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ESSENTIAL NORMS OF INTEGRAL OPERATORS

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Abstract. We estimate the essential norms of Volterra-type integral operators V_g and I_g, and multiplication operators M_g with holomorphic symbols g on a large class of generalized Fock spaces on the complex plane $\mathbb{C}$. The weights defining these spaces are radial and subjected to a mild smoothness conditions. In addition, we assume that the weights decay faster than the classical Gaussian weight. Our main result estimates the essential norms of V_g in terms of an asymptotic upper bound of a quantity involving the inducing symbol g and the weight function, while the essential norms of M_g and I_g are shown to be comparable to their operator norms. As a means to prove our main results, we first characterized the compact composition operators acting on the spaces which is interest of its own.

1. Introduction

The theory of integral operators constitutes a significant part of modern functional analysis, see for example [6, 9, 10, 16] and references therein for some overviews on the subject. The operators arise in many branches of mathematics, physics, engineering, biology, and economics [3, 6, 9, 10, 16], and often used in modelling real-world situations. A typical examples of these operators include the integral operators of Fredholm, Volterra, Hammerstein and Urysohn type. In this paper, we study the essential norms of linear integral operators of Volterra-type. More specifically, for a holomorphic function g, we consider the Volterra-type integral operator V_g and its companion I_g defined by

$$V_g f(z) = \int_0^z f(w)g'(w)dw \quad \text{and} \quad I_g f(z) = \int_0^z f'(w)g(w)dw.$$

Applying integration by parts in any of the above integrals gives the relation

$$V_g f + I_g f = M_g f - f(0)g(0), \tag{1.1}$$

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where $M_g f = gf$ is the multiplication operator of symbol g. These operators have been studied extensively with various contexts by several mathematicians. For more information on the subject, we may refer to [1, 4, 12, 13, 20] and the related references therein.

In [17], J. Pau and J. Peláez studied among others some properties of the operator V_g on weighted Bergman spaces A^p_w in the unit disk $\mathbb{D}$ when w belongs to a large class of rapidly decreasing weights. In an interesting and technical paper, Constantin and Pelàez [5] fortified the approach in [17] and considered generalized Fock spaces $\mathcal{F}_p^\psi$ over $\mathbb{C}$ when the corresponding weight decays faster than the classical Gaussian weight $e^{-\frac{|z|^2}{2}}$. They reported several results including a complete characterization of the bounded and compact V_g acting between these spaces. Their results show that there exists a much richer structure of V_g on $\mathcal{F}_p^\psi$ in contrast to its action on the classical Fock spaces $\mathcal{F}_p$.

In [15], we continued that line of research and studied the boundedness and compactness of I_g, and M_g on the spaces $\mathcal{F}_p^\psi$, and also V_g for the case where it was not considered in [5]. Unlike the operator V_g, the results in [15] showed that there exists no richer structure of I_g and M_g when they act between two different generalized spaces of these type than on the classical setting. In some cases, the structure of the operators rather gets poorer in contrast to the case on the classical setting.

The purpose of this note is to continue those lines of research in [5, 15] and estimate the essential norms of the operators V_g, I_g, and M_g when they act between the spaces $\mathcal{F}_p^\psi$. Our main result expresses the essential norms of V_g as an asymptotic upper bound of a quantity involving the inducing map g and the weight function ψ. On the other hand, the essential norm of I_g and M_g are shown to be comparable to their operator norms and expressed only in terms of the growth of the inducing symbol g.

We shall first recall the setting. We consider a twice continuously differentiable function $\psi : [0,\infty) \to [0,\infty)$ which we extend it to the whole complex plane by setting $\psi(z) = \psi(|z|)$. We further assume that the Laplacian $\Delta\psi$ is positive and set

$$\tau(z) \overset{1}{\simeq} \begin{cases} 1, & 0 \le |z| < 1, \\ (\Delta\psi(z))^{-1/2}, & |z| \ge 1, \end{cases} \tag{1.2}$$

where τ is a radial differentiable function satisfying the conditions

$$\lim_{r\to\infty} \tau(r) = 0 \quad \text{and} \quad \lim_{r\to\infty} \tau'(r) = 0. \tag{1.3}$$

[1]The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant C such that $U(z) \le CV(z)$ holds for all z in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$.

In addition, we require that either there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty} \tau'(r)\log\frac{1}{\tau(r)} = 0.$$

Throughout the paper we will assume that ψ and τ satisfy all the above mentioned growth and smoothness admissibility conditions. It is worth noting that there are many canonical examples of weight functions ψ that satisfy these conditions. The power functions $\psi_\alpha(r) = r^\alpha$, $\alpha > 2$, the exponential type functions such as $\psi_\beta(r) = e^{\beta r}$, $\beta > 0$, and the supper exponential functions $\psi(r) = e^{e^r}$ are all typical examples of such weights. At the end of this section, we will specialize our main results to one of such examples.

