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LEHMER'S GENERALIZED EULER NUMBERS IN HYPERGEOMETRIC FUNCTIONS

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Abstract. In 1935, D. H. Lehmer introduced and investigated generalized Euler numbers W_n, defined by

$$\frac{3}{e^t + e^{\omega t} + e^{\omega^2 t}} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!},$$

where ω is a complex root of $x^2+x+1=0$. In 1875, Glaisher gave several interesting determinant expressions of numbers, including Bernoulli and Euler numbers. These concepts can be generalized to the hypergeometric Bernoulli and Euler numbers by several authors, including Ohno and the second author. In this paper, we study more general numbers in terms of determinants, which involve Bernoulli, Euler and Lehmer's generalized Euler numbers. The motivations and backgrounds of the definition are in an operator related to Graph theory. We also give several expressions and identities by Trudi's and inversion formulae.

1. Introduction

In 1935, D. H. Lehmer [17] introduced and investigated generalized Euler numbers W_n, defined by the generating function

$$\frac{3}{e^t + e^{\omega t} + e^{\omega^2 t}} = \sum_{n=0}^{\infty} W_n \frac{t^n}{n!}, \tag{1}$$

where $\omega = \frac{-1+\sqrt{-3}}{2}$ and $\omega^2 = \bar{\omega} = \frac{-1-\sqrt{-3}}{2}$ are the primitive cube roots of unity. Notice that $W_n = 0$ unless $n \equiv 0 \pmod 3$. The sequence of these numbers is given by

$$\begin{aligned}\{W_{3n}\}_{n\ge 0} = 1, -1, 19, -1513, 315523, -136085041, 105261234643,\\ -132705221399353, 254604707462013571, \dots\end{aligned}$$

and the sequence of these absolute values is recorded in [19, A002115]. In [14], the complementary numbers $W_n^{(j)}$ $(j = 0, 1, 2)$ to Lehmer's Euler numbers are

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defined by the generating function

$$\sum_{n=0}^\infty W_n^{(j)}\frac{t^n}{n!}=\left(1+\sum_{l=1}^\infty\frac{t^{3l}}{(3l+j)!}\right)^{-1}. \tag{2}$$

Notice that $W_n^{(j)}=0$ unless $n\equiv 0 \pmod 3$. When $j=0$, $W_n=W_n^{(0)}$ are the original Lehmer's Euler numbers. When $j=1$, we also have

$$\sum_{n=0}^\infty W_n^{(1)}\frac{t^n}{n!}=\frac{3t}{e^t+\omega^2 e^{\omega t}+\omega e^{\omega^2 t}}. \tag{3}$$

Lehmer's Euler numbers and their complementary numbers $W_n^{(j)}$ can be considered analogous of the classical Euler numbers E_n and their complementary Euler numbers $\widehat{E}_n$ ([11, 16]). For, their generating functions are given by

$$\sum_{n=0}^\infty E_n\frac{t^n}{n!}=\frac{1}{\cosh t}=\frac{2}{e^t+e^{-t}}=\left(\sum_{l=0}^\infty\frac{t^{2l}}{(2l)!}\right)^{-1} \tag{4}$$

and

$$\sum_{n=0}^\infty \widehat{E}_n\frac{t^n}{n!}=\frac{t}{\sinh t}=\frac{2t}{e^t-e^{-t}}=\left(\sum_{l=0}^\infty\frac{t^{2l}}{(2l+1)!}\right)^{-1}, \tag{5}$$

respectively. For $N\ge 0$ *hypergeometric Euler numbers* $E_{N,n}$ ([11, 16]) are defined by

$$\begin{aligned}\frac{1}{{}_1F_2(1;N+1,(2N+1)/2;t^2/4)}&=\frac{t^{2N}/(2N)!}{\cosh t-\sum_{n=0}^{N-1}t^{2n}/(2n)!}\\&:=\sum_{n=0}^\infty E_{N,n}\frac{t^n}{n!},\end{aligned}$$

where ${}_1F_2(a;b,c;z)$ is the hypergeometric function defined by

$${}_1F_2(a;b,c;z)=\sum_{n=0}^\infty\frac{(a)^{(n)}}{(b)^{(n)}(c)^{(n)}}\frac{z^n}{n!}.$$

Here $(x)^{(n)}$ denotes the rising factorial, defined by $(x)^{(n)}=x(x+1)\cdots(x+n-1)$ ($n\ge 1$) with $(x)^{(0)}=1$. When $N=0$, $E_n=E_{0,n}$ are the original Euler numbers. Similarly, *hypergeometric Euler numbers of the second kind* $\widehat{E}_{N,n}$ ([11, 16]) are defined by

$$\begin{aligned}\frac{1}{{}_1F_2(1;N+1,(2N+3)/2;t^2/4)}&=\frac{t^{2N+1}/(2N+1)!}{\sinh t-\sum_{n=0}^{N-1}t^{2n+1}/(2n+1)!}\\&:=\sum_{n=0}^\infty \widehat{E}_{N,n}\frac{t^n}{n!}.\end{aligned}$$

When $N = 0$, $\widehat{E}_n = \widehat{E}_{0,n}$ are the original complementary Euler numbers. There are many kinds of generalizations of Euler numbers, but hypergeometric Euler numbers have advantages as natural extensions in terms of determinant expressions ([11, 16]). For $N \ge 0$ and $n \ge 1$, we have

$$(6)\quad E_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N)!}{(2N+2)!} & 1 & & & \\ \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(2N)!}{(2N+2n-2)!} & \frac{(2N)!}{(2N+2n-4)!} & \cdots & \frac{(2N)!}{(2N+2)!} & 1 \\ \frac{(2N)!}{(2N+2n)!} & \frac{(2N)!}{(2N+2n-2)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} \end{vmatrix}$$

and

$$(7)\quad \widehat{E}_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N+1)!}{(2N+3)!} & 1 & & & \\ \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(2N+1)!}{(2N+2n-1)!} & \frac{(2N+1)!}{(2N+2n-3)!} & \cdots & \frac{(2N+1)!}{(2N+3)!} & 1 \\ \frac{(2N+1)!}{(2N+2n+1)!} & \frac{(2N+1)!}{(2N+2n-1)!} & \cdots & \frac{(2N+1)!}{(2N+5)!} & \frac{(2N+1)!}{(2N+3)!} \end{vmatrix} .$$

When $N = 0$ in (6), we have a famous determinant expression of Euler numbers discovered by Glaisher in 1875 ([3, p. 52]).

