If the top element of L (which we shall denote by 1) is compact and L has FIP, then L is called *coherent*. A frame homomorphism between algebraic frames is called a *coherent map* if it maps compact elements to compact elements. The lattice $\mathrm{RId}(A)$ of radical ideals of A, ordered by inclusion, is a coherent frame (see [6]). Its compact elements are the finitely generated radical ideals.

A *polar* of L is an element of the form

$$z^{\perp} = \bigvee \{x \in L \mid x \wedge z = 0\}.$$

For any $a, b \in L$, $a \leq b$ implies $b^{\perp} \leq a^{\perp}$. If $a^{\perp} \vee b = 1$, it is said that a is *rather below* b. If every element of L is the join of elements rather below it, then L is *regular*.

Let L be an algebraic frame with FIP. The *d-nucleus* on L is defined by

$$d(a) = \bigvee \{c^{\perp\perp} \mid c \in \mathfrak{k}(L) \text{ and } c \leq a\}.$$

The resulting quotient frame (which is, of course, a sublocale of L) is denoted by dL, and its elements are called *d-elements*. It is an algebraic frame with FIP. It follows immediately from the definition that, for any $a \in L$, a is a d-element if and only if $c \leq a$ and c compact imply $c^{\perp\perp} \leq a$. We write $d_L \colon L \to dL$ for the frame homomorphism induced by d.

2. Lattices of d-ideals and ζ-ideals

Let A be a ring. In [8] it is claimed that the d-elements of the frame $\mathrm{RId}(A)$ are precisely the d-ideals of A. The argument purporting to verify this, however, has a gap in that it is claimed on page 4 of that paper that, for any d-ideal I,

$$\bigvee \{K^{\perp\perp} \mid K \in \mathfrak{k}(\mathrm{RId}(A)), K \subseteq I\} \subseteq I.$$

Since the compact elements of $\mathrm{RId}(A)$ are exactly the finitely generated ideals of A, this statement is true provided $\mathrm{Ann}^2(a_1, \ldots, a_n) \subseteq I$ for any finite set $\{a_1, \ldots, a_n\} \subseteq I$, if I is a d-ideal in A. This certainly holds for rings that have the *annihilator condition* of Henriksen and Jerison [12], which states that for any $a, b \in A$, there exists $c \in A$ such that $\mathrm{Ann}(a,b) = \mathrm{Ann}(c)$. It thus holds for reduced f-rings; which is what was intended in [8].

Let us identify the ideals of A that constitute the frame $d(\mathrm{RId}(A))$. In [2], an ideal I of A is called a *ζ-ideal* in case $\mathrm{Ann}^2(a_1,\dots,a_n) \subseteq I$ for any $\{a_1,\dots,a_n\} \subseteq I$. To be sure, this is not the definition used in [2]; however, as Mason shows in [19, Theorem 2.1], the condition stated above is equivalent to the definition given in [2]. It is clear that every ζ-ideal is a radical ideal, so that it belongs to the frame $\mathrm{RId}(A)$. Since A is reduced, for any finite set $\{a_1,\dots,a_n\} \subseteq A$, the double polar of the compact element $\sqrt{\langle a_1,\dots,a_n\rangle}$ of $\mathrm{RId}(A)$ is

$$\sqrt{\langle a_1,\dots,a_n\rangle}^{\perp\perp} = \mathrm{Ann}^2(a_1,\dots,a_n).$$

Since the compact elements of $\mathrm{RId}(A)$ are precisely the ideals of the form $\sqrt{\langle a_1,\dots,a_n\rangle}$, we have the following result.

Proposition 2.1. *The d-elements of* $\mathrm{RId}(A)$ *are precisely the ζ-ideals of A.*

We shall now show that the lattice of d-ideals of A is a frame. We utilize the notion of a prenucleus – an artefact that was introduced by Banaschewski [4]. Let us recall it.

For any frame L, a *prenucleus* on L is a mapping $k_0 \colon L \to L$ such that, for all $x, y \in L$:

(a) $x \le k_0(x)$,
(b) $x \le y$ implies $k_0(x) \le k_0(y)$, and
(c) $k_0(x) \wedge y \le k_0(x \wedge y)$.

The set $\mathrm{Fix}(k_0) = \{t \in L \mid k_0(t) = t\}$ is then a frame, and the mapping $k \colon L \to L$ given by

$$k(x) = \bigwedge\{t \in L \mid x \le t = k_0(t)\}$$

is a nucleus on L with $\mathrm{Fix}(k) = \mathrm{Fix}(k_0)$. We prove that $\mathrm{DId}(A)$ is a frame by exhibiting a prenucleus on $\mathrm{RId}(A)$ whose set of fixed elements is $\mathrm{DId}(A)$. As already mentioned, for any $a \in A$ the double polar $[a]^{\perp\perp}$ in the frame $\mathrm{RId}(A)$ is precisely $\mathrm{Ann}^2(a)$ since A is reduced.

In [4], Banaschewski calls a nucleus $j \colon L \to L$ *finitary* if it preserves joins of (upwards) directed sets; that is, if $j(\bigvee D) = \bigvee j[D]$ for each directed $D \subseteq L$. This is the case precisely when $\bigvee D \in \mathrm{Fix}(j)$ whenever $D \subseteq \mathrm{Fix}(j)$ is directed, with the join calculated in L.

Theorem 2.2. *For any reduced ring A, the lattice* $\mathrm{DId}(A)$ *is a coherent frame.*

Proof. (i) Let us show first that $\mathrm{DId}(A)$ is a frame. Define a mapping $k_0 : \mathrm{RId}(A) \to \mathrm{RId}(A)$ by

$$k_0(I) = \bigvee\{\mathrm{Ann}^2(a) \mid a \in I\}.$$

We claim that k_0 is a prenucleus on $\mathrm{RId}(A)$. It is clear that, for any $I, J \in \mathrm{RId}(A)$, $I \subseteq k_0(I)$, and $I \subseteq J$ implies $k_0(I) \subseteq k_0(J)$. Now,

$$J \cap k_0(I) = J \cap \bigvee\{\mathrm{Ann}^2(a) \mid a \in I\} = \bigvee\{J \cap \mathrm{Ann}^2(a) \mid a \in I\},$$

and since $J = \bigvee\{[u] \mid u \in J\}$, we have, for any $a \in I$,

$$\begin{aligned}
\mathrm{Ann}^2(a) \cap J &= \mathrm{Ann}^2(a) \cap \bigvee\{[u] \mid u \in J\} \\
&\subseteq \mathrm{Ann}^2(a) \cap \bigvee\{[u]^{\perp\perp} \mid u \in J\} \\
&= \bigvee\{\mathrm{Ann}^2(a) \cap \mathrm{Ann}^2(u) \mid u \in J\} \\
&= \bigvee\{\mathrm{Ann}^2(au) \mid u \in J\} \\
&\subseteq \bigvee\{\mathrm{Ann}^2(t) \mid t \in I \cap J\} \text{ since } au \in I \cap J \text{ for each } u \in J \\
&= k_0(I \cap J),
\end{aligned}$$

which leads to $J \cap k_0(I) \subseteq k_0(I \cap J)$, thus showing that k_0 is a prenucleus. It is easy to see that $\mathrm{Fix}(k_0) = \mathrm{DId}(A)$. Consequently, $\mathrm{DId}(A)$ is a frame.