The generalized Fock spaces $\mathcal{F}_p^\psi$ consist of all entire functions f for which

$$\|f\|_{\mathcal{F}_p^\psi}^p = \int_{\mathbb{C}} |f(z)|^p e^{-p\psi(z)} dm(z) < \infty,$$

where $0 < p < \infty$, and dm denotes the usual Lebesgue area measure on $\mathbb{C}$. For $p = \infty$, the corresponding growth type generalized space $\mathcal{F}_\infty^\psi$ consist of all such functions f for which

$$\|f\|_{\mathcal{F}_\infty^\psi} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\psi(z)} < \infty.$$

An important concept in the theory of operators has been the notion of essential norm which we define it as follows. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Banach spaces. Then, the essential norm $\|T\|_e$ of a bounded linear operator $T : \mathscr{H}_1 \to \mathscr{H}_2$ is defined as the distance from T to the space of compact operators from $\mathscr{H}_1$ to $\mathscr{H}_2$:

$$\|T\|_e = \inf_K \big\{\|T - K\|;\ K : \mathscr{H}_1 \to \mathscr{H}_2 \text{ is a compact operator}\big\}.$$

In particular, $\|T\|_e \le \|T\|$ and T is compact if and only if its essential norm is zero. This means that the essential norm of an operator provides a useful measure for the noncompactness of the operator. We refer to [7,8,13,14,19,21, 22] for some examples on estimations of such norms for various operators on Hardy spaces, Bergman spaces, L^p, and Fock spaces. We prove the following estimates for V_g, I_g, and M_g on the generalized spaces.

Theorem 1.1. *Let g be an entire function on $\mathbb{C}$ and $1 \le p \le q \le \infty$. If*

(i) *$V_g : \mathcal{F}_p^\psi \to \mathcal{F}_q^\psi$ is bounded, then*

$$\|V_g\|_e \simeq \begin{cases} \limsup_{|z|\to\infty} \frac{|g'(z)|}{1+\psi'(z)}, & p = q = \infty, \\ \limsup_{|z|\to\infty} \frac{|g'(z)|\big(\Delta\psi(z)\big)^{\frac{1}{p}}}{1+\psi'(z)}, & 1 \le p < q = \infty, \\ \limsup_{|z|\to\infty} \frac{|g'(z)|\big(\Delta\psi(z)\big)^{\frac{q-p}{pq}}}{1+\psi'(z)}, & 1 \le p \le q < \infty. \end{cases} \tag{1.4}$$

(ii) *I_g or $M_g : \mathcal{F}_p^\psi \to \mathcal{F}_p^\psi$ is bounded, then*

$$\|I_g\|_e \simeq \|I_g\| \simeq \|M_g\| \simeq \|M_g\|_e. \tag{1.5}$$

We note that if $1 \le p = q < \infty$, then the third part of the estimate in (1.4) simplifies to

$$\|V_g\|_e \simeq \limsup_{|z|\to\infty} \frac{|g'(z)|}{1+\psi'(z)},$$

and involves no exponent p as in the first part. This shows a significance difference with the corresponding estimate when $1 \le p < q < \infty$. It has been known that such a difference does not exist in the classical Fock spaces setting [13, Theorem 3]. A similar difference has been observed on conditions describing the boundedness of both V_g and I_g; see [5, Theorem 3] and [15, Theorem 1.2]. On the other hand, the appearance of such a difference when we move from the classical to the general setting with a fast decaying weight is not totally unexpected; since in the classical Fock spaces, the monotonicity property in the sense of inclusion $\mathcal{F}_p \subseteq \mathcal{F}_q$ whenever $0 < p \le q \le \infty$, holds [11] while as seen from Corollary 2 of [5], this fails to hold for the family of generalized Fock spaces $\mathcal{F}_p^\psi$. For finite p and q such that $p \neq q$, it has been, in addition, proved that $\mathcal{F}_p^\psi \setminus \mathcal{F}_q^\psi \neq \emptyset$ and $\mathcal{F}_q^\psi \setminus \mathcal{F}_p^\psi \neq \emptyset$.

It should also be mentioned that if $0 < q < p \le \infty$, then $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_q^\psi$ is bounded if and only if it is compact [5, Theorem 3] and [15, Theorem 1.1]. Thus, its essential norm vanishes in this case. The same conclusion holds for I_g and M_g when they act between $\mathcal{F}_p^\psi$ and $\mathcal{F}_q^\psi$ for which $p \neq q$ because of Theorem 1.2 of [15]. By such a theorem, we in addition, have that I_g or $M_g : \mathcal{F}_p^\psi \to \mathcal{F}_p^\psi$ is bounded if and only if g is a constant function. This implies that the essential norms in (1.5) above are simply comparable with the value of the function g.

As pointed out earlier, the functions $\psi_\alpha(z) = |z|^\alpha, \alpha > 2$, $\psi_\beta(z) = e^{\beta|z|}$, $\beta > 0$ and $\psi(z) = e^{e^{|z|}}$ satisfy all the growth and smoothness admissibility conditions mentioned above. For such weights, one can apply Corollaries 25-27 of [5] and Theorem 1.1 of [15] to simplify further the estimates in Theorem 1.1. For instance for the case $\psi_\alpha(z) = |z|^\alpha$ we have the following.

Corollary 1.2. *Let g be an entire function on $\mathbb{C}$, $1 \le p \le q \le \infty$, $\psi(z) = \psi_\alpha(z) = |z|^\alpha$, $\alpha > 2$ and $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_q^\psi$ be a bounded linear operator. Then, if*

(i) *$q < \infty$, and $1 + (\alpha-2)\big(1 - \frac{1}{p} + \frac{1}{q}\big) < 0$, then $\|V_g\|_e = 0$;*

(ii) *$q < \infty$, and $1 + (\alpha-2)\big(1 - \frac{1}{p} + \frac{1}{q}\big) \ge 0$, then*

$$\|V_g\|_e \simeq \limsup_{|z|\to\infty} |g'(z)||z|^{1-\alpha+\frac{(\alpha-2)(q-p)}{pq}},$$

where g is a polynomial of degree not exceeding $2 + (\alpha-2)\big(1 - \frac{1}{p} + \frac{1}{q}\big)$.