Similar hypergeometric numbers are *hypergeometric Bernoulli numbers* $B_{N,n}$ ([4–9]), defined by

$$(8)\qquad \frac{1}{{}_1F_1(1; N+1; t)} = \frac{t^N/N!}{e^t - \sum_{n=0}^{N-1} t^n/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{t^n}{n!} ,$$

where ${}_1F_1(a; b; z)$ is the confluent hypergeometric function defined by

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!} .$$

The determinant expression of hypergeometric Bernoulli numbers are given by

$$(9)\qquad B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix} .$$

When $N = 1$, $B_n = B_{1,n}$ are the classical Bernoulli numbers defined by

$$(10)\qquad \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} .$$

The determinant expression for the classical Bernoulli numbers was discovered by Glaisher ([3, p. 53]).

In this paper, we introduce and study the hypergeometric Euler numbers of higher grade are introduced as total generalizations of hypergeometric Euler numbers, hypergeometric Euler numbers of the second kind, hypergeometric Bernoulli numbers as well as Lehmer's generalized Euler numbers.

2. Hypergeometric Lehmer-Euler numbers of higher grade

For $N, n \geq 0$, define *hypergeometric Lehmer-Euler numbers* $W_{N,n,r}^{(j)}$ $(j = 0, 1)$ of grade r by

$$
\begin{aligned}
(11) \qquad & \sum_{n=0}^{\infty} W_{N,n,r}^{(j)} \frac{t^n}{n!} \\
& = \left({}_1F_r \left(1; \frac{rN+j+1}{r}, \frac{rN+j+2}{r}, \cdots, \frac{rN+j+r}{r}; \left(\frac{t}{r}\right)^r \right) \right)^{-1},
\end{aligned}
$$

where ${}_1F_r(a; b_1, \dots, b_r; z)$ is the hypergeometric function, defined by

$$
{}_1F_r(a; b_1, \dots, b_r; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b_1)^{(n)} \cdots (b_r)^{(n)}} \frac{z^n}{n!}.
$$

From the definition, $W_{N,n,r}^{(j)} \equiv 0 \pmod r$ unless $n \equiv 0 \pmod r$. When $N = 0$ and $r = 3$ in (11), $W_n^{(j)} = W_{0,n,3}^{(j)}$ are the Lehmer's generalized Euler numbers ($j = 0$) in (1) and their complementary numbers ($j = 1$) in (3). These numbers where $r = 3$ have been extensively studied by Ohno and the second author ([14]), including congruence properties. When $N = 0$ and $r = 2$ in (11), $E_n = W_{0,n,2}^{(j)}$ are the classical Euler numbers ($j = 0$) in (4) and their complementary numbers ($j = 1$) in (5). When $r = 1$ and $j = 0$ in (11), $B_{N,n} = W_{N,n,1}^{(0)}$ are the hypergeometric Bernoulli numbers. When $N = r = 1$ and $j = 0$ in (11), $B_n = W_{1,n,1}^{(0)}$ are the classical Bernoulli numbers in (10).

We can write (11) as

$$
\begin{aligned}
(12) \qquad & {}_1F_r \left(1; \frac{rN+j+1}{r}, \frac{rN+j+2}{r}, \dots, \frac{rN+j+r}{r}; \left(\frac{t}{r}\right)^r \right) \\
& = \sum_{n=0}^{\infty} \frac{t^{rn}}{(rN+j+1)(rN+j+2)\cdots(rN+j+rn)} \\
& = \sum_{n=0}^{\infty} \frac{(rN+j)!}{(rN+rn+j)!} t^{rn} = 1 + \sum_{n=1}^{\infty} \frac{(rN+j)!}{(rN+rn+j)!} t^{rn}.
\end{aligned}
$$

When $N=0$ in (11), $W_{n,r}^{(j)}=W_{0,n,r}^{(j)}$ are the *Lehmer-Euler numbers of grade* r, defined by

$$\sum_{n=0}^{\infty} W_{n,r}^{(j)}\frac{t^n}{n!}=\left({}_1F_r\left(1;\frac{j+1}{r},\frac{j+2}{r},\dots,\frac{j+r}{r};\left(\frac{t}{r}\right)^r\right)\right)^{-1}\,. \tag{13}$$

If $r=p$ is prime, the generating functions of Lehmer-Euler numbers of degree p can be expressed by

$$\sum_{n=0}^{\infty} W_{n,p}^{(0)}\frac{t^n}{n!}=\frac{r}{\sum_{l=0}^{p-1} e^{\zeta_p^l t}}$$

and

$$\sum_{n=0}^{\infty} W_{n,p}^{(1)}\frac{t^n}{n!}=\frac{rt}{\sum_{l=0}^{p-1} \zeta_p^{p-l} e^{\zeta_p^l t}}\,,$$

where ζ_p is the (primitive) p-th root of unity.

The definition (11) with (12) may be obvious or artificial for the readers with different backgrounds. However, there are motivations from Combinatorics, in particular, graph theory. In 1989, Cameron [2] considered the operator A defined on the set of sequences of non-negative integers as follows: for $x=\{x_n\}_{n\ge 1}$ and $z=\{z_n\}_{n\ge 1}$, set $Ax=z$, where

$$1+\sum_{n=1}^{\infty} z_n t^n=\left(1-\sum_{n=1}^{\infty} x_n t^n\right)^{-1}\,. \tag{14}$$

Suppose that x enumerates a class C. Then Ax enumerates the class of disjoint unions of members of C, where the order of the component members of C is significant. The operator A also plays an important role for free associative (non-commutative) algebras. More motivations and background together with many concrete examples (in particular, in the aspects of Graph theory) by this operator can be seen in [2]. In the sense of Cameron's operator A, we have the following relations:

$$A\left\{-\frac{N!}{(N+n)!}\right\}=\left\{\frac{B_{N,n}}{n!}\right\},$$

$$A\left\{-\frac{(2N)!}{(2N+2n)!}\right\}=\left\{\frac{E_{N,2n}}{(2n)!}\right\},$$

$$A\left\{-\frac{(2N+1)!}{(2N+2n+1)!}\right\}=\left\{\frac{\widehat{E}_{N,2n}}{(2n)!}\right\},$$

$$A\left\{-\frac{(3N+j)!}{(3N+3n+j)!}\right\}=\left\{\frac{W_{N,3n,3}^{(j)}}{(3n)!}\right\}.$$

These relations are interchangeable in the sense of determinants too. See the Section 5 about Trudi's formula.

We have the following recurrence relation.

Proposition 2.1. *For $N \ge 0$ and $j = 0, 1$, we have*

$$W^{(j)}_{N,rn,r} = -\sum_{k=0}^{n-1} \frac{(rn)!(rN+j)!}{(rN+rn-rk+j)!(rk)!} W^{(j)}_{N,rk,r} \quad (n \ge 1)$$

with $W^{(j)}_{N,0,r} = 1$.