(ii) We show next that $\mathrm{DId}(A)$ is compact. Let k_0 be the prenucleus on $\mathrm{RId}(A)$ defined above, and let k be the associated nucleus. Since $\mathrm{Fix}(k) = \mathrm{Fix}(k_0)$, to conclude that $\mathrm{DId}(A)$ is compact, it suffices, by the compactness criterion of Banaschewski mentioned above, to show that $\mathrm{DId}(A)$ is closed under directed joins taken in $\mathrm{RId}(A)$. Clearly, if $\{I_\alpha\}$ is a directed collection of d-ideals, then $\bigcup_\alpha I_\alpha$ is a d-ideal. So the result follows.

(iii) Finally, we prove the algebraic property and coherence. All joins mentioned in this part of the proof are calculated in $\mathrm{DId}(A)$. Now observe that, for any $J \in \mathrm{DId}(A)$,

$$J = \bigvee\{\mathrm{Ann}^2(u) \mid u \in J\}.$$

We claim that the compact elements of $\mathrm{DId}(A)$ are precisely the ideals of the form

$$\mathrm{Ann}^2(a_1) \vee \cdots \vee \mathrm{Ann}^2(a_n)$$

for finitely many elements $a_1, \ldots, a_n$ in A. To see that each such element of $\mathrm{DId}(A)$ is compact, it suffices to show that $\mathrm{Ann}^2(x)$ is compact for every $x \in A$. This however is true because if $\mathrm{Ann}^2(x)$ is below the join of some directed set $\mathcal{S} \subseteq \mathrm{DId}(A)$, then $x \in I$ for some $I \in \mathcal{S}$ since $\bigvee\mathcal{S} = \bigcup\mathcal{S}$, and hence $\mathrm{Ann}^2(x) \subseteq I$ since I is a d-ideal. On the other hand, let J be a compact element in $\mathrm{DId}(A)$. Since $J = \bigvee\{\mathrm{Ann}^2(u) \mid u \in J\}$, there are finitely many $u_1, \ldots, u_n$ in J such that $J = \mathrm{Ann}^2(u_1) \vee \cdots \vee \mathrm{Ann}^2(u_n)$.

We have thus far described $\mathfrak{k}(\mathrm{DId}(A))$, and shown that $\mathrm{DId}(A)$ is an algebraic frame. To show coherence, let $I = \mathrm{Ann}^2(a_1) \vee \cdots \vee \mathrm{Ann}^2(a_n)$ and $J = \mathrm{Ann}^2(b_1) \vee \cdots \vee \mathrm{Ann}^2(b_m)$ be any two compact elements. Then

$$I \wedge J = \Big(\mathrm{Ann}^2(a_1) \cap \mathrm{Ann}^2(b_1)\Big) \vee \cdots \vee \Big(\mathrm{Ann}^2(a_n) \cap \mathrm{Ann}^2(b_m)\Big)$$

$$= \mathrm{Ann}^2(a_1b_1) \vee \cdots \vee \mathrm{Ann}^2(a_nb_m),$$

which is a compact element. □

For any radical ideal I of A, let I_d denote the smallest d-ideal containing I. Define the mapping $k_A\colon \mathrm{RId}(A) \to \mathrm{DId}(A)$ by $k_A(I) = I_d$. Then of course k_A is the surjective frame homomorphism induced by the nucleus k above.

Since the compact elements of $\mathrm{RId}(A)$ are precisely the joins of finitely many ideals of the form $[a]$, and the compact elements of $\mathrm{DId}(A)$ are precisely the joins of finitely many ideals of the form $\mathrm{Ann}^2(a)$, we shall call these ideals *basic compact elements*. Note that the smallest d-ideal containing $[a]$ is $\mathrm{Ann}^2(a)$, so that $k_A([a]) = \mathrm{Ann}^2(a)$. Thus, k_A sends a basic compact element to a compact element. We therefore have the following corollary. Recall that a *dense* frame homomorphism is one that maps only the bottom to the bottom.

Corollary 2.3. *The mapping $k_A\colon \mathrm{RId}(A) \to \mathrm{DId}(A)$ is a dense onto coherent map. It is an isomorphism if and only if every radical ideal of A is a d-ideal.*

Remark 2.4. The mapping $k_A\colon \mathrm{RId}(A) \to \mathrm{DId}(A)$ is not *codense*; that is, the top is not the only element it sends to the top. Indeed, let A be any reduced ring with a non-divisor of zero a which is not invertible. Then $[a]$ is not the top element of the frame $\mathrm{RId}(A)$, but $k_A([a]) = \mathrm{Ann}^2(a) = A$, the top element of $\mathrm{DId}(A)$.

We saw in Proposition 2.1 that the d-elements of $\mathrm{RId}(A)$ are exactly the ζ-ideals of A. We show now that the same holds for $\mathrm{DId}(A)$.

Proposition 2.5. *If A is a reduced ring, then $d(\mathrm{RId}(A)) = d(\mathrm{DId}(A))$, and we have the following commutative diagram*:

$$\begin{tikzcd}
\mathrm{RId}(A) \arrow[r, "k_A"] \arrow[d, "d_{\mathrm{RId}(A)}"'] & \mathrm{DId}(A) \arrow[d, "d_{\mathrm{DId}(A)}"] \\
d(\mathrm{RId}(A)) \arrow[r, equal] & d(\mathrm{DId}(A))
\end{tikzcd} \tag{2.1}$$

Proof. (i) We must show that the d-elements of $\mathrm{DId}(A)$ are exactly the ζ-ideals of A. We observed in the discussion leading to Proposition 2.1 that the double polars of compact elements of $\mathrm{RId}(A)$ are precisely the ideals $\mathrm{Ann}^2(a_1, \ldots, a_n)$, for some finitely many elements a_i of A. Let I be a ζ-ideal of A. Then, of course, I is a d-ideal, so that it is an element of $\mathrm{DId}(A)$. We show that it is a d-element of this frame. Consider any compact element $K = \mathrm{Ann}^2(a_1) \vee \cdots \vee \mathrm{Ann}^2(a_n)$ in $\mathrm{DId}(A)$. We claim that the double polar of K in $\mathrm{DId}(A)$ is $\mathrm{Ann}^2(a_1, \ldots, a_n)$.

Indeed,

$$\begin{aligned} K^{\perp} = (\mathrm{Ann}^2(a_1) \vee \cdots \vee \mathrm{Ann}^2(a_n))^{\perp} &= \mathrm{Ann}^2(a_1)^{\perp} \cap \cdots \cap \mathrm{Ann}^2(a_n)^{\perp} \\ &= \mathrm{Ann}(a_1) \cap \cdots \cap \mathrm{Ann}(a_n) \\ &= \mathrm{Ann}(a_1, \ldots, a_n), \end{aligned}$$

which then implies $K^{\perp\perp} = \mathrm{Ann}^2(a_1, \ldots, a_n)$, as claimed. Now if $K \subseteq I$, then the elements $a_1, \ldots, a_n$ belong to I, and since I is a ζ-ideal we have $\mathrm{Ann}^2(a_1, \ldots, a_n) \subseteq I$, which then shows that I is a d-element in $\mathrm{DId}(A)$. That every d-element of $\mathrm{DId}(A)$ is a ζ-ideal is shown similarly.