(iii) *$q = \infty$, then*

$$\|V_g\|_e \simeq \begin{cases} \limsup_{|z|\to\infty} |g'(z)||z|^{1-\alpha}, & p = \infty, \\ \limsup_{|z|\to\infty} |g'(z)||z|^{\frac{\alpha(1-p)+(p-2)}{p}}, & 1 \le p < \infty, \end{cases}$$

where g is a complex polynomial of degree at most α when $p = q = \infty$ and $(\alpha(p-1)+2)/p$ whenever $1 \leq p < q = \infty$.

2. Preliminaries

In this section, we collect some basic facts which will be used to prove our main result in the next section. By Proposition A and Corollary 8 of [5], for a sufficiently large positive number R, there exists a number $\eta(R)$ such that for any $w \in \mathbb{C}$ with $|w| > \eta(R)$, there exists an entire function $f_{(w,R)}$ such that

$$|f_{(w,R)}(z)|e^{-\psi(z)} \leq C \min\left\{1, \left(\frac{\min\{\tau(w), \tau(z)\}}{|z-w|}\right)^{\frac{R^2}{2}}\right\} \tag{2.1}$$

for all z in $\mathbb{C}$, and for some constant C that depends on ψ and R. In particular, when z belongs to $D(w, R\tau(w))$, the estimate becomes

$$|f_{(w,R)}(z)|e^{-\psi(z)} \simeq 1, \tag{2.2}$$

where $D(a,r)$ denotes the Euclidean disk centered at a and radius $r > 0$. Furthermore, the functions $f_{(w,R)}$ belong to $\mathcal{F}_p^\psi$ with norms estimated by

$$\|f_{(w,R)}\|^p_{\mathcal{F}_p^\psi} \simeq \tau(w)^2, \quad \eta(R) \leq |w| \tag{2.3}$$

for all p in the range $0 < p < \infty$. On the other hand, because of (2.1) and (2.2), we observe that $f_{(w,R)}$ also belong to $\mathcal{F}_\infty^\psi$ and

$$\|f_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \simeq 1 \tag{2.4}$$

for all $w \in \mathbb{C}$. The sequence of functions $f_{(w,R)}$ will serve as a test function in our subsequent considerations replacing the roll of the sequence of the reproducing kernels in the classical Fock space setting. An explicit expression for the kernel function is still an open problem in the current setting.

Another important ingredient in proving our results is the Littlewood–Paley type formula for functions in $\mathcal{F}_p^\psi$. For $p = \infty$, the formula is

$$\|f\|_{\mathcal{F}_\infty^\psi} \simeq |f(0)| + \sup_{z\in\mathbb{C}} \frac{|f'(z)|e^{-\psi(z)}}{1+\psi'(z)}, \tag{2.5}$$

which was proved recently in [15]. The corresponding formula in $\mathcal{F}_p^\psi$ for finite p was obtained in [5] and reads

$$\|f\|^p_{\mathcal{F}_p^\psi} \simeq |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p \frac{e^{-p\psi(z)}}{\big(1+\psi'(z)\big)^p} dm(z). \tag{2.6}$$

Another useful fact is the pointwise local estimate for subharmonic functions f, namely that

$$|f(z)|^p e^{-\beta\psi(z)} \lesssim \frac{1}{\delta^2\tau(z)^2} \int_{D(z,\delta\tau(z))} |f(w)|^p e^{-\beta\psi(w)} dm(w) \tag{2.7}$$

for all finite exponent p, any real number β, and a small positive number δ: see Lemma 7 of [5] for more details.

The composition operator is one of the classical and well studied notions at the interface of operator theory and function theory. Yet the notion pops up at various instances due to its multifaced applications and continues being a point of interest. In this note we prove the following compactness result when it acts on generalized Fock spaces. Only the sufficiency part of the result will be used in proving our main results. But we rather formulate the statement in general since it is interest of its own as pointed before.

Proposition 2.1. *Let $0 < p \le q \le \infty$ and Φ be a nonconstant entire function on $\mathbb{C}$. Then the composition operator $C_\Phi : \mathcal{F}_p^\psi \to \mathcal{F}_q^\psi$ is compact if and only if $\Phi(z) = az + b$ for some complex numbers a and b such that $|a| < 1$.*

This result is similar to its counterpart in the classical setting which has been studied independently by several authors for example in [2, 12, 13, 22]. The faster decaying weights used to define the spaces $\mathcal{F}_p^\psi$ fail to provide a richer structure for the operator C_Φ as well.

Proof of Proposition 2.1. We begin with the proof of the necessity. We assume that C_Φ is compact and observe that the normalized sequence $f^*_{(w,R)} = \|f_{(w,R)}\|^{-1}_{\mathcal{F}_p^\psi} f_{(w,R)}$, as described from (2.1)-(2.4), converges to zero as $|w| \to \infty$, and the convergence is uniform on compact subset of $\mathbb{C}$. If $p < \infty$ and $q = \infty$, then C_Φ applied to such a sequence and subsequently invoking (2.3) imply

$$\begin{aligned} 0 &= \lim_{|w|\to\infty} \|C_\Phi f^*_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \\ &= \lim_{|w|\to\infty} \tau(w)^{\frac{-2}{p}} \sup_{z\in\mathbb{C}} |f_{(w,R)}(\Phi(z))| e^{-\psi(z)} \\ &\ge \lim_{|w|\to\infty} \tau(w)^{\frac{-2}{p}} \big|f_{(w,R)}(\Phi(z))\big| e^{-\psi(\Phi(z))} e^{\psi(\Phi(z))-\psi(z)} \end{aligned}$$

for all $z, w \in \mathbb{C}$. In particular, it follows from setting $w = \Phi(z)$ and applying (2.2) that

$$\begin{aligned} 0 &= \lim_{|\Phi(z)|\to\infty} \tau(\Phi(z))^{\frac{-2}{p}} e^{\psi(\Phi(z))-\psi(z)} \\ &= \lim_{|\Phi(z)|\to\infty} e^{\psi(\Phi(z))-\psi(z)-\frac{2}{p}\log(1+\tau(\Phi(z)))} \end{aligned} \tag{2.8}$$

from which we claim that

$$\lim_{|\Phi(z)|\to\infty} \big(\psi(\Phi(z)) - \psi(z) - \frac{2}{p}\log(1+\tau(\Phi(z)))\big) < 0. \tag{2.9}$$