Proof. By (11), we get

$$\begin{aligned} 1 &= \left(1 + \sum_{l=1}^{\infty} \frac{(rN+j)!}{(rN+rl+j)!} t^{rl}\right) \left(\sum_{n=0}^{\infty} W^{(j)}_{N,rn,r} \frac{t^{rn}}{(rn)!}\right) \\ &= \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r} \frac{t^{rn}}{(rn)!} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \frac{(rN+j)! W^{(j)}_{N,rk,r}}{(rN+rn-rk+j)!(rk)!} t^{rn} . \end{aligned}$$

Comparing the coefficient on both sides, we obtain

$$\frac{W^{(j)}_{N,rn,r}}{(rn)!} + \sum_{k=0}^{n-1} \frac{(rN+j)! W^{(j)}_{N,rk,r}}{(rN+rn-rk+j)!(rk)!} = 0 \quad (n \ge 1) .$$

□

We have an explicit expression of $W^{(j)}_{N,n,r}$.

Theorem 2.2. *Let $j = 0, 1$. For $n \ge 1$,*

$$W^{(j)}_{N,rn,r} = (rn)! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \ge 1}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)! \cdots (rN+ri_k+j)!} .$$

Proof. The proof is done by induction for n. From Proposition 2.1 with $n = 1$,

$$W^{(j)}_{N,r,r} = -\frac{r!(rN+j)!}{(rN+j+r)!} W^{(j)}_{N,0,r} = -\frac{r!(rN+j)!}{(rN+j+r)!} .$$

This matches the result when $n = 1$. Assume that the result is valid up to $n-1$. Then by Proposition 2.1

$$\begin{aligned} \frac{W^{(j)}_{N,rn,r}}{(rn)!} &= -\sum_{l=0}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!} \frac{W^{(j)}_{N,rl,r}}{(rl)!} \\ &= -\sum_{l=1}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!} \sum_{k=1}^{l} (-1)^k \\ &\quad \times \sum_{\substack{i_1+\cdots+i_k=l \\ i_1,\dots,i_k \ge 1}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)! \cdots (rN+ri_k+j)!} \\ &\quad - \frac{(rN+j)!}{(rN+rn+j)!} \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=1}^{n-1}(-1)^k \sum_{l=k}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!}\\
&\quad \times \sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\dots,i_k\ge 1}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
&\quad - \frac{(rN+j)!}{(rN+rn+j)!}\\
&= -\sum_{k=2}^{n}(-1)^{k-1} \sum_{l=k-1}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!}\\
&\quad \times \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\dots,i_{k-1}\ge 1}} \frac{\big((rN+j)!\big)^{k-1}}{(rN+ri_1+j)!\cdots(rN+ri_{k-1}+j)!}\\
&\quad - \frac{(rN+j)!}{(rN+rn+j)!}\\
&= \sum_{k=2}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
&\quad - \frac{(rN+j)!}{(rN+rn+j)!} \qquad (n-l=i_k)\\
&= \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 1}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\,.
\end{aligned}
$$

□

There is an alternative form of $W_{N,n,r}^{(j)}$ by using binomial coefficients. The proof is similar to that of Theorem 2.2 and is omitted.

Theorem 2.3. *For $n \ge 1$,*

$$
\begin{aligned}
&W_{N,rn,r}^{(j)}\\
&= (rn)!\sum_{k=1}^{n}(-1)^k \binom{n+1}{k+1} \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\dots,i_k\ge 0}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\,.
\end{aligned}
$$

3. Determinant expressions

In this section, we shall show an expression in terms of determinants. This result is a generalization of those of Bernoulli, Euler numbers and Lehmer's Euler numbers of grade 3.

Theorem 3.1. *For $n \ge 1$,*

$$W_{N,rn,r}^{(j)} = (-1)^n (rn)!$$

$$\times \begin{vmatrix} \frac{(rN+j)!}{(rN+j+r)!} & 1 & & & \\ \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(rN+j)!}{(rN+rn+j-r)!} & \frac{(rN+j)!}{(rN+rn+j-2r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} & 1 \\ \frac{(rN+j)!}{(rN+rn+j)!} & \frac{(rN+j)!}{(rN+rn+j-r)!} & \cdots & \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} \end{vmatrix}.$$

Proof. For simplicity, put $\tilde{W}_{N,n} = (-1)^{n/r} W_{N,n,r}^{(j)}/n!$. Then, we shall prove that for any $n \ge 1$

$$\tilde{W}_{N,rn} = \begin{vmatrix} \frac{(rN+j)!}{(rN+j+r)!} & 1 & & & \\ \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(rN+j)!}{(rN+rn+j-r)!} & \frac{(rN+j)!}{(rN+rn+j-2r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} & 1 \\ \frac{(rN+j)!}{(rN+rn+j)!} & \frac{(rN+j)!}{(rN+rn+j-r)!} & \cdots & \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} \end{vmatrix}. \tag{15}$$

When $n = 1$, (15) is valid because by Theorem 2.2

$$\tilde{W}_{N,r} = \frac{(rN+j)!}{(rN+j+r)!}\,.$$

Assume that (15) is valid up to $n-1$. Notice that by Proposition 2.1, we have

$$\tilde{W}_{N,rn} = \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(rN+j)!}{(rN+rn-rk+j)!}\tilde{W}_{N,rk}\,.$$

Thus, by expanding the first row of the right-hand side (15), it is equal to

$$\begin{aligned}
&\frac{(rN+j)!}{(rN+j+r)!}\tilde{W}_{N,rn-r} \\
&- \begin{vmatrix} \frac{(rN+j)!}{(rN+j+2r)!} & 1 & & & \\ \frac{(rN+j)!}{(rN+j+3r)!} & \frac{(rN+j)!}{(rN+j+r)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{(rN+j)!}{(rN+rn+j-r)!} & \frac{(rN+j)!}{(rN+rn+j-3r)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} & 1 \\ \frac{(rN+j)!}{(rN+rn+j)!} & \frac{(rN+j)!}{(rN+rn+j-2r)!} & \cdots & \frac{(rN+j)!}{(rN+j+2r)!} & \frac{(rN+j)!}{(rN+j+r)!} \end{vmatrix} \\
= &\frac{(rN+j)!}{(rN+j+r)!}\tilde{W}_{N,rn-r} - \frac{(rN+j)!}{(rN+j+2r)!}\tilde{W}_{N,rn-2r} \\
&+ \cdots + (-1)^n \begin{vmatrix} \frac{(rN+j)!}{(rN+rn+j-r)!} & 1 \\ \frac{(rN+j)!}{(rN+rn+j)!} & \frac{(rN+j)!}{(rN+j+r)!} \end{vmatrix} \\
= &\sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}(rN+j)!}{(rN+rn-rk+j)!}\tilde{W}_{N,rk} = \tilde{W}_{N,rn}\,.
\end{aligned}$$