(ii) To prove commutativity of the diagram, let us write J_ζ to designate the smallest ζ-ideal containing an ideal J. Let $I \in \mathrm{RId}(A)$. We must show that $d_{\mathrm{DId}(A)}(k_A(I)) = d_{\mathrm{RId}(A)}(I)$, that is, $d_{\mathrm{DId}(A)}(I_d) = I_\zeta$. Since $d_{\mathrm{DId}(A)}(I_d)$ is a ζ-ideal, by the first part, it is a d-ideal, and since it contains I, we have $I_d \subseteq I_\zeta$. Thus, I_ζ is a d-element in $\mathrm{DId}(A)$ containing I_d. Since $d_{\mathrm{DId}(A)}(I_d)$ is the smallest such, we have $d_{\mathrm{DId}(A)}(I_d) \subseteq I_\zeta$. On the other hand, $d_{\mathrm{DId}(A)}(I_d)$ is a ζ-ideal containing I_d, and hence I, and so, I_ζ being the smallest such, we have $I_\zeta \subseteq d_{\mathrm{DId}(A)}(I_d)$, and hence the desired equality. □

Remark 2.6. The reader may wonder why we did not depict diagram (2.1) as a triangle, with the bottom horizontal edge reduced to a single vertex. Here is the reason. Recall that a frame homomorphism $h\colon L \to M$ is *skeletal* if $a^\perp = b^\perp$ in L implies $h(a)^\perp = h(b)^\perp$. In [11], it is shown that every skeletal coherent map $h\colon L \to M$ between algebraic frames with FIP induces a coherent map $d(h)\colon dL \to dM$ that makes the diagram:

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "d_L"'] & M \arrow[d, "d_M"] \\ dL \arrow[r, "d(h)"] & dM \end{tikzcd}$$

commute. Since k_A is clearly skeletal, it induces a map $d(k_A)\colon d(\mathrm{RId}(A)) \to d(\mathrm{DId}(A))$. Using the calculation in the proof of Proposition 2.5, it is easy to see that this map is the identity map on $d(\mathrm{RId}(A))$.

3. Characterizing Baer rings in terms of DId

We are aiming for the characterization in Theorem 3.5 below. En route, we establish some intermediate results that we need, starting with the following lemma, which will frequently be used for other purposes as well. Recall that by an *sd*-ring we mean a ring in which the sum of two d-ideals is a d-ideal.

Lemma 3.1. *Every Baer ring is an sd-ring.*

Proof. Let I and J be d-ideals in a Baer ring A. Let $x \in I + J$, and pick $u \in I$ and $v \in J$ such that $x = u + v$. Now, $\mathrm{Ann}(u) \cap \mathrm{Ann}(v) \subseteq \mathrm{Ann}(x)$. Let e and f be idempotents such that $\mathrm{Ann}(u) = \langle e \rangle$ and $\mathrm{Ann}(v) = \langle f \rangle$. Then $\mathrm{Ann}(u) = \mathrm{Ann}^2(e)$ and $\mathrm{Ann}(v) = \mathrm{Ann}^2(f)$, and consequently

$$\mathrm{Ann}^2(ef) = \mathrm{Ann}^2(e) \cap \mathrm{Ann}^2(f) \subseteq \mathrm{Ann}(x),$$

whence $\mathrm{Ann}^2(x) \subseteq \mathrm{Ann}(ef) = \langle 1 - ef \rangle$. But $\mathrm{Ann}(u) = \langle e \rangle$ implies $1 - e \in \mathrm{Ann}^2(u) \subseteq I$, and $\mathrm{Ann}(v) = \langle f \rangle$ implies $1 - f \in \mathrm{Ann}^2(v) \subseteq J$, so that $1 - ef = f(1 - e) + (1 - f) \in I + J$, and hence $\mathrm{Ann}^2(x) \subseteq I + J$. Therefore $I + J$ is a d-ideal. $\square$

Remark 3.2. Observe that, in any ring, if the sum of two ideals is a d-ideal, then, in fact, the sum of any collection of d-ideals is a d-ideal. For, if $\{I_\alpha\}$ is a collection of d-ideals and $x \in \sum_\alpha I_\alpha$, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $x = x_{\alpha_1} + \cdots + x_{\alpha_n}$ for some $x_{\alpha_i} \in I_{\alpha_i}$. But now $I_{\alpha_1} + \cdots + I_{\alpha_n}$ is a d-ideal (by a simple induction argument), and so $\mathrm{Ann}^2(a) \subseteq I_{\alpha_1} + \cdots + I_{\alpha_n} \subseteq \sum_\alpha I_\alpha$.

Another preliminary result towards the main goal is, in fact, itself a characterization of Baer rings. Recall that an algebraic frame is *projectable* if $c^{\perp\perp} \vee c^{\perp} = 1$ for every $c \in \mathfrak{k}(L)$. It is known (see, for instance, [11, 4.2(c)]) that a ring A is Baer if and only if $\mathrm{RId}(A)$ is projectable.

Henceforth, we shall write the top of the frames $\mathrm{RId}(A)$ and $\mathrm{DId}(A)$ as $\top$, so as not to overstretch the usage of the symbols 1. Also, we will at times write $a^{\perp}$ and $a^{\perp\perp}$ for $\mathrm{Ann}(a)$ and $\mathrm{Ann}^2(a)$ when we view these ideals as elements in the frames $\mathrm{RId}(A)$ and $\mathrm{DId}(A)$.

Proposition 3.3. *The following are equivalent for a reduced ring A.*

(1) *A is a Baer ring.*
(2) *A is an sd-ring and* $\mathrm{DId}(A)$ *is regular.*
(3) *A is an sd-ring and* $\mathrm{DId}(A)$ *is projectable.*

Proof. (1) $\Rightarrow$ (2): Assume that A is a Baer ring. As shown in Lemma 3.1, A is an *sd*-ring. To show that $\mathrm{DId}(A)$ is regular, it suffices to show that every basic compact element is rather below itself. If K is a basic compact element in $\mathrm{DId}(A)$, then $K = \langle e \rangle$ for some idempotent $e \in A$. But clearly $\langle e \rangle \prec \langle e \rangle$.

(2) $\Rightarrow$ (3): In any algebraic frame, regularity implies projectability.

(3) $\Rightarrow$ (1): Assume that (3) holds. Let $a \in A$. Then $a^{\perp\perp} \vee a^{\perp} = \top$, which implies $\mathrm{Ann}^2(a) + \mathrm{Ann}(a) = A$, since A is an *sd*-ring. Therefore A is a Baer ring. $\square$

Remark 3.4. From a topological perspective, projectability is a weaker form of extremal disconnectedness. A frame L is *extremally disconnected* if $x^{\perp} \vee x^{\perp\perp} = 1$ for every $x \in L$. Having used the name Baer as we did, let us say a ring is *strongly Baer* if every annihilator ideal is generated by a single idempotent. A proof as above shows that a reduced ring A is strongly Baer if and only if it is an *sd*-ring and $\mathrm{DId}(A)$ is extremally disconnected.

We write $\mathrm{Idp}(A)$ for the Boolean algebra of idempotents of a ring A. It is perhaps worth recalling the operations of $\mathrm{Idp}(A)$:

$$u \wedge v = uv, \qquad u \vee v = u + v - uv, \qquad u' = 1 - u$$

for any $u, v \in \mathrm{Idp}(A)$. Thus, $e \leq f$ if and only if $e = ef$. Note that if I is an ideal in a ring A, then $I \cap \mathrm{Idp}(A)$ is a lattice-ideal in $\mathrm{Idp}(A)$. Note also that every element in a d-ideal of a Baer ring is a multiple of some idempotent belonging to the ideal. We write $\mathfrak{J}(\mathrm{Idp}(A))$ for the frame of ideals of $\mathrm{Idp}(A)$.