If not, taking the limit further in the right-hand side of (2.8) and applying the admissibility assumptions on (1.3), and the fact that Φ is a nonconstant entire function, we get

$$0 = e^{\lim_{|\Phi(z)|\to\infty} \left(\psi(\Phi(z))-\psi(z)-\frac{2}{p}\log(1+\tau(\Phi(z)))\right)} \ge 1,$$

which gives a contradiction. By the growth assumption on ψ and (2.9) we now easily see that $\Phi(z) = az + b$ for some a, b in $\mathbb{C}$ and $|a| < 1$.

On the other hand, if $p = q = \infty$, then using (2.2) and (2.4), and arguing as above, we have

$$0 = \lim_{|w|\to\infty} \|C_\Phi f_{(w,R)}\|_{\mathcal{F}^\psi_\infty} \geq \lim_{|\Phi(z)|\to\infty} e^{\psi(\Phi(z))-\psi(z)} \tag{2.10}$$

from which we again see that $\Phi(z) = az + b$ for some $|a| < 1$.

If $1 \leq p \leq q < \infty$, then we may first reformulate the task in terms of embedding maps between $\mathcal{F}^\psi_p$ and $\mathcal{F}^\psi_q$. We set a pullback measure μ_Φ on $\mathbb{C}$ as

$$\mu_{(\Phi,q)}(E) = \int_{\Phi^{-1}(E)} e^{-q\psi(w)} dm(w) \tag{2.11}$$

for every Borel subset E of $\mathbb{C}$. Then we write

$$\|C_\Phi f\|^q_{\mathcal{F}^\psi_q} = \int_{\mathbb{C}} |f(\Phi(z))|^q e^{-q\psi(z)} dm(z) = \int_{\mathbb{C}} |f(z)|^q d\mu_{(\Phi,q)}(z). \tag{2.12}$$

From this, it follows that $C_\Phi : \mathcal{F}^\psi_p \to \mathcal{F}^\psi_q$ is compact if and only if the embedding map $i_d : \mathcal{F}^\psi_p \to L^q(\mu_{(\Phi,q)})$ is compact. By Theorem 1 of [5], the latter holds if and only if for some $\delta > 0$,

$$\lim_{|w|\to\infty} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q\psi(z)} d\mu_{(\Phi,q)}(z) = 0.$$

Using (2.11), this condition simplifies further to

$$\begin{aligned} 0 &= \lim_{|w|\to\infty} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q\psi(z)} d\mu_{(\Phi,q)}(z) \\ &= \lim_{|w|\to\infty} \frac{1}{\tau(w)^{2q/p}} \int_{D(w,\delta\tau(w))} e^{q(\psi(z)-\psi(\Phi^{-1}(z))} dm(\Phi^{-1}(z)). \end{aligned} \tag{2.13}$$

Let us first assume that (2.13) holds and show that $\Phi(z) = az + b$ for some $|a| < 1$. An application of (1.3) and estimating further on the right-hand side of (2.13) gives

$$\begin{aligned} 0 &\geq \lim_{|w|\to\infty} \tau(w)^{2-\frac{2q}{p}} e^{q\left(\psi((w))-\psi(\Phi^{-1}(w))\right)} \\ &= \lim_{|\Phi(z)|\to\infty} \tau(\Phi(z))^{2-\frac{2q}{p}} e^{q\left(\psi((\Phi(z)))-\psi(z))\right)} \\ &= \lim_{|\Phi(z)|\to\infty} e^{q\psi((\Phi(z)))-q\psi(z))+2\frac{p-q}{p}\log(1+\tau(\Phi(z)))} \end{aligned}$$

from which and after arguing as those in (2.9), our assertion follows.

To show the converse, we assume that $\Phi(z) = az + b$, $|a| < 1$ and proceed to show that the right-hand side of (2.13) vanishes. To this end, we have

$$\begin{aligned} &\lim_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2q}{p}}} \int_{D(w,\delta\tau(w))} e^{q(\psi(z)-\psi(\Phi^{-1}(z))} dm(\Phi^{-1}(z)) \\ &= \lim_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2q}{p}}} \int_{D(w,\delta\tau(w))} e^{q\left(\psi(z)-\psi(\frac{z-b}{a}\right)} dm\Big(\frac{z-b}{a}\Big) \end{aligned}$$

$$\lesssim \lim_{|w|\to\infty} e^{q\left(\psi(w)-\psi(\frac{w-b}{a})\right)+2\frac{p-q}{p}\log(1+\tau(w))}.$$

From our assumption (1.2), and since $|a|<1$, it follows that

$$\lim_{|w|\to\infty} e^{q(\psi(w)-\psi(\frac{w-b}{a}))+2\frac{p-q}{p}\log(1+\tau(w))}=0.$$