Note that $\tilde{W}_{N,r} = \frac{(rN+j)!}{(rN+j+r)!}$ and $\tilde{W}_{N,0} = 1$. $\square$

Remark 3.2. When $N = 0$, $r = 3$ and $j = 0, 1$, we have an determinant expression of the Lehmer's Euler numbers and their complementary numbers $W_{3n}^{(j)} = W_{0,3n,3}^{(j)}$ ([14]):

$$W_{3n}^{(j)} = (-1)^n (3n)! \begin{vmatrix} \frac{1}{(j+3)!} & 1 & & & \\ \frac{1}{(j+6)!} & \frac{1}{(j+3)!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{(3n+j-3)!} & \frac{1}{(3n+j-6)!} & \cdots & \frac{1}{(j+3)!} & 1 \\ \frac{1}{(3n+j)!} & \frac{1}{(3n+j-3)!} & \cdots & \frac{1}{(j+6)!} & \frac{1}{(j+3)!} \end{vmatrix}.$$

However, $W_{0,3n,3}^{(2)} \neq W_{3n}^{(2)}$ because these generating functions are given by

$$\sum_{n=0}^{\infty} W_{0,3n,3}^{(2)} \frac{t^{3n}}{(3n)!} = \left(1 + \sum_{l=1}^{\infty} \frac{2t^{3l}}{(3l+2)!}\right)^{-1}$$

and

$$\sum_{n=0}^{\infty} W_{3n}^{(2)} \frac{t^{3n}}{(3n)!} = \left(1 + \sum_{l=1}^{\infty} \frac{t^{3l}}{(3l+2)!}\right)^{-1},$$

respectively.

4. Incomplete Lehmer-Euler numbers

In order to generalize the hypergeometric numbers $W_{N,n,r}^{(j)}$, we shall introduce two kinds of incomplete Lehmer-Euler numbers. Similar but slightly different kinds of incomplete numbers are considered in [10,12,13,15]. For $j = 0, 1$ and $n \geq m \geq 1$, define the *restricted* hypergeometric Lehmer-Euler numbers $W_{N,n,r,\leq m}^{(j)}$ of grade r by

$$\sum_{n=0}^{\infty} W_{N,n,r,\leq m}^{(j)} \frac{t^n}{n!} = \left(1 + \sum_{l=1}^{m} \frac{(rN+j)!}{(rN+rl+j)!} t^{rl}\right)^{-1} \tag{16}$$

and the *associated* hypergeometric Lehmer-Euler numbers $W_{N,n,r,\geq m}^{(j)}$ of grade r by

$$\sum_{n=0}^{\infty} W_{N,n,r,\geq m}^{(j)} \frac{t^n}{n!} = \left(1 + \sum_{l=m}^{\infty} \frac{(rN+j)!}{(rN+rl+j)!} t^{rl}\right)^{-1}. \tag{17}$$

When $m \to \infty$ in (16) and $m = 1$ in (17), $W_{N,n,r}^{(j)} = W_{N,n,r,\leq\infty}^{(j)} = W_{N,n,r,\geq 1}^{(j)}$ are the original hypergeometric Lehmer-Euler numbers of grade r, defined in (11) with (12). Hence, both incomplete numbers are reduced to the hypergeometric Lehmer-Euler numbers too.

Notice that $W_{N,n,r,\leq m}^{(j)} = W_{N,n,r,\geq m}^{(j)} = 0$ unless $n \equiv 0 \pmod r$.

The restricted hypergeometric Lehmer-Euler numbers satisfy the following recurrence relation.

Proposition 4.1. *For $j = 0, 1$, we have*

$$W^{(j)}_{N,rn,r,\leq m} = -\sum_{k=\max\{n-m,0\}}^{n-1} \frac{(rn)!(rN+j)!}{(rN+rn-rk+j)!(rk)!} W^{(j)}_{N,rk,r,\leq m} \quad (n \geq 1)$$

with $W^{(j)}_{N,0,r,\leq m} = 1$.

The associated hypergeometric Lehmer-Euler numbers satisfy the following recurrence relation.

Proposition 4.2.

$$W^{(j)}_{N,rn,r,\geq m} = -\sum_{k=0}^{n-m} \frac{(rn)!(rN+j)!}{(rN+rn-rk+j)!(rk)!} W^{(j)}_{N,rk,r,\geq m} \quad (n \geq m)$$

with $W^{(j)}_{N,0,r,\geq m} = 1$ and $W^{(j)}_{N,r,r,\geq m} = \cdots = W^{(j)}_{N,r(m-1),r,\geq m} = 0$.

Proof of Proposition 4.1. By the definition (16), we get

$$\begin{aligned}1 &= \left(1 + \sum_{l=1}^{m} \frac{(rN+j)!t^{rl}}{(rN+rl+j)!}\right)\left(\sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\leq m} \frac{t^{rn}}{(rn)!}\right)\\ &= \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\leq m} \frac{t^{rn}}{(rn)!} + \sum_{n=1}^{\infty} \sum_{k=\max\{n-m,0\}}^{n-1} \frac{(rN+j)! W^{(j)}_{N,rk,r,\leq m}}{(rN+rn-rk+j)!(rk)!} t^{rn}.\end{aligned}$$

Comparing the coefficient on both sides, we obtain the first identity. $\square$

Proof of Proposition 4.2. By the definition (17), we get

$$\begin{aligned}1 &= \left(1 + \sum_{l=m}^{\infty} \frac{(rN+j)!t^{rl}}{(rl+j)!}\right)\left(\sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\geq m} \frac{t^{rn}}{(rn)!}\right)\\ &= \sum_{n=0}^{\infty} W^{(j)}_{N,rn,r,\geq m} \frac{t^{rn}}{(rn)!} + \sum_{n=m}^{\infty} \sum_{k=0}^{n-m} \frac{(rN+j)! W^{(j)}_{N,rk,r,\geq m}}{(rN+rn-rk+j)!(rk)!} t^{rn}.\end{aligned}$$

Comparing the coefficient on both sides, we obtain the desired result. $\square$

The restricted and associated hypergeometric Lehmer-Euler numbers have the following expressions in terms of determinants. All elements in some more bands become 0, in the expression in Theorem 3.1.