Theorem 3.5. *A reduced ring A is a Baer ring if and only if it is an sd-ring and the frames $\mathrm{DId}(A)$ and $\mathfrak{J}(\mathrm{Idp}(A))$ are isomorphic.*

Proof. ($\Rightarrow$) Suppose A is a Baer ring. Then A is an *sd*-ring, by Lemma 3.1. Define a mapping $g\colon \mathrm{DId}(A) \to \mathfrak{J}(\mathrm{Idp}(A))$ by

$$g(I) = I \cap \mathrm{Idp}(A).$$

This mapping clearly preserves the bottom element, the top element, and order. It also preserves binary meets because they are intersections in both frames. For a directed family $\{I_\alpha\} \subseteq \mathrm{DId}(A)$, we have

$$g\Big(\bigvee_\alpha I_\alpha\Big) = g\Big(\bigcup_\alpha I_\alpha\Big) = \mathrm{Idp}(A) \cap \bigcup_\alpha I_\alpha = \bigcup_\alpha \big(\mathrm{Idp}(A) \cap I_\alpha\big) = \bigcup_\alpha g(I_\alpha) = \bigvee_\alpha g(I_\alpha).$$

Now let $I, J \in \mathrm{DId}(A)$, and pick any $e \in g(I \vee J)$, that is, $e \in g(I + J)$. Then e is an idempotent belonging to $I + J$. We aim to show that $e \in g(I) \vee g(J)$. Since I and J are d-ideals in a Baer ring, there are elements $s, t \in A$ and idempotents $u \in I$ and $v \in J$ such that $e = su + tv$. Observe that $u \vee v \in g(I) \vee g(J)$. Now,

$$\begin{aligned} e \wedge (u \vee v) &= (su + tv)(u + v - uv) \\ &= su + suv - suv + tvu + tv - tvu \\ &= su + tv \\ &= e, \end{aligned}$$

which says $e \leq u \vee v$ in $\mathrm{Idp}(A)$, and hence e belongs to the ideal $g(I) \vee g(J)$ of $\mathrm{Idp}(A)$; whence $g(I \vee J) \subseteq g(I) \vee g(J)$, and hence equality. Therefore g is a frame homomorphism.

The principal ideals $\downarrow e$, for $e \in \mathrm{Idp}(A)$, generate the frame $\mathfrak{J}(\mathrm{Idp}(A))$. Since $g(\langle e \rangle) = \downarrow e$ (as one checks quickly), it follows that g is onto. Since the frame $\mathrm{DId}(A)$ is regular, by Proposition 3.3, and $\mathfrak{J}(\mathrm{Idp}(A))$ is compact, if we can show that g is dense, it will follow from [20, Proposition VII 2.2.1] that g is one-one. If $g(I) = \{0\}$, then 0 is the only idempotent of A belonging to I, and this makes I the zero ideal since I is a d-ideal. Therefore g is dense. In all then, g is an isomorphism.

($\Leftarrow$) Suppose A is an *sd*-ring, and that the frames $\mathrm{DId}(A)$ and $\mathfrak{J}(\mathrm{Idp}(A))$ are isomorphic. Let $h\colon \mathrm{DId}(A) \to \mathfrak{J}(\mathrm{Idp}(A))$ be a frame isomorphism. Consider any $a \in A$. Since $a^{\perp\perp}$ is a compact element in $\mathrm{DId}(A)$, $h(a^{\perp\perp})$ is a compact element in $\mathfrak{J}(\mathrm{Idp}(A))$. But the compact elements of this latter frame are precisely the principal ideals; so there is an idempotent $e \in A$ such that $h(a^{\perp\perp}) = {\downarrow}e$. The pseudocomplement of $a^{\perp\perp}$ in $\mathrm{DId}(A)$ is $a^{\perp}$, and it is mapped by h to the pseudocomplement of ${\downarrow}e$, which is, in fact, the complement ${\downarrow}(1-e)$ of ${\downarrow}e$. In light of h being an isomorphism, this says $a^{\perp}$ is the complement of $a^{\perp\perp}$ in $\mathrm{DId}(A)$. Since A is an sd-ring,

$$A = \top = a^{\perp\perp} \vee a^{\perp} = a^{\perp\perp} + a^{\perp} = \mathrm{Ann}^2(a) + \mathrm{Ann}(a),$$

which proves that A is a Baer ring. □

Remark 3.6. The mapping $Q \mapsto Q \cap \mathrm{Idp}(A)$, restricted to minimal prime ideals, is employed by Kist [16] to prove that a reduced ring A is a Baer ring if and only if this mapping is a homeomorphism $\mathrm{Min}(A) \to \mathrm{Min}(\mathfrak{J}(\mathrm{Idp}(A)))$.

The remaining results in this section are not characterizations of Baer rings, but their inclusion is justified by their nature. As mentioned in the Introduction, it is known that if the Axiom of Choice is assumed, then, for any ring A, the frame $\mathrm{RId}(A)$ is isomorphic to the frame $\mathfrak{O}(\mathrm{Spec}(A))$, where $\mathrm{Spec}(A)$ designates the spectrum of A with the Zariski topology. Since $\mathrm{Min}(A)$ is a subspace of $\mathrm{Spec}(A)$, we may reasonably expect that (in perhaps some cases) the frame $\mathfrak{O}(\mathrm{Min}(A))$ is isomorphic to some quotient on $\mathrm{RId}(A)$. For Baer rings, this is indeed the case.

Theorem 3.7. *If A is a Baer ring, then the frame $\mathfrak{O}(\mathrm{Min}(A))$ is isomorphic to $\mathrm{DId}(A)$.*

Proof. Define a mapping $g\colon \mathrm{DId}(A) \to \mathfrak{O}(\mathrm{Min}(A))$ by

$$g(I) = \{P \in \mathrm{Min}(A) \mid P \not\supseteq I\}.$$

Then clearly g preserves the bottom element, the top element, and order. Using the last-mentioned property, one checks easily that for any $I, J \in \mathrm{DId}(A)$, $g(I) \cap g(J) \subseteq g(I \cap J)$, so that g preserves binary meets. Since joins are sums, it is again easy to establish the containment $g\big(\sum_\alpha I_\alpha\big) \subseteq \bigcup_\alpha g(I_\alpha)$, which then delivers preservation of joins. Since the sets of the form $\{P \in \mathrm{Min}(A) \mid a \notin P\}$, for $a \in A$, form a base for $\mathrm{Min}(A)$, and since, for any minimal prime ideal P and $a \in A$, $a \notin P$ if and only if $\mathrm{Ann}^2(a) \nsubseteq P$, it follows that g is onto. To see that g is one-one, we recall from [21, Theorem 5.1] that in a Baer ring every d-ideal equals the intersection of the minimal prime ideals containing it. Thus, if $g(I) = g(J)$, then

$$I = \bigcap\{P \in \mathrm{Min}(A) \mid I \subseteq P\} = \bigcap\big(\mathrm{Min}(A) \smallsetminus g(I)\big) = \bigcap\big(\mathrm{Min}(A) \smallsetminus g(J)\big) = J,$$