Next, we prove the sufficiency of the condition when $q=\infty$, and let f_n be a uniformly bounded sequence of functions in $\mathcal{F}_p^\psi$ that converge uniformly to zero on compact subsets of $\mathbb{C}$. Then

$$\begin{aligned}
\|C_\Phi f_n\|_{\mathcal{F}_\infty^\psi} &= \sup_{z\in\mathbb{C}} |f_n(\Phi(z))|e^{-\psi(z)}\\
&= \sup_{z\in\mathbb{C}} |f_n(az+b)|e^{-\psi(z)}\\
&\simeq \sup_{|z|>r} |f_n(az+b)|e^{-\psi(az+b)}e^{\psi(az+b)-\psi(z)} + \sup_{|z|\le r}|f_n(az+b)|e^{-\psi(z)}\\
&\lesssim \|f_n\|_{\mathcal{F}_p^\psi}\sup_{|z|>r}\frac{e^{\psi(az+b)-\psi(z)}}{\tau(az+b)^{\frac{2}{p}}} + \sup_{|z|\le r}|f_n(az+b)|, \qquad (2.14)
\end{aligned}$$

where in the last inequality we used the pointwise estimate (2.7). Since $\|f_n\|_{\mathcal{F}_p^\psi}$ is uniformly bounded, $|a|<1$ and by the growth assumption on τ, the first summand in (2.14) goes to zero as $r\to\infty$ and the second goes to zero when $n\to\infty$. This implies $\|C_\Phi f_n\|_{\mathcal{F}_\infty^\psi}\to 0$ as $n\to\infty$ from which our assertion follows, and completes the proof of the proposition. □

We will also need the following covering lemma from [5].

Lemma 2.2. *Let τ be as above. Then there exists a sequence of points z_j in $\mathbb{C}$ satisfying the following conditions*:

(i) $z_j \notin D(z_k,\tau(z_k))$, $j\neq k$;
(ii) $\mathbb{C}=\bigcup_j D(z_j,\tau(z_j))$;
(iii) $\bigcup_{z\in D(z_j,\tau(z_j))} D(z,\tau(z))\subset D(z_j,3\tau(z_j))$;
(iv) *The sequence $D(z_j,3\tau(z_j))$ is a covering of $\mathbb{C}$ with finite multiplicity N.*

3. Proof of the main result

We now turn to the proof of our main results. In many of related earlier works on spaces of analytic functions, a classical approach in proving results of these kinds has been that a sequence of finite rank operators which map a given function f to its nth partial sum of its Taylor series was used. Such a sequence is uniformly bounded for $p>1$, which is known to be false for the case $p=1$. Due to this, several known results on essential norms of operators do not include the functional space for $p=1$; see [7,8,13,14,18,21,22] for some examples. The nobility of the approach here is that we do not use such Taylor series techniques and the proof works fine for $p=1$ as well.

3.1. Proof of the lower estimates in (1.4)

A classical approach to estimate lower bounds for the essential norm is to find a suitable weakly null sequence of functions f_n and use the fact that

$$\|V_g\|_e \geq \limsup_{n\to\infty} \|V_g f_n\|_{\mathcal{F}_q^\psi}. \tag{3.1}$$

On classical Fock spaces, the sequence of the reproducing kernels does this job. Since no explicit expression is known for the kernel function in our current setting, we will instead use the sequence of functions $f^*_{(w,R)} = f_{(w,R)}/\|f_{(w,R)}\|_{\mathcal{F}_p^\psi}$ as described in (2.1), (2.2), (2.3), and (2.4). With this we proceed to make further estimates on the right-hand side of the norm in (3.1).

If $p = q = \infty$, then applying (2.2), (2.4), and (2.5) we have

$$\begin{aligned}\|V_g\|_e &\geq \limsup_{|w|\to\infty} \|V_g f^*_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \simeq \limsup_{|w|\to\infty} \sup_{z\in\mathbb{C}} \frac{|g'(z)||f_{(w,R)}(z)|e^{-\psi(z)}}{1+\psi'(z)}\\ &\geq \limsup_{|w|\to\infty} \frac{|g'(w)||f_{(w,R)}(w)|e^{-\psi(w)}}{1+\psi'(w)} \simeq \limsup_{|w|\to\infty} \frac{|g'(w)|}{1+\psi'(w)}\end{aligned}$$

from which our assertion follows.

Seemingly, when $1 \leq p < q = \infty$, an application of (2.2), (2.3), and (2.5) again leads to the estimate

$$\begin{aligned}\|V_g\|_e &\geq \limsup_{|w|\to\infty} \|V_g f^*_{(w,R)}\|_{\mathcal{F}_\infty^\psi} \geq \limsup_{|w|\to\infty} \frac{|g'(w)||f^*_{(w,R)}(w)|e^{-\psi(w)}}{1+\psi'(w)}\\ &\simeq \limsup_{|w|\to\infty} \frac{|g'(w)|}{(1+\psi'(w))\tau(w)^{\frac{2}{p}}} = \limsup_{|w|\to\infty} \frac{|g'(w)|(\Delta\psi(w))^{\frac{1}{p}}}{1+\psi'(w)}.\end{aligned} \tag{3.2}$$

It remains to show when $1 \leq p \leq q < \infty$. In this case, making use of (2.3) and (2.6), we estimate

$$\begin{aligned}\|V_g\|_e &\geq \limsup_{|w|\to\infty} \left\|V_g f^*_{(w,R)}\right\|_{\mathcal{F}_q^\psi}\\ &\simeq \limsup_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{\mathbb{C}} \frac{|g'(z)|^q|f_{(w,R)}(z)|^q e^{-q\psi(z)}}{(1+\psi'(z))^q} dm(z)\right)^{\frac{1}{q}}\\ &\geq \limsup_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q|f_{(w,R)}(z)|^q e^{-q\psi(z)}}{(1+\psi'(z))^q} dm(z)\right)^{\frac{1}{q}}\end{aligned}$$

for some small positive number δ. By (2.2), the last term above is comparable to

$$\limsup_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2}{p}}} \left(\int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q}{(1+\psi'(z))^q} dm(z)\right)^{\frac{1}{q}}.$$

