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WEAK LAWS OF LARGE NUMBERS FOR WEIGHTED COORDINATEWISE PAIRWISE NQD RANDOM VECTORS IN HILBERT SPACES

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Abstract. In this paper, we investigate weak laws of large numbers for weighted coordinatewise pairwise negative quadrant dependence random vectors in Hilbert spaces in the case that the decay order of tail probability is r for some $0 < r < 2$. Moreover, we extend results concerning Pareto-Zipf distributions and St. Petersburg game.

1. Introduction

Let $\{X_n; n \geq 1\}$ be the player's gains in a sequence of independent repetitions of the St. Petersburg game, i.e., a sequence of iid random variables with the common distribution

$$P(X = 2^k) = 2^{-k} \quad \text{for} \quad k = 1, 2, \ldots.$$

For the total gain, $S_n := X_1 + \cdots + X_n$ in n games, Feller [4] proved that

$$S_n/(n \log n) \to 1 \text{ in probability as } n \to \infty.$$

A. Gut [6] gave weak laws of large numbers for a generalized St. Petersburg game. Note that these games are formulated by nonnegative random variables with infinite means. On the other hand, Adler [1] got rid of the identically distributed condition with respect to independent Pareto-Zipf distributions. He studied weighted laws of large numbers for each model, respectively. Recently, Nakata [16] obtained some weak laws of large numbers for weighted sums of independent random variables in the case that the decay order of tail probability is r $(0 < r \leq 1)$. Dung et al. [3] gave weak laws of large numbers for sequences of independent random variables with infinite rth moments $(0 < r < 2)$.

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Lehmann [12] introduced the notion of negative quadrant dependence (NQD): Two random variables X_1 and X_2 is called NQD if

$$P(X_1 > x_1, X_2 > x_2) \leq P(X_1 > x_1)P(X_2 > x_2)$$

for all real numbers x_1, x_2. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be pairwise NQD if every pair of random variables in the sequence is NQD. It is easy to see that a pairwise NQD sequence of random variables is much weaker than the NA one [9]. In many statistics and mechanic models, a pairwise NQD assumption among the random variables in the models is more reasonable than an independence assumption, so many statisticians have investigated this topic with interest (see [13–15, 17]).

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$ and let $\{e_j, j \in B\}$ be an orthonormal basis in H. In this paper, we investigate weak laws of large numbers for the weighted sum

$$S_n = \sum_{j=1}^{m_n} a_{nj} X_j, \tag{1}$$

where $(a_{nj}; 1 \leq j \leq m_n, n \geq 1)$ is an array of real numbers, $\{X_n; n \geq 1\}$ is a sequence of coordinatewise pairwise NQD random vectors in Hilbert spaces which satisfies that

$$\sum_{j \in B} P(|\langle X_n, e_j\rangle| > x) \asymp x^{-r} \quad \text{for a fixed } \; 0 < r < 2. \tag{2}$$

It is easy to see that if the cardinality $|B|$ of B is finite, random vectors X_n that fulfill (2), then $E\|X_n\|^r = \infty$. Moreover, (2) is equivalent to

$$P(\|X_n\| > x) \asymp x^{-r} \quad \text{for a fixed } \; 0 < r < 2.$$

Many random evolutions and also statistical procedures such as parametric or nonparametric estimation of regression with fixed design, produce statistics of type (1). One example is the nonlinear regression model

$$y(x) = f(x) + \xi(x),$$

where $f(x)$ is an unknown function and $\xi(x)$ is the noise. Now, we fix the design points $x_{n1}, \ldots, x_{nn}$ and we get

$$y_{ni} = f(x_{ni}) + \xi_i,$$

where ξ_i is a centered sequence of random variables. The nonparametric estimator of $f(x)$ is defined to be $\widehat{f}_n(x) = \sum_{i=1}^n w_{ni}(x) y_{ni}$, where the weight functions $w_{ni}(x) = w_{ni}(x, \mathbf{x_n})$ depend both on x and the design points $\mathbf{x_n} = \{x_{n1}, \ldots, x_{nn}\}$. It is obvious that $\widehat{f}_n(x) - E\widehat{f}_n(x)$ is of the type (1).

We shall also see that the asymptotic behavior of the sum of variables of the form

$$X_k = \sum_{i=-\infty}^{\infty} a_{k+i}\xi_i$$

can be obtained by studying the sum of the type (1).

Methods for Hilbert space valued random vectors might also help to analyze nonlinear statistics of real valued data.

For example, we will consider general bivariate and degenerate von Mises-statistics (V-statistics). Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a symmetric, measurable function. We call

$$V_n := \frac{1}{b_n^2} \sum_{i,j=1}^{n} h(X_i, X_j)$$

be V-statistic with kernel h. The kernel and related V-statistic are called degenerate, if $E(h(x, X_i)) = 0$ for all $x \in \mathbb{R}$. Furthermore, we assume that h is Lipschitz-continuous and positive definite. By Sun's version of Mercers theorem, we have under these conditions a representation

$$h(x, y) = \sum_{l=1}^{\infty} \lambda \phi_l(x)\phi_l(y)$$

for orthonormal eigenfuctions $(\phi_l)_{l\in\mathbb{N}}$ with the following properties: $E\phi_l(X_n) = 0$, $E\phi_l^2(X_n) = 1$ for all $l \in \mathbb{N}$, and $\lambda_l \geq 0$ for all $l \in \mathbb{N}$, $\sum_{l=1}^{\infty} \lambda_l < \infty$ (see [2]).

Let H be Hibert space