which shows that g is one-one. □

Remark 3.8. Recall that a *point* of a frame is an element $p < 1$ such that whenever $x \wedge y \le p$, then $x \le p$ or $y \le p$. For a Baer ring A, the points of $\mathrm{DId}(A)$ are precisely the minimal prime ideals of A. Indeed, if P is minimal prime, then it is a d-ideal, and if $I \cap J \subseteq P$ for $I, J \in \mathrm{DId}(A)$, then $IJ \subseteq P$, so that $I \subseteq P$ or $J \subseteq P$. On the other hand, if P is a point in $\mathrm{DId}(A)$, then for any $a, b \in A$ with $ab \in P$, we have $\mathrm{Ann}^2(a) \cap \mathrm{Ann}^2(b) = \mathrm{Ann}^2(ab) \subseteq P$, so that $\mathrm{Ann}^2(a) \subseteq P$ or $\mathrm{Ann}^2(b) \subseteq P$, whence $a \in P$ or $b \in P$, showing that P is a prime ideal. It is minimal prime because if $x \in P$, then $\mathrm{Ann}^2(x) = \langle e \rangle$ for some idempotent e, whence $1 - e$ is a non-member of P annihilating x.

In their study of regularity in algebraic frames, Martínez and Zenk [18] introduce what they call the Reg-properties, which, listed from the strongest to the weakest are:

Reg(1) Each element of L is the join of elements rather below it (i.e., L is regular).
Reg(2) Each d-element of L is the join of elements rather below it.
Reg(3) Each polar of L is the join of elements rather below it.
Reg(4) Each $c^{\perp}$, with c compact, is the join of elements rather below it.

They show in [18, Theorem 2.4] that Reg(1) is equivalent to $c \vee c^{\perp} = 1$ for every $c \in \mathfrak{k}(L)$; Reg(2) and Reg(3) are equivalent, and each is equivalent to the condition $c^{\perp\perp} \vee c^{\perp} = 1$ for every $c \in \mathfrak{k}(L)$; and Reg(4) is equivalent to the condition that $c^{\perp} \vee d^{\perp} = 1$ whenever $c \wedge d = 0$ in $\mathfrak{k}(L)$.

So, Proposition 3.3 tells that Baer rings are precisely the objects of **SdRng** that satisfy each of the first three Reg-properties. Regarding the fourth, we recall that a ring A is called an *mp-ring* in case every maximal ideal contains a unique minimal prime ideal. In [1, Lemma β], the authors give two characterizations of *mp*-rings. The one we shall use says a reduced ring A is an *mp*-ring if and only whenever $ab = 0$ in A, then $\mathrm{Ann}(a) + \mathrm{Ann}(b) = A$.

Proposition 3.9. *An sd-ring A is an mp-ring if and only if* $\mathrm{DId}(A)$ *satisfies* Reg(4).

Proof. Assume that $\mathrm{DId}(A)$ satisfies Reg(4). Let $a, b \in A$ be such that $ab = 0$. Then, $a^{\perp\perp}$ and $b^{\perp\perp}$ are compact elements of $\mathrm{DId}(A)$ with $a^{\perp\perp} \cap b^{\perp\perp} = (ab)^{\perp\perp} = \{0\}$. Since $\mathrm{DId}(A)$ satisfies Reg(4), $a^{\perp} \vee b^{\perp} = \top$, which implies that $\mathrm{Ann}(a) + \mathrm{Ann}(b) = A$. Therefore A is an *mp*-ring.

Conversely, assume that A is an *mp*-ring. Let K and H be compact elements in $\mathrm{DId}(A)$ with $K \wedge H = \{0\}$. Pick elements $a_1, \dots, a_n$ and $b_1, \dots, b_m$ in A such that $K = a_1^{\perp\perp} \vee \cdots \vee a_n^{\perp\perp}$ and $H = b_1^{\perp\perp} \vee \cdots \vee b_m^{\perp\perp}$. For any pair $(i, j) \in \{1, \dots, n\} \times \{1, \dots, m\}$, we have $a_i^{\perp\perp} \cap b_j^{\perp\perp} = \{0\}$, so that $a_i b_j = 0$, and hence $\mathrm{Ann}(a_i) + \mathrm{Ann}(b_j) = A$, since A is an *mp*-ring, whence $a_i^{\perp} \vee b_j^{\perp} = \top$ in $\mathrm{DId}(A)$. Consequently,

$$
\begin{aligned}
K^{\perp} \vee H^{\perp} &= (a_1^{\perp} \wedge \cdots \wedge a_n^{\perp}) \vee (b_1^{\perp} \wedge \cdots \wedge b_m^{\perp}) \\
&= \bigwedge \Big\{ a_i^{\perp} \vee b_j^{\perp} \mid (i, j) \in \{1, \dots, n\} \times \{1, \dots, m\} \Big\} \\
&= \top.
\end{aligned}
$$

Therefore $\mathrm{DId}(A)$ satisfies Reg(4). $\square$

In [6], Banaschewski calls a ring A *Gelfand* if whenever $a+b=1$ in A, there exist $r,s\in A$ such that $(1+ra)(1+sb)=0$. He then proves that a ring A is Gelfand if and only if $\mathrm{RId}(A)$ is normal. We show that, for rings in **SdRng**, the normality of $\mathrm{DId}(A)$ is a necessary but not sufficient condition for A to be Gelfand.

Proposition 3.10. *If A is a Gelfand sd-ring, then* $\mathrm{DId}(A)$ *is normal. The converse fails.*

Proof. Let $I,J\in\mathrm{DId}(A)$ be such that $I\vee J=\top$. So $I+J=A$, and hence there exist $u\in I$ and $v\in J$ such that $u+v=1$. Therefore $[u]\vee[v]=\top$ in $\mathrm{RId}(A)$. Since $\mathrm{RId}(A)$ is normal, by the result of Banaschewski mentioned above, there are elements $S,T\in\mathrm{RId}(A)$ such that

$$S\cap T=\{0\}\qquad\text{and}\qquad S\vee[u]=\top=T\vee[v].$$

The latter ensures us elements $s\in S$ and $t\in T$ such that $[s]\vee[u]=\top=[t]\vee[v]$. Applying the homomorphism $k_A\colon \mathrm{RId}(A)\to\mathrm{DId}(A)$ to this yields $s^{\perp\perp}\vee u^{\perp\perp}=\top=t^{\perp\perp}\vee v^{\perp\perp}$. Now, since I and J are d-ideals, $u^{\perp\perp}\subseteq I$ and $v^{\perp\perp}\subseteq J$, and since $S\cap T=\{0\}$, $(st)^{\perp\perp}=\{0\}$. Consequently,

$$s^{\perp\perp}\cap t^{\perp\perp}=\{0\}\qquad\text{and}\qquad s^{\perp\perp}\vee I=\top=t^{\perp\perp}\vee J,$$

which shows that $\mathrm{DId}(A)$ is normal. $\square$

Here is an example showing that the converse fails.

Example 3.11. In any ring, every proper d-ideal consists entirely of zero-divisors. So the Baer ring $\mathbb{Z}$ has only two d-ideals; namely, the zero ideal and the whole ring. Thus, $\mathbb{Z}$ is an *sd*-ring with $\mathrm{DId}(\mathbb{Z})$ a two-element frame, which is normal. But of course $\mathbb{Z}$ is not a Gelfand ring.