On the other hand, since $|g'|^q$ is subharmonic, it follows from (2.7) that

$$\begin{aligned}
&\limsup_{|w|\to\infty} \frac{1}{\tau(w)^{\frac{2}{p}}}\bigg(\int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q}{(1+\psi'(z))^q} dm(z)\bigg)^{\frac{1}{q}}\\
&\gtrsim \limsup_{|w|\to\infty} \frac{\tau(w)^{\frac{2}{q}}|g'(w)|}{\tau(w)^{\frac{2}{p}}(1+\psi'(w))} = \limsup_{|w|\to\infty} \frac{|g'(w)|\big(\Delta\psi(w)\big)^{\frac{q-p}{pq}}}{1+\psi'(w)},
\end{aligned}$$

and this completes the proof of the lower estimate in (1.4).

3.2. Proof of the upper estimates in (1.4)

For this, we may consider a sequence of maps Φ_k given by $\Phi_k(z) = \frac{k}{k+1}z$ for each $k \in \mathbb{N}$. By Proposition 2.1, C_{Φ_k} constitutes a sequence of compact composition operators on $\mathcal{F}_p^{\psi}$ for all $p \geq 1$. On the other hand, if V_g is bounded, then $V_g \circ C_{\Phi_k} : \mathcal{F}_p^{\psi} \to \mathcal{F}_q^{\psi}$ also constitutes a sequence of compact operators. We may consider two different cases.

Case 1: If $q = \infty$, then making use of (2.5) we have

$$\begin{aligned}
\|V_g\|_e &\leq \|V_g - V_g \circ C_{\Phi_k}\|\\
&= \sup_{\|f\|_{\mathcal{F}_p^{\psi}}\leq 1} \|(V_g - V_g\circ C_{\Phi_k})f\|_{\mathcal{F}_\infty^{\psi}}\\
&\simeq \sup_{\|f\|_{\mathcal{F}_p^{\psi}}\leq 1}\sup_{z\in\mathbb{C}} \frac{|g'(z)|\Big|f(z) - f(\Phi_k(z))\Big|}{1+\psi'(z)} e^{-\psi(z)}\\
&\simeq \sup_{\|f\|_{\mathcal{F}_p^{\psi}}\leq 1}\sup_{|z|>r} \frac{|g'(z)|}{1+\psi'(z)}\Big|f(z) - f(\Phi_k(z))\Big|e^{-\psi(z)}\\
&\quad + \sup_{\|f\|_{\mathcal{F}_p^{\psi}}\leq 1}\sup_{|z|\leq r} \frac{|g'(z)|}{1+\psi'(z)}\Big|f(z) - f(\Phi_k(z))\Big|e^{-\psi(z)} \qquad (3.3)
\end{aligned}$$

for a certain fixed positive number r. Next, we analyze the two summands above separately. If $p = \infty$ as well, then the first summand above can be estimated as

$$\begin{aligned}
&\sup_{\|f\|_{\mathcal{F}_\infty^{\psi}}\leq 1}\sup_{|z|>r}\bigg(\frac{|g'(z)|}{1+\psi'(z)}\bigg)\sup_{|z|>r}\bigg(\Big|f(z) - f(\Phi_k(z))\Big|e^{-\psi(z)}\bigg)\\
&\leq \sup_{\|f\|_{\mathcal{F}_\infty^{\psi}}\leq 1}\sup_{|z|>r}\bigg(\frac{|g'(z)|}{1+\psi'(z)}\bigg)\|f\|_{\mathcal{F}_\infty^{\psi}} \leq \sup_{|z|>r}\frac{|g'(z)|}{1+\psi'(z)}. \qquad (3.4)
\end{aligned}$$

On the other hand, if $1 \leq p < \infty$, then (2.7) implies the first summand in (3.3) is bounded by

$$\sup_{\|f\|_{\mathcal{F}_p^{\psi}}\leq 1}\sup_{|z|>r} \frac{|g'(z)|}{(1+\psi'(z))}\frac{\delta^{-\frac{2}{p}}}{\tau(z)^{\frac{2}{p}}}\bigg(\int_{D(z,\delta\tau(z))}\frac{\big|f(w) - f(\Phi_k(w))\big|^p}{e^{p\psi(w)}}dm(w)\bigg)^{\frac{1}{p}}$$

$$\lesssim \sup_{\|f\|_{\mathcal{F}_p^\psi}\leq 1} \sup_{|z|>r} \frac{|g'(z)|}{(1+\psi'(z))} \frac{\|f\|_{\mathcal{F}_p^\psi}}{\tau(z)^{\frac{2}{p}}} \leq \sup_{|z|>r} \frac{|g'(z)|(\Delta\psi(z))^{\frac{1}{p}}}{1+\psi'(z)}. \tag{3.5}$$

As for the second summand in (3.3), we observe that by integrating the function f' along the radial segment $[\frac{kz}{k+1}z, z]$ we find

$$\left|f(z) - f\Big(\frac{k}{k+1}z\Big)\right| \leq \frac{|z||f'(z^*)|}{k+1} \tag{3.6}$$

for some z^* in the radial segment $[\frac{kz}{k+1}z, z]$. By Cauchy estimate's for f', we also have

$$|f'(z^*)| \leq \frac{1}{r} \max_{|z|=2r} |f(z)|,$$

and hence

$$\left|f(z) - f\Big(\frac{k}{k+1}z\Big)\right| \leq \frac{|z|}{r(k+1)} \max_{|z|=2r} |f(z)|. \tag{3.7}$$