Theorem 4.3. *For integers n and m with $n \geq m \geq 1$, we have*

$W^{(j)}_{N,rn,r,\leq m} = (-1)^n (rn)!$

$$\times\left|\begin{array}{cccccc} \frac{(rN+j)!}{(rN+j+r)!} & 1 & 0 & & & \\ \vdots & & \ddots & \ddots & & \\ \frac{(rN+j)!}{(rN+rm+j)!} & & & & \ddots & \\ 0 & \ddots & & & \ddots & 0 \\ & & \ddots & & & 1 \\ & & 0 & \frac{(rN+j)!}{(rN+rm+j)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} \\ \multicolumn{3}{c}{\underbrace{\hspace{12em}}_{n-m}} & \multicolumn{3}{c}{\underbrace{\hspace{14em}}_{m}} \end{array}\right|.$$

Theorem 4.4. *For integers n and m with $n \ge m \ge 1$, we have*

$$W^{(j)}_{N,rn,r,\ge m} = (-1)^n (rn)! \left|\begin{array}{cccccc} 0 & 1 & 0 & & & \\ \vdots & & \ddots & \ddots & & \\ 0 & & & & \ddots & \\ \frac{(rN+j)!}{(rN+rm+j)!} & \ddots & & & \ddots & 0 \\ \vdots & & \ddots & & & 1 \\ \frac{(rN+j)!}{(rN+rn+j)!} & \cdots & \frac{(rN+j)!}{(rN+rm+j)!} & 0 & \cdots & 0 \\ \multicolumn{3}{c}{\underbrace{\hspace{16em}}_{n-m+1}} & \multicolumn{3}{c}{\underbrace{\hspace{7em}}_{m-1}} \end{array}\right|.$$

Proof of Theorem 4.3. For simplicity, put $\tilde{W}_{N,rn,\le m} = (-1)^n W^{(j)}_{N,rn,r,\le m}/(rn)!$ and prove that for $n \ge m \ge 1$

(18)

$$\tilde{W}_{N,rn,\le m} = \left|\begin{array}{cccccc} \frac{(rN+j)!}{(rN+j+r)!} & 1 & 0 & & & \\ \vdots & & \ddots & \ddots & & \\ \frac{(rN+j)!}{(rN+rm+j)!} & & & & \ddots & \\ 0 & \ddots & & & \ddots & 0 \\ & & \ddots & & & 1 \\ & & 0 & \frac{(rN+j)!}{(rN+rm+j)!} & \cdots & \frac{(rN+j)!}{(rN+j+r)!} \\ \multicolumn{3}{c}{\underbrace{\hspace{12em}}_{n-m}} & \multicolumn{3}{c}{\underbrace{\hspace{14em}}_{m}} \end{array}\right|.$$

When $n = m$, we have $\tilde{W}_{N,rm,\le m} = \tilde{W}_{N,rm}$, which is reduced to Theorem 3.1. Assume that (18) is valid up to $n-1$. If $n \ge 2m$, then the determinant on right-hand side of (18) is equal to

$$\frac{\tilde{W}_{N,rn-r,\le m}}{(rN+j+r)!} - \frac{\tilde{W}_{N,rn-2r,\le m}}{(rN+j+2r)!} + \cdots$$

$$
+(-1)^{m-1}\begin{vmatrix}
\frac{(rN+j)!}{(rN+rm+j)!} & 1 & 0 & & & \\
0 & \frac{(rN+j)!}{(rN+r+j)!} & 1 & & & \\
 & \vdots & & & & \\
 & \frac{(rN+j)!}{(rN+rm+j)!} & & & & \\
 & & \ddots & & & 1 \\
 & & & \frac{(rN+j)!}{(rN+rm+j)!} & \cdots & \frac{(rN+j)!}{(rN+r+j)!}
\end{vmatrix}
$$

$$
\begin{aligned}
&= \frac{\tilde W_{N,rn-r,\le m}}{(rN+r+j)!} - \frac{\tilde W_{N,rn-2r,\le m}}{(rN+2r+j)!} + \cdots + (-1)^{m-1}\frac{\tilde W_{N,rn-rm,\le m}}{(rN+rm+j)!}\\
&= \tilde W_{N,rn,\le m}\,.
\end{aligned}
$$

If $m<n<2m$, then the determinant on right-hand side of (18) is equal to

$$
\frac{\tilde W^{(j)}_{N,rn-r,\le m}}{(rN+r+j)!} - \frac{\tilde W^{(j)}_{N,rn-2r,\le m}}{(rN+2r+j)!} + \cdots
$$

$$
+(-1)^{m-n-1}\begin{vmatrix}
\frac{(rN+j)!}{(rN+rn-rm+j)!} & 1 & 0 & & & \\
\vdots & \vdots & & & & \\
\frac{(rN+j)!}{(rN+rm+j)!} & \frac{(rN+j)!}{(rN+2rm-rn+j)!} & & & & \\
0 & \vdots & & & & \\
\vdots & & & & & 1 \\
0 & \frac{(rN+j)!}{(rN+rm+j)!} & & \cdots & \cdots & \frac{(rN+j)!}{(rN+r+j)!}
\end{vmatrix}
$$

$$
\begin{aligned}
&= \frac{\tilde W_{N,rn-r,\le m}}{(rN+r+j)!} - \frac{\tilde W_{N,rn-2r,\le m}}{(rN+2r+j)!} + \cdots + (-1)^{n-m-1}\frac{\tilde W_{N,rm,\le m}}{(rN+rn-rm+j)!}\\
&= \frac{\tilde W_{N,rn-r,\le m}}{(rN+r+j)!} - \frac{\tilde W_{N,rn-2r,\le m}}{(rN+2r+j)!} + \cdots + (-1)^{m-1}\frac{\tilde W_{N,rn-rm,\le m}}{(rN+rm+j)!}\\
&= \tilde W_{N,rn,\le m}\,. \qquad \square
\end{aligned}
$$

Proof of Theorem 4.4. For simplicity, put $\tilde W_{N,rn,\ge m}=(-1)^n W_{N,rn,r,\ge m}/(rn)!$ and prove that

$$
\tilde W_{N,rn,\ge m} = \begin{vmatrix}
0 & 1 & 0 & & & \\
\vdots & & \ddots & \ddots & & \\
0 & & & & \ddots & \\
\frac{(rN+j)!}{(rN+rm+j)!} & \ddots & & & \ddots & 0 \\
\vdots & & \ddots & & & 1 \\
\frac{(rN+j)!}{(rN+rn+j)!} & \cdots & \frac{(rN+j)!}{(rN+rm+j)!} & 0 & \cdots & 0 \\
\multicolumn{3}{c}{\underbrace{\hphantom{\frac{(rN+j)!}{(rN+rn+j)!}\cdots\frac{(rN+j)!}{(rN+rm+j)!}}}_{n-m+1}} & \multicolumn{3}{c}{\underbrace{\hphantom{0\cdots 0}}_{m-1}}
\end{vmatrix}\,. \tag{19}
$$