4. Making DId into a functor

We are going to show that DId defines a functor $\mathrm{DId}\colon \mathbf{BRng}\to\mathbf{CohFrm}$. We will then extend the domain of DId to include more objects than just the Baer rings. Let us explain. We saw in Lemma 3.1 that $\mathbf{BRng}\subseteq\mathbf{SdRng}$. The following example shows that the inclusion is proper.

Example 4.1. Recall that a completely regular Hausdorff space X is called an *F-space* [10] if any finitely generated ideal in the ring $C(X)$ is a principal ideal. If X is an F-space, then the sum of two d-ideals in $C(X)$ is a d-ideal. This is a consequence of [13, Theorem 4.4] and [14, Theorem 10.5]. Now, as shown in [3, Theorem 3.3], $C(X)$ is a Baer ring precisely when X is basically disconnected (meaning that the closure of every cozero set is open). It is known that $\beta\mathbb{R}\smallsetminus\mathbb{R}$ is an F-space which is not basically disconnected (see [10, Chapter 14]). Thus, $C(\beta\mathbb{R}\smallsetminus\mathbb{R})$ is an object in **SdRng** but not in **BRng**.

In the course of the upcoming proof we shall write $S+T=\{s+t \mid s \in S, t \in T\}$ for any subsets S and T of a ring, even if they are not ideals. Since $e^{\perp\perp}=\langle e\rangle$ for any idempotent element e, the basic compact elements in $\mathrm{DId}(A)$, with A a Baer ring, are precisely the d-ideals $e^{\perp\perp}$ for e an idempotent in A.

Lemma 4.2. *For any morphism $\phi\colon A \to B$ in* **SdRng**, *the mapping $\tilde{\phi}$: $\mathrm{DId}(A) \to \mathrm{DId}(B)$ defined by*

$$\tilde{\phi}(I)=\bigvee\{\phi(a)^{\perp\perp} \mid a \in I\}$$

is a frame homomorphism. Furthermore, if A is a Baer ring, then $\tilde{\phi}$ is a coherent map.

Proof. (i) We prove first that $\tilde{\phi}$ is a frame homomorphism. Observe that, for any d-ideal I of A, $\tilde{\phi}(I)$ is the smallest d-ideal of B containing the set $\phi[I]$. It is clear that $\tilde{\phi}$ preserves the bottom and the top elements of $\mathrm{DId}(A)$. Also, $\tilde{\phi}$ preserves order. Let us show that $\tilde{\phi}$ preserves meets. So let $I, J \in \mathrm{DId}(A)$. Then,

$$\begin{aligned}
\tilde{\phi}(I) \cap \tilde{\phi}(J) &= \bigvee\{\phi(a)^{\perp\perp} \mid a \in I\} \cap \bigvee\{\phi(b)^{\perp\perp} \mid b \in J\}\\
&= \bigvee\{\phi(a)^{\perp\perp} \cap \phi(b)^{\perp\perp} \mid a \in I, b \in J\}\\
&= \bigvee\{\big(\phi(a)\phi(b)\big)^{\perp\perp} \mid a \in I, b \in J\}\\
&= \bigvee\{\phi(ab)^{\perp\perp} \mid a \in I, b \in J\}\\
&\subseteq \bigvee\{\phi(t)^{\perp\perp} \mid t \in I \cap J\}\\
&= \tilde{\phi}(I \cap J),
\end{aligned}$$

and hence we have the desired equality since $\tilde{\phi}$ preserves order.

Next, we show that $\tilde{\phi}$ preserves directed joins, which are of course unions. So let $\{I_\alpha\}$ be a directed collection of d-ideals of A. Since

$$\phi\Big[\bigcup_\alpha I_\lambda\Big] \subseteq \bigcup_\alpha \phi[I_\alpha] \subseteq \bigvee_\alpha \tilde{\phi}(I_\alpha),$$

and since $\tilde{\phi}\Big(\bigcup_\alpha I_\alpha\Big)$ is the smallest d-ideal of B containing the set $\phi\Big[\bigcup_\alpha I_\alpha\Big]$, it follows that $\tilde{\phi}\Big(\bigvee_\alpha I_\alpha\Big) \subseteq \bigvee_\alpha \tilde{\phi}(I_\alpha)$, and hence equality. To conclude that $\tilde{\phi}$ is a frame homomorphism, we are left with showing that it preserves binary joins. Let $I, J \in \mathrm{DId}(A)$. Since A is an *sd*-ring, $I \vee J = I + J$, so that $\tilde{\phi}(I \vee J)=\bigvee\{\phi(x)^{\perp\perp} \mid x \in I+J\}$. Now, if $x \in I+J$, then

$$\phi(x) \in \phi[I]+\phi[(J)] \subseteq \tilde{\phi}(I)+\tilde{\phi}(J)=\tilde{\phi}(I) \vee \tilde{\phi}(J),$$

and hence $\phi(x)^{\perp\perp} \subseteq \tilde{\phi}(I) \vee \tilde{\phi}(J)$, since the latter is a d-ideal. Thus, $\tilde{\phi}(I \vee J) \subseteq \tilde{\phi}(I) \vee \tilde{\phi}(J)$, and hence equality. Therefore $\tilde{\phi}$ is a frame homomorphism.

(ii) Now we show that if A is a Baer ring, then $\tilde{\phi}$ is a coherent map. We need only show that it sends basic compact elements to compact elements. Consider then any idempotent $e \in A$. For any $x \in e^{\perp\perp}$, we have $x \in \langle e \rangle$, so that $x = xe$, whence $\phi(x) = \phi(x)\phi(e)$, and hence

$$\phi(x)^{\perp\perp} = \big(\phi(x)\phi(e)\big)^{\perp\perp} = \phi(x)^{\perp\perp} \cap \phi(e)^{\perp\perp} \subseteq \phi(x)^{\perp\perp}.$$

Since $e \in e^{\perp\perp}$, we then have

$$\phi(e)^{\perp\perp} \subseteq \bigvee\{\phi(x)^{\perp\perp} \mid x \in e^{\perp\perp}\} \subseteq \phi(e)^{\perp\perp},$$

which shows that $\tilde{\phi}(e^{\perp\perp}) = \phi(e)^{\perp\perp}$. Therefore $\tilde{\phi}$ is a coherent map. □

Theorem 4.3. $\mathrm{DId}\colon \mathbf{BRng} \to \mathbf{CohFrm}$ *is a functor sending an object A to* $\mathrm{DId}(A)$*, and a morphism $\phi\colon A \to B$ to $\tilde{\phi}\colon \mathrm{DId}(A) \to \mathrm{DId}(B)$.*

Proof. We need only show that DId preserves identity maps and composition. The former is immediate since $I = \bigvee\{x^{\perp\perp} \mid x \in I\}$ for every d-ideal. So we are left with showing that $\mathrm{DId}(\psi \cdot \phi) = \mathrm{DId}(\psi) \cdot \mathrm{DId}(\phi)$, for any two morphisms $\phi\colon A \to B$ and $\psi\colon B \to C$ in **BRng**. By coherence of the frames involved, it suffices to show that these composites agree on basic compact elements. So let e be an idempotent in A. We have observed in the proof of the lemma above that if $\tau\colon R \to S$ is a morphism in **SdRng** whose domain is a Baer ring, then $\tilde{\tau}(u^{\perp\perp}) = \tau(u)^{\perp\perp}$ for every idempotent u in A. Since the image of an idempotent under a ring homomorphism is an idempotent,

$$\begin{aligned}
\big(\mathrm{DId}(\psi) \cdot \mathrm{DId}(\phi)\big)(e^{\perp\perp}) &= \tilde{\psi}\big(\phi(e)^{\perp\perp}\big) \\
&= \psi\big(\phi(e)\big)^{\perp\perp} \\
&= (\psi \cdot \phi)(e)^{\perp\perp} \\
&= \mathrm{DId}(\psi \cdot \phi)(e^{\perp\perp}),
\end{aligned}$$

which shows that $\mathrm{DId}(\psi \cdot \phi) = \mathrm{DId}(\psi) \cdot \mathrm{DId}(\phi)$. □

Recall the mapping $k_A\colon \mathrm{RId}(A) \to \mathrm{DId}(A)$ from Section 2. It is natural to wonder if the system $\eta = (k_A)_{A \in \mathbf{BRng}}$ is a natural transformation $\eta\colon \mathrm{RId} \longrightarrow \mathrm{DId}$. We shall see that if we prune the class of morphisms a little (in a way that is necessary), then η is a natural transformation. Recall that, for any $a \in A$, $k_A([a]) = a^{\perp\perp}$.