The above estimates ensure that

$$\begin{aligned} &\frac{|g'(z)|}{1+\psi'(z)}\left|f(z) - f\left(\frac{k}{k+1}z\right)\right|e^{-\psi(z)} \\ &\leq \frac{|z|}{r(k+1)} \sup_{z\in\mathbb{C}} \left(\frac{|g'(z)|}{1+\psi'(z)}e^{-\psi(z)}\right) \max_{|z|=2r} |f(z)|. \end{aligned}$$

By our admissibility assumption, the weight function ψ grows faster than the classical Gaussian weight function $\frac{|z|^2}{2}$. Consequently, the function$f_0 = 1$ belongs to $\mathcal{F}_p^\psi$ for all p. This together with the boundedness of V_g implies

$$\|V_g f_0\|_{\mathcal{F}_\infty^\psi} = \sup_{z\in\mathbb{C}} \frac{|g'(z)|}{1+\psi'(z)} e^{-\psi(z)} < \infty.$$

By our growth assumption on ψ and (2.7) again, we further estimate

$$\begin{aligned} \max_{|z|=2r} |f(z)| &\lesssim \max_{|z|=2r} \frac{\delta^{-\frac{2}{p}} e^{\psi(z)}}{(\tau(z))^{\frac{2}{p}}} \left(\int_{D(z,\delta\tau(z))} |f(w)|^p e^{-p\psi(w)} dm(w)\right)^{\frac{1}{p}} \\ &\lesssim \|f\|_{\mathcal{F}_p^\psi} \max_{|z|=2r} \frac{\delta^{-\frac{1}{p}} e^{\psi(z)}}{\tau(z)^{\frac{1}{p}}} \lesssim \|f\|_{\mathcal{F}_p^\psi} e^{\psi(2r)} (\Delta\psi(2r))^{\frac{1}{p}}. \end{aligned}$$

Now combining all the above estimates, we find that the second piece of the sum in (3.3) is bounded by

$$\begin{aligned} &\sup_{\|f\|_{\mathcal{F}_p^\psi}\leq 1} \sup_{|z|\leq r} \frac{|g'(z)|}{1+\psi'(z)} \big|f(z) - f(\Phi_k(z))\big| e^{-\psi(z)} \\ &\lesssim \frac{1}{k+1} \sup_{\|f\|_{\mathcal{F}_p^\psi}\leq 1} \|f\|_{\mathcal{F}_p^\psi} e^{\psi(2r)} \leq \frac{1}{k+1} e^{\psi(2r)} \to 0 \text{ as } k\to\infty, \end{aligned}$$

from which, (3.4), (3.5) and since r is arbitrary, we deduce

$$\|V_g\|_e \lesssim \sup_{|z|>r} \frac{|g'(z)|}{1+\psi'(z)} = \limsup_{|z|\to\infty} \frac{|g'(z)|}{1+\psi'(z)}$$

as asserted.

Case 2: When $1 \le p \le q < \infty$, applying (2.6) we have

$$\|V_g\|_e \le \|V_g - V_g \circ C_{\Phi_k}\| = \sup_{\|f\|_{\mathcal{F}_p^\psi}\le 1} \|(V_g - V_g\circ C_{\Phi_k})f\|_{\mathcal{F}_q^\psi}$$

$$\simeq \sup_{\|f\|_{\mathcal{F}_p^\psi}\le 1} \bigg(\int_{\mathbb{C}} \big|f(z) - f(\Phi_k(z))\big|^q \frac{|g'(z)|^q}{\big(1+\psi'(z)\big)^q} e^{-q\psi(z)} dm(z)\bigg)^{\frac{1}{q}}. \tag{3.8}$$

Setting

$$d\mu_{(g,q)}(z) = \frac{|g'(z)|^q e^{-q\psi(z)}}{\big(1+\psi'(z)\big)^q} dm(z)$$

and applying Lemma 2.2 and estimate (2.7), we get

$$\begin{aligned}
&\int_{\mathbb{C}} \big|f(z) - f(\Phi_k(z))\big|^q d\mu_{(g,q)}(z)\\
&\le \sum_j \int_{D(z_j,\delta\tau(z_j))} \big|f(z) - f(\Phi_k(z))\big|^q d\mu_{(g,q)}(z)\\
&\lesssim \sum_j \int_{D(z_j,\delta\tau(z_j))} \bigg(\int_{D(z,\delta\tau(z))} \frac{\big|f(w) - f(\Phi_k(w))\big|^p}{e^{p\psi(w)}} dm(w)\bigg)^{\frac{q}{p}} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z)\\
&\lesssim \sum_j \bigg(\int_{D(z_j,3\delta\tau(z_j))} \frac{\big|f(w) - f(\Phi_k(w))\big|^p}{e^{p\psi(w)}} dm(w)\bigg)^{\frac{q}{p}} \int_{D(z_j,\delta\tau(z_j))} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z).
\end{aligned}$$