If $m \le n < 2m$, the determinant on the right-hand side of (19) is equal to

$$
\left|\begin{array}{c c}
0 & \begin{array}{cccc} 1 & 0 & & \\ 0 & \ddots & & \\ \vdots & & & \end{array} \\
\vdots & \\
0 & \\
\frac{(rN+j)!}{(rN+rm+j)!} & \begin{array}{cccc} & & \ddots & 0 \end{array} \\
\vdots & \\
\frac{(rN+j)!}{(rN+rn+j)!} & \underbrace{\begin{array}{cccc} \vdots & & & 1 \\ 0 & \cdots & \cdots & 0 \end{array}}_{m-1}
\end{array}\right|
$$

$$
= (-1)^m(-1)^{m+1}\frac{(rN+j)!}{(rN+rn+j)!}\left|\begin{array}{cccc} 1 & 0 & & \\ 0 & \ddots & & \\ & & \ddots & \\ 0 & & 0 & 1 \end{array}\right| = -\frac{(rN+j)!}{(rN+rn+j)!}\,.
$$

Since only the term for $k=0$ does not vanish in Proposition 4.2, we have

$$
W_{N,rn,\ge m} = -\frac{(rN+j)!}{(rN+rn+j)!}\,.
$$

If $n \ge 2m$, the determinant on the right-hand side of (19) is equal to

$$
(-1)^{m-1}\left|\begin{array}{cccccccc}
\frac{(rN+j)!}{(rN+rm+j)!} & 1 & 0 & & & & & \\
\vdots & 0 & \ddots & & & & & \\
 & \vdots & \ddots & & & & & \\
 & 0 & & & & & & \\
 & \frac{(rN+j)!}{(rN+rm+j)!} & & & & & \ddots & 0 \\
\vdots & \vdots & \ddots & & & & \ddots & 1 \\
\frac{(rN+j)!}{(rN+rn+j)!} & \multicolumn{3}{c}{\underbrace{\frac{(rN+j)!}{(r(N+n-m)+j)!} \quad \cdots \quad \frac{(rN+j)!}{(r(N+m)+j)!}}_{n-2m+1}} & \multicolumn{4}{c}{\underbrace{0 \quad \cdots \quad 0}_{m-1}}
\end{array}\right|
$$

$$
= (-1)^{m-1}\frac{\tilde W^{(j)}_{N,rn-rm,\ge m}}{(rN+rm+j)!}
$$

$$
+ (-1)^{m}\left|\begin{array}{cccccccc}
\frac{(rN+j)!}{(rN+rm+r+j)!} & 1 & 0 & & & & & \\
\vdots & 0 & \ddots & & & & & \\
 & \vdots & \ddots & & & & & \\
 & 0 & & & & & & \\
 & \frac{(rN+j)!}{(rN+rm+j)!} & & & & & \ddots & 0 \\
\vdots & \vdots & \ddots & & & & \ddots & 1 \\
\frac{(rN+j)!}{(rN+rn+j)!} & \multicolumn{3}{c}{\underbrace{\frac{(rN+j)!}{(r(N+n-m-1)+j)!} \quad \cdots \quad \frac{(rN+j)!}{(r(N+m)+j)!}}_{n-2m}} & \multicolumn{4}{c}{\underbrace{0 \quad \cdots \quad 0}_{m-1}}
\end{array}\right|
$$

$$= \cdots$$

$$= -\sum_{k=m}^{n-m} \frac{(-1)^{n-k}\tilde{W}_{N,rk,\geq m}}{(r(N+n-k)+j)!} = \tilde{W}_{N,rn,\geq m}\,.$$

Here, we used Proposition 4.2 again. □

There exist explicit expressions for both incomplete Lehmer-Euler numbers.

Theorem 4.5. *For $n, m \geq 1$,*

$$W_{N,rn,r,\leq m}^{(j)} = (rn)! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+\cdots+i_k=n \\ 1\leq i_1,\dots,i_k\leq m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\,.$$

Theorem 4.6. *For $n, m \geq 1$,*

$$W_{N,rn,r,\geq m}^{(j)} = (rn)! \sum_{k=1}^{n} (-1)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k\geq m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\,.$$

Proof of Theorem 4.5. When $n \leq m$, the proof is similar to that of Proposition 2.1. Note that in the proof of Proposition 2.1,

$$1 \leq n-l = i_k \leq n-k+1 \leq n\,.$$

Let $n \geq m+1$. By Proposition 4.1

$$\begin{aligned}
&\frac{W_{N,rn,r,\leq m}^{(j)}}{(rn)!} \\
&= -\sum_{l=n-m}^{n-1} \frac{(rN+j)! W_{N,rn,r,\leq m}^{(j)}}{(rN+rn-rl+j)!(rl)!} \\
&= -\sum_{l=n-m}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!} \sum_{k=1}^{l} (-1)^k \\
&\quad \times \sum_{\substack{i_1+\cdots+i_k=l \\ 1\leq i_1,\dots,i_k\leq m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!} \\
&= -\sum_{l=1}^{n-1} \frac{(rN+j)!}{(rN+rn-rl+j)!} \sum_{k=1}^{l} (-1)^k \\
&\quad \times \sum_{\substack{i_1+\cdots+i_k=l \\ 1\leq i_1,\dots,i_k\leq m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=1}^{n-m-1}\frac{(rN+j)!}{(rN+rn-rl+j)!}\sum_{k=1}^{l}(-1)^k\\
&\times\sum_{\substack{i_1+\cdots+i_k=l\\1\le i_1,\dots,i_k\le m}}\frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
=\;&-\sum_{k=1}^{n-1}(-1)^k\sum_{l=k}^{n-1}\frac{(rN+j)!}{(rN+rn-rl+j)!}\\
&\times\sum_{\substack{i_1+\cdots+i_k=l\\1\le i_1,\dots,i_k\le m}}\frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
&+\sum_{k=1}^{n-m-1}(-1)^k\sum_{l=k}^{n-m-1}\frac{(rN+j)!}{(rN+rn-rl+j)!}\\
&\times\sum_{\substack{i_1+\cdots+i_k=l\\1\le i_1,\dots,i_k\le m}}\frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
=\;&\sum_{k=2}^{n}(-1)^k\sum_{l=k-1}^{n-1}\frac{(rN+j)!}{(rN+rn-rl+j)!}\\
&\times\sum_{\substack{i_1+\cdots+i_{k-1}=l\\1\le i_1,\dots,i_{k-1}\le m}}\frac{\big((rN+j)!\big)^{k-1}}{(rN+ri_1+j)!\cdots(rN+ri_{k-1}+j)!}\\
&-\sum_{k=2}^{n-m}(-1)^k\sum_{l=k-1}^{n-m-1}\frac{(rN+j)!}{(rN+rn-rl+j)!}\\
&\times\sum_{\substack{i_1+\cdots+i_{k-1}=l\\1\le i_1,\dots,i_{k-1}\le m}}\frac{\big((rN+j)!\big)^{k-1}}{(rN+ri_1+j)!\cdots(rN+ri_{k-1}+j)!}\\
=\;&\sum_{k=n-m+1}^{n}(-1)^k\sum_{l=k-1}^{n-1}\frac{(rN+j)!}{(rn-rl+j)!}\\
&\times\sum_{\substack{i_1+\cdots+i_{k-1}=l\\1\le i_1,\dots,i_{k-1}\le m}}\frac{\big((rN+j)!\big)^{k-1}}{(rN+ri_1+j)!\cdots(rN+ri_{k-1}+j)!}\\
&+\sum_{k=2}^{n-m}(-1)^k\sum_{l=n-m}^{n-1}\frac{(rN+j)!}{(rN+rn-rl+j)!}
\end{aligned}
$$