Lemma 4.4. *For any morphism $\phi\colon A \to B$ in* **SdRng***, the three statements below are equivalent. Furthermore, they imply that $\tilde{\phi}\colon \mathrm{DId}(A) \to \mathrm{DId}(B)$ is a coherent map.*

(1) *$\tilde{\phi}(x^{\perp\perp}) = \phi(x)^{\perp\perp}$ for every $x \in A$.*
(2) *For any $u, v \in A$, $u^{\perp\perp} = v^{\perp\perp}$ implies $\phi(u)^{\perp\perp} = \phi(v)^{\perp\perp}$.*

(3) *The diagram*

$$\begin{tikzcd} \mathrm{RId}(A) \arrow[r, "k_A"] \arrow[d, "\mathrm{RId}(\phi)"'] & \mathrm{DId}(A) \arrow[d, "\mathrm{DId}(\phi)"] \\ \mathrm{RId}(B) \arrow[r, "k_B"] & \mathrm{DId}(B) \end{tikzcd}$$

commutes, where, as before, $\mathrm{DId}(\phi) = \tilde{\phi}$.

Proof. (1) $\Leftrightarrow$ (2): Assume that (1) holds, and consider any $u, v \in A$ with $u^{\perp\perp} = v^{\perp\perp}$. Then

$$\phi(u)^{\perp\perp} = \tilde{\phi}(u^{\perp\perp}) = \tilde{\phi}(v^{\perp\perp}) = \phi(v)^{\perp\perp},$$

which shows that (2) holds.

Conversely, assume (2) holds, and let $x \in A$. If $z \in x^{\perp\perp}$, then $z^{\perp\perp} \subseteq x^{\perp\perp}$, and hence $z^{\perp\perp} = z^{\perp\perp} \cap x^{\perp\perp} = (zx)^{\perp\perp}$, whence, by (2),

$$\phi(z)^{\perp\perp} = \phi(zx)^{\perp\perp} = \big(\phi(z)\phi(x)\big)^{\perp\perp} = \phi(z)^{\perp\perp} \cap \phi(x)^{\perp\perp},$$

which implies $\phi(z)^{\perp\perp} \subseteq \phi(x)^{\perp\perp}$. Consequently,

$$\phi(x)^{\perp\perp} \subseteq \bigvee\{\phi(z)^{\perp\perp} \mid z \in x^{\perp\perp}\} \subseteq \phi(x)^{\perp\perp},$$

which proves that $\tilde{\phi}(x^{\perp\perp}) = \phi(x)^{\perp\perp}$.

(1) $\Leftrightarrow$ (3): Since $\mathrm{RId}(A)$ is an algebraic frame, the diagram commutes if and only if the composites $\mathrm{DId}(\phi) \cdot k_A$ and $k_B \cdot \mathrm{RId}(\phi)$ agree on basic compact elements. Since, for any $x \in A$, $\mathrm{RId}(\phi)([x]) = [\phi(x)]$, we have

$$\mathrm{DId}(\phi)(k_A([x])) = k_B(\mathrm{RId}(\phi)([x])) \quad\iff\quad \tilde{\phi}(x^{\perp\perp}) = \phi(x)^{\perp\perp},$$

which shows that (1) and (3) are equivalent.

Finally, we show that (3) implies $\tilde{\phi}$ is a coherent map. We prove this via a more general observation. Suppose $h\colon L \to M$ and $g\colon M \to N$ are frame homomorphisms between algebraic frames with h onto, and h and gh coherent. Then g is coherent. To see this, let $b \in \mathfrak{k}(M)$. Since h is coherent and onto, there is an $a \in \mathfrak{k}(L)$ with $h(a) = b$. Since gh is coherent and $g(b) = gh(a)$, it follows that $g(b)$ is compact. Thus g is coherent. The result at hand then follows since k_A is an onto coherent map, and, by (3), $\mathrm{DId}(\phi) \cdot k_A = k_B \cdot \mathrm{RId}(\phi)$, which is a coherent map. $\square$

Remark 4.5. Ring homomorphisms between Baer rings that have the property described by condition (2) in the foregoing lemma were named "R-compatible" by Speed [21]. These homomorphisms were also considered by Contessa in [7]. She named them "Baer homomorphisms". It is rather curious that they should arise in the current work in the form they did. We give them a name slightly modified from the one Speed used because we do not wish to restrict them to Baer rings.

Definition 4.6. A ring homomorphism $\phi\colon A \to B$ is *compatible* if, for any $u, v \in A$, $u^{\perp\perp} = v^{\perp\perp}$ implies $\phi(u)^{\perp\perp} = \phi(v)^{\perp\perp}$. Composites of compatible homomorphisms are compatible. We let $\mathbf{SdRng}_{\mathrm{c}}$ be the subcategory of $\mathbf{SdRng}$ whose morphisms are the compatible ring homomorphisms.

If $\phi\colon A \to B$ is an arbitrary ring homomorphism between $\mathbf{SdRng}$-objects, all we know is that $\tilde{\phi}$ is a frame homomorphisms (Lemma 4.2). We are not guaranteed that it is a coherent map. But now, thanks to Lemma 4.4, we know that if ϕ is a compatible homomorphism, then $\tilde{\phi}$ is a coherent map. In proving that DId preserves composition (in Theorem 4.3), we specifically used Baerness. But now, once again thanks to Lemma 4.4, if $\phi\colon A \to B$ and $\psi\colon B \to C$ are compatible homomorphisms, then $\widetilde{\psi \cdot \phi} = \tilde{\psi} \cdot \tilde{\phi}$, as one checks quickly. In all then, we have the following theorem.

Theorem 4.7. DId: $\mathbf{SdRng}_{\mathrm{c}} \to \mathbf{CohFrm}$ *is functor, and the system* $\eta = (k_A)_{A \in \mathbf{SdRng}_{\mathrm{c}}}$ *is a natural transformation* $\eta\colon \mathrm{RId} \longrightarrow \mathrm{DId}$*, with the domain category of* RId *restricted to* $\mathbf{SdRng}_{\mathrm{c}}$*.*

DId is not faithful. Let $\mathbb{C}$ denote the complex field, $\iota\colon \mathbb{C} \to \mathbb{C}$ be the identity map, and $\phi\colon \mathbb{C} \to \mathbb{C}$ the homomorphism that sends each complex number to its conjugate. Since $\mathrm{DId}(\mathbb{C})$ is the two-element frame, $\mathrm{DId}(\iota) = \mathrm{DId}(\phi)$.