We spilt the above sum as

$$\sum_j = \sum_{j:|z_j|>r} + \sum_{j:|z_j|\le r} \tag{3.9}$$

for some fixed positive number r again. Then since $q \ge p$, applying Minkowski inequality and the finite multiplicity N of the covering sequence $D(z_j, 3\delta\tau(z_j))$, the first sum is bounded by

$$\begin{aligned}
&\sup_{j:|z_j|>r} \bigg(\int_{D(z_j,\delta\tau(z_j))} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z)\bigg)\\
&\quad\times \bigg(\sum_{|z_j|>r} \int_{D(z_j,3\delta\tau(z_j))} \frac{\big|f(w) - f(\Phi_k(w))\big|^p}{e^{p\psi(w)}} dm(w)\bigg)^{\frac{q}{p}}\\
&\lesssim \sup_{|z_j|>r} \|f\|_{\mathcal{F}_p^\psi}^q \int_{D(z_j,\delta\tau(z_j))} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z).
\end{aligned}$$

In particular, for $\|f\|_{\mathcal{F}_p^\psi} \leq 1$, then the right-hand quantity above is bounded by

$$\begin{aligned}
&\sup_{j:|z_j|>r} \int_{D(z_j,\delta\tau(z_j))} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z)\\
&= \sup_{j:|z_j|>r} \int_{D(z_j,\delta\tau(z_j))} \frac{|g'(z)|^q \tau(z)^{-\frac{2q}{p}}}{\big(1+\psi'(z)\big)^q} dm(z)\\
&\leq \sup_{|w|>r} \int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q \tau(z)^{-\frac{2q}{p}}}{\big(1+\psi'(z)\big)^q} dm(z)\\
&\simeq \sup_{|w|>r} \tau(w)^{-2} \int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q \tau(z)^{-\frac{2q}{p}+2}}{\big(1+\psi'(z)\big)^q} dm(z),
\end{aligned}$$

here the last estimate follows by Lemma 5 of [5], where it was proved that $\tau(w) \simeq \tau(z)$ whenever z belongs to $D(w, \delta\tau(w))$. In addition, as $V_g : \mathcal{F}_p^\psi \to \mathcal{F}_q^\psi$ is a bounded operator, Theorem 3 of [5] again ensures that the integrand in the above last integral is uniformly bounded over $\mathbb{C}$. Thus,

$$\begin{aligned}
&\sup_{|w|>r} \tau(w)^{-2} \int_{D(w,\delta\tau(w))} \frac{|g'(z)|^q \tau(z)^{-\frac{2q}{p}+2}}{\big(1+\psi'(z)\big)^q} dm(z)\\
&\lesssim \sup_{|w|>r} \frac{|g'(w)|^q \tau(w)^{-\frac{2q}{p}+2}}{\big(1+\psi'(w)\big)^q} = \sup_{|w|>r} \frac{|g'(w)|^q \big(\Delta\psi(w)\big)^{\frac{q-p}{p}}}{\big(1+\psi'(w)\big)^q}. \qquad (3.10)
\end{aligned}$$

We plan to show that the second sum in (3.9) tends to zero when $k \to \infty$. Then since r is arbitrary, our upper estimate will follow from the series of estimates we made from (3.8) to (3.10). To this end, as done before, making use of (3.7) and Minkowski inequality, we proceed to estimate

$$\begin{aligned}
&\sum_{j:|z_j|\leq r} \bigg(\int_{D(z_j,3\delta\tau(z_j))} \frac{\big|f(w) - f(\Phi_k(w))\big|^p}{e^{p\psi(w)}} dm(w) \bigg)^{\frac{q}{p}}\\
&\quad \times \int_{D(z_j,\delta\tau(z_j))} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z)\\
&\lesssim \bigg(\sum_{j:|z_j|\leq r} \int_{D(z_j,3\delta\tau(z_j))} \frac{|w|^p \big(\max_{|w|=2r} |f(w)|\big)^p}{r(k+1)^p e^{p\psi(w)}} dm(w) \bigg)^{\frac{q}{p}}\\
&\quad \times \int_{D(z_j,\delta\tau(z_j))} \frac{e^{q\psi(z)}}{\tau(z)^{\frac{2q}{p}}} d\mu_{(g,q)}(z).
\end{aligned}$$

Using the assumption on τ, we have

$$|w| \leq |w - z_j| + |z_j| \leq r + \delta\tau(z_j) \leq r + \delta \sup_{z_j} \tau(z_j) \leq Mr$$

for some $M > 0$, from which we have that the preceding sum is bounded by

$$\begin{aligned}
&\frac{M^q \|f\|_{\mathcal{F}_p^\psi}^q}{(1+k)^q} \sup_{|z_j|\le r} \int_{D(z_j,\delta\tau(z_j))} \frac{|g'(z)|^q}{\tau(z)^{\frac{2q}{p}}(1+\psi'(z))^q} dm(z)\\
&\lesssim \frac{\tau(r)^{-2}}{(1+k)^q} \sup_{|z_j|\le r} \int_{D(z_j,\delta\tau(z_j))} \frac{\tau(z)^2|g'(z)|^q}{\tau(z)^{\frac{2q}{p}}(1+\psi'(z))^q} dm(z)\\
&\lesssim \frac{\tau(r)^{-2}}{(1+k)^q} \sup_{z\in\mathbb{C}} \left(\frac{\tau(z)^2|g'(z)|^q}{\tau(z)^{\frac{2q}{p}}(1+\psi'(z))^q}\right) \sup_{|z_j|\le r} \int_{D(z_j,\delta\tau(z_j))} dm(z)\\
&\lesssim \frac{\tau(r)^{-2}}{(1+k)^q} \sup_{|z_j|\le r} \tau(z_j)^2 \le \frac{1}{(1+k)^q} \to 0 \quad \text{as} \quad k\to\infty,
\end{aligned}$$

and, this completes the proof of part (i) of the theorem.

The proof of part (ii) is a simple variant of the proof of part (i). Thus, we omit it, and leave it to interested readers.

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