$$\times \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ 1\le i_1,\dots,i_{k-1}\le m}} \frac{\bigl((rN+j)!\bigr)^{k-1}}{(rN+ri_1+j)!\cdots(rN+ri_{k-1}+j)!}\,.$$

By putting $i_k = n-l$, in the first term by $n-1 \ge l \ge k-1 \ge n-m$, in the second term by $n-1 \ge l \ge n-m$, we have

$$1 \le n-l = i_k \le m\,.$$

Therefore,

$$\begin{aligned}
\frac{W_{N,rn,r,\le m}^{(j)}}{(rn)!} &= \sum_{k=n-m+1}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}} \frac{\bigl((rN+j)!\bigr)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
&\quad + \sum_{k=2}^{n-m}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}} \frac{\bigl((rN+j)!\bigr)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\\
&= \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ 1\le i_1,\dots,i_k\le m}} \frac{\bigl((rN+j)!\bigr)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\,.
\end{aligned}$$

Note that the term is vanished for $k=1$ as $n>m$. $\square$

Proof of Theorem 4.6. Since the set

$$\{(i_1,\dots,i_k)|i_1+\cdots+i_k=n,\ i_1,\dots,i_k\ge m\}$$

is empty for $n=1,\dots,m-1$, we have $W_{N,r,r,\ge m}^{(j)} = \cdots = W_{N,rm-r,r,\ge m}^{(j)} = 0$. For $n=m$, by Theorem 4.4

$$\begin{aligned}
W_{N,rm,r,\ge m}^{(j)} &= (-1)^m (rm)! \begin{vmatrix} 0 & 1 & & \\ \vdots & & & \\ 0 & & & 1\\ \frac{(rN+j)!}{(rN+rm+j)!} & 0 & \cdots & 0 \end{vmatrix}\\
&= (-1)^m (rm)!(-1)^{m-1}\frac{(rN+j)!}{(rN+rm+j)!} = -\frac{(rN+j)!}{(rN+rm+j)!}\,,
\end{aligned}$$

which matches the result for $n=m$. Assume that the result is valid up to $n-1(\ge m)$. Then by Proposition 4.2

$$\begin{aligned}
\frac{W_{N,rn,r,\ge m}}{(rn)!} &= -\sum_{l=0}^{n-m} \frac{(rN+j)!}{(rN+rn-rl+j)!(rl)!} W_{N,rl,r,\ge m}^{(j)}\\
&= -\frac{(rN+j)!}{(rN+rn+j)!}\\
&\quad - \sum_{l=1}^{n-m} \frac{(rN+j)!}{(rN+rn-rl+j)!} \sum_{k=1}^{l}(-1)^k
\end{aligned}$$

$$\times \sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\ldots,i_k\ge m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}$$

$$= -\frac{(rN+j)!}{(rN+rn+j)!}$$

$$- \sum_{k=1}^{n-m}(-1)^k \sum_{l=k}^{n-m} \frac{(rN+j)!}{(rN+rn-rl+j)!}$$

$$\times \sum_{\substack{i_1+\cdots+i_k=l\\ i_1,\ldots,i_k\ge m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}$$

$$= -\frac{(rN+j)!}{(rN+rn+j)!}$$

$$+ \sum_{k=2}^{n-m+1}(-1)^k \sum_{l=k-1}^{n-m} \frac{(rN+j)!}{(rN+rn-rl+j)!}$$

$$\times \sum_{\substack{i_1+\cdots+i_{k-1}=l\\ i_1,\ldots,i_{k-1}\ge m}} \frac{\big((rN+j)!\big)^{k-1}}{(rN+ri_1+j)!\cdots(rN+ri_{k-1}+j)!}$$

$$= -\frac{(rN+j)!}{(rN+rn+j)!}$$

$$+ \sum_{k=2}^{n-m+1}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\ge m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}$$

$$(i_k = n-l)$$

$$= \sum_{k=1}^{n-m+1}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\ge m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}$$

$$= \sum_{k=1}^{n}(-1)^k \sum_{\substack{i_1+\cdots+i_k=n\\ i_1,\ldots,i_k\ge m}} \frac{\big((rN+j)!\big)^k}{(rN+ri_1+j)!\cdots(rN+ri_k+j)!}\,.$$

Note that $i_k = n-l \ge m$ as $l \le n-m$. As $1 \le m \le n-1$, we have

$$n < (n-m+2)m \le km \le n = i_1+\cdots+i_k\,,$$

so the set

$$\{(i_1,\ldots,i_k) \mid i_1+\cdots+i_k = n,\ i_1,\ldots,i_k \ge m\}$$

is empty for $n-m+2 \le k \le n$. □

5. Applications by the Trudi's formula

We shall use the Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $W_{N,n,r}^{(j)}$.

Lemma 5.1. *For a positive integer n, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix}$$
$$= \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ *are the multinomial coefficients.*

This relation is known as Trudi's formula [18, Vol. 3, p. 214], [20] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [1], [18, Vol. 3, pp. 208–209].

In addition, there exists the following inversion formula (see, e.g. [15]), which is based upon the relation

$$\sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n-k) = 0 \quad (n \ge 1)$$

or Cameron's operator in (14).

Lemma 5.2. *If $\{\alpha_n\}_{n\ge 0}$ is a sequence defined by $\alpha_0 = 1$ and*

$$\alpha_n = \begin{vmatrix} D(1) & 1 & & \\ D(2) & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix}, \text{ then } D(n) = \begin{vmatrix} \alpha_1 & 1 & & \\ \alpha_2 & \ddots & \ddots & \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n}.$$

By applying these lemmata to Theorem 4.3 and Theorem 4.4, we obtain an explicit expression for the hypergeometric Lehmer-Euler numbers.