5. A categorical note on $\mathbf{SdRng}_{\mathrm{c}}$

We end by showing that the category $\mathbf{SdRng}_{\mathrm{c}}$ has finite products. We will establish this via two lemmas, the first of which is stated more generally than is necessary to arrive at the main result.

Given a family $\{A_\alpha \mid \alpha \in F\}$ of rings, we denote, as usual, by $p_\beta\colon \prod A_\alpha \to A_\beta$ the βth projection.

Lemma 5.1. *For any family* $\{A_\alpha \mid \alpha \in F\}$ *of rings, each projection is a compatible homomorphism.*

Proof. Fix an index $\beta \in F$. Let $a = (a_\alpha)$ and $b = (b_\alpha)$ be elements of $\prod A_\alpha$ with $a^\perp = b^\perp$. We claim that $a_\beta^\perp = b_\beta^\perp$. Indeed, if w is an element of A_β with $wa_\beta = 0$, then the element $\overline{w}$ of $\prod A_\alpha$ whose αth coordinate is given by

$$w_\alpha = \begin{cases} 0 & \text{if} \quad \alpha \neq \beta, \\ w & \text{if} \quad \alpha = \beta, \end{cases}$$

annihilates a, and hence b, which then implies $wb_\beta = 0$, whence $a_\beta^\perp \subseteq b_\beta^\perp$, thence equality by symmetry. Since $p_\beta(a) = a_\beta$ and $p_\beta(b) = b_\beta$, it follows that p_β is a compatible homomorphism. $\square$

Lemma 5.2. *Let* $\{A_\alpha \mid \alpha \in F\}$ *be a family of rings. Suppose, for each* $\alpha \in F$*,* I_α *and* J_α *are ideals of* A_α*.*

(1) $\prod I_\alpha$ *is a d-ideal in* $\prod A_\alpha$ *if and only if each* I_α *is a d-ideal in* A_α.
(2) *If* $I_\alpha + J_\alpha$ *is a d-ideal for each* α, *then* $\prod I_\alpha + \prod J_\alpha$ *is a d-ideal in* $\prod A_\alpha$.

Proof. (1) Assume first that each I_α is a d-ideal. Let $a = (a_\alpha)$ and $b = (b_\alpha)$ be elements of $\prod A_\alpha$ with $a^\perp = b^\perp$ and $a \in \prod I_\alpha$. As observed in the previous proof, $a_\alpha^\perp = b_\alpha^\perp$ for each α. Since $a_\alpha \in I_\alpha$ and I_α is a d-ideal, $b_\alpha \in I_\alpha$, and hence $b \in \prod I_\alpha$. Therefore $\prod I_\alpha$ is a d-ideal.

Conversely, assume that $\prod I_\alpha$ is a d-ideal. Fix an index $\beta \in F$. Let $u, v \in A_\beta$ be such that $u^\perp = v^\perp$ and $u \in I_\beta$. Let $\overline{u}$ and $\overline{v}$ be elements of $\prod A_\alpha$ constructed the same way $\overline{w}$ was in the proof of the previous lemma. It is easy to see that any element of $\prod A_\alpha$ annihilates $\overline{u}$ if and only if it annihilates $\overline{v}$; so that $\overline{u}^\perp = \overline{v}^\perp$. Since $u \in I_\beta$, we have $\overline{u} \in \prod I_\alpha$, and hence $\overline{v} \in \prod I_\alpha$, which implies $v \in I_\beta$. Therefore I_β is d-ideal.

(2) A routine calculation shows that $\prod I_\alpha + \prod J_\alpha = \prod (I_\alpha + J_\alpha)$, and hence the result follows from the first part. $\square$

We now have enough material to prove the result we aimed for.

Theorem 5.3. *The category* $\mathbf{SdRng}_{\mathrm{c}}$ *has finite products.*

Proof. Let $A_1, \dots, A_n$ be sd-rings, and denote by A their direct product $A_1 \times \cdots \times A_n$ (calculated in **Rng**). Let I and J be d-ideals of A. Since each A_k (for $k \in \{1, \dots, n\}$) has identity, there are ideals $I_k \subseteq A_k$ and $J_k \subseteq A_k$ such that

$$I = I_1 \times \cdots \times I_n \qquad \text{and} \qquad J = J_1 \times \cdots \times J_n.$$

Then, by Lemma 5.2(1), each I_k and each J_k is a d-ideal, and hence $I_k + J_k$ is a d-ideal because A_k is an sd-ring. Therefore, by Lemma 5.2(2), $I + J$ is a d-ideal in A, which shows that A is an object in the appropriate category. By Lemma 5.1, the homomorphisms $p_k \colon A \to A_k$ are morphisms in the category $\mathbf{SdRng}_{\mathrm{c}}$. We show that the pair $(A, (p_k)_k)$ is the desired product. Consider any sd-ring B with morphisms $\phi_k \colon B \to A_k$, for $k \in \{1, \dots, n\}$, belonging to $\mathbf{SdRng}_{\mathrm{c}}$. We must produce a unique compatible ring homomorphism $\phi \colon B \to A$ such that, for each k, the diagram

$$\begin{tikzcd} B \arrow[r, dashed, "\phi"] \arrow[dr, "\phi_k"'] & A \arrow[d, "p_k"] \\ & A_k \end{tikzcd}$$

commutes. As in the **Rng** case, let $\phi \colon B \to A$ be the ring homomorphism defined by

$$\phi(b) = (\phi_1(b), \dots, \phi_n(b)).$$

Then ϕ is a unique ring homomorphism making the diagram commute. To see that it is compatible, let $x, y \in B$ be such that $x^\perp = y^\perp$. The compatibility of each ϕ_k ensures that $\phi_k(x)^\perp = \phi_k(y)^\perp$. Now if $(a_1, \ldots, a_n) \in A$ annihilates $\phi(x)$, then each a_k is in $\phi_k(x)^\perp = \phi_k(y)^\perp$, which shows that $(a_1, \ldots, a_n)$ annihilates $\phi(y)$. From this we can derive that $\phi(x)^\perp = \phi(y)^\perp$, thus showing that ϕ is a compatible homomorphism. □

The reason we cannot go the whole hog and claim all products in $\mathbf{SdRng}_{\mathrm{c}}$ is that, in an arbitrary product, we are not guaranteed that each d-ideal is of the form $\prod I_\alpha$.

By the way, the example of $\mathbb{Z}$ shows how spectacularly DId: $\mathbf{SdRng}_{\mathrm{c}} \to \mathbf{CohFrm}$ fails to turn finite products into coproducts. As remarked in Example 3.11, $\mathrm{DId}(\mathbb{Z}) = \mathbf{2}$, the two-element frame. In the ring $\mathbb{Z}\times\mathbb{Z}$, the ideal generated by $(1, 0)$ is a nonzero proper d-ideal. Thus, $\mathrm{DId}(\mathbb{Z}\times\mathbb{Z}) \ncong \mathrm{DId}(\mathbb{Z})\oplus\mathrm{DId}(\mathbb{Z})$, since $\mathrm{DId}(\mathbb{Z}) \oplus \mathrm{DId}(\mathbb{Z}) = \mathbf{2} \oplus \mathbf{2} = \mathbf{2}$.

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