Theorem 5.3. *For $n \ge m \ge 1$, we have*

$$W_{N,rn,r,\le m}^{(j)} = (rn)! \sum_{t_1+2t_2+\cdots+mt_m=n} \binom{t_1+\cdots+t_m}{t_1,\dots,t_m}$$

$$\times (-1)^{t_1+\cdots+t_m}\left(\frac{(rN+j)!}{(rN+j+r)!}\right)^{t_1}\cdots\left(\frac{(rN+j)!}{(rN+rm+j)!}\right)^{t_m}$$

and

$$\begin{aligned}
&W^{(j)}_{N,rn,r,\ge m}\\
&= (rn)! \sum_{mt_m+(m+1)t_{m+1}+\cdots+nt_n=n}\binom{t_m+t_{m+1}+\cdots+t_n}{t_m,t_{m+1},\dots,t_n}\\
&\quad\times(-1)^{t_m+t_{m+1}+\cdots+t_n}\left(\frac{(rN+j)!}{(rN+rm+j)!}\right)^{t_m}\left(\frac{(rN+j)!}{(rN+rm+j+r)!}\right)^{t_{m+1}}\\
&\quad\cdots\left(\frac{(rN+j)!}{(rN+rn+j)!}\right)^{t_n}.
\end{aligned}$$

As a special case of Theorem 5.3, we can obtain the expressions for the original hypergeometric Lehmer-Euler numbers.

Corollary 5.4. *For $n\ge 1$, we have*

$$\begin{aligned}
W^{(j)}_{N,rn,r} &= (rn)! \sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}\\
&\qquad\times(-1)^{t_1+\cdots+t_n}\left(\frac{(rN+j)!}{(rN+j+r)!}\right)^{t_1}\cdots\left(\frac{(rN+j)!}{(rN+rn+j)!}\right)^{t_n}.
\end{aligned}$$

By applying the inversion relation in Lemma 5.2 to Theorem 3.1, we have the following.

Theorem 5.5. *Let $j=0,1$. For $n\ge 1$, we have*

$$\frac{(-1)^n(rN+j)!}{(rN+rn+j)!}=\begin{vmatrix}
\frac{W^{(j)}_{N,r,r}}{r!} & 1 & & & \\
\frac{W^{(j)}_{N,2r,r}}{(2r)!} & \frac{W^{(j)}_{N,r,r}}{r!} & & & \\
\vdots & \vdots & \ddots & 1 & \\
\frac{W^{(j)}_{N,rn-r,r}}{(rn-r)!} & \frac{W^{(j)}_{N,rn-2r,r}}{(rn-2r)!} & \cdots & \frac{W^{(j)}_{N,r,r}}{r!} & 1\\
\frac{W^{(j)}_{N,rn,r}}{(rn)!} & \frac{W^{(j)}_{N,rn-r,r}}{(rn-r)!} & \cdots & \frac{W^{(j)}_{N,2r,r}}{(2r)!} & \frac{W^{(j)}_{N,r,r}}{r!}
\end{vmatrix}.$$

In this sense, we have the inversion relation of Corollary 5.4 too.

Corollary 5.6. *For $n\ge 1$, we have*

$$\frac{(rN+j)!}{(rN+rn+j)!}=\sum_{t_1+2t_2+\cdots+nt_n=n}\binom{t_1+\cdots+t_n}{t_1,\dots,t_n}$$

$$\times (-1)^{t_1+\cdots+t_n} \left(\frac{W_{N,r,r}^{(j)}}{r!} \right)^{t_1} \cdots \left(\frac{W_{N,rn,r}^{(j)}}{(rn)!} \right)^{t_n} .$$

6. Additional comments

The hypergeometric Lehmer-Euler numbers of higher order $W_{N,n,r}^{(j)}$ includes hypergeometric Bernoulli and Euler numbers, and the classical Bernoulli, Euler and Lehmer's Euler numbers as special cases. However, the numbers $W_{N,n,r}^{(j)}$ do not include some famous generalized numbers, for example, poly-Bernoulli and Euler numbers, Apostol-Bernoulli and Euler numbers and some p-adic numbers and q-numbers, because they do not satisfy the relation (14) as definitions.

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References

[1] F. Brioschi, *Sulle funzioni Bernoulliane ed Euleriane*, Annali de Mat., i. (1858), 260–263; Opere Mat., i. pp. 343–347.

[2] P. J. Cameron, *Some sequences of integers*, Discrete Math. **75** (1989), no. 1-3, 89–102.

[3] J. W. L. Glaisher, *Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers etc. as determinants*, Messenger (2) **6** (1875), 49–63.

[4] A. Hassen and H. D. Nguyen, *Hypergeometric Bernoulli polynomials and Appell sequences*, Int. J. Number Theory **4** (2008), no. 5, 767–774.

[5] ———, *Hypergeometric zeta functions*, Int. J. Number Theory **6** (2010), no. 1, 99–126.

[6] F. T. Howard, *A sequence of numbers related to the exponential function*, Duke Math. J. **34** (1967), 599–615.

[7] ———, *Some sequences of rational numbers related to the exponential function*, Duke Math. J. **34** (1967), 701–716.

[8] S. Hu and M.-S. Kim, *On hypergeometric Bernoulli numbers and polynomials*, Acta Math. Hungar. **154** (2018), no. 1, 134–146.

[9] K. Kamano, *Sums of products of hypergeometric Bernoulli numbers*, J. Number Theory **130** (2010), no. 10, 2259–2271.

[10] T. Komatsu, *Incomplete poly-Cauchy numbers*, Monatsh. Math. **180** (2016), no. 2, 271–288.

[11] ———, *Complementary Euler numbers*, Period. Math. Hungar. **75** (2017), no. 2, 302–314.

[12] ———, *Incomplete multi-poly-Bernoulli numbers and multiple zeta values*, Bull. Malays. Math. Sci. Soc. **41** (2018), 2029–2040.

[13] T. Komatsu, K. Liptai, and I. Mező, *Incomplete Cauchy numbers*, Acta Math. Hungar. **149** (2016), no. 2, 306–323.

[14] T. Komatsu and Y. Ohno, *Lehmer's generalized Euler numbers*, preprint.

[15] T. Komatsu and J. L. Ramirez, *Some determinants involving incomplete Fubini numbers*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **26** (2018), no. 3, 143–170.

[16] T. Komatsu and H. Zhu, *Hypergeometric Euler numbers*, preprint. arXiv:1612.06210.

[17] D. H. Lehmer, *Lacunary recurrence formulas for the numbers of Bernoulli and Euler*, Ann. of Math. (2) **36** (1935), no. 3, 637–649.

[18] T. Muir, *The Theory of Determinants in the Historical Order of Development*, Four volumes, Dover Publications, New York, 1960.

[19] N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, available at oeis.org.

[20] N. Trudi, *Intorno ad alcune formole di sviluppo*, Rendic. dell' Accad. Napoli (1862), 135–143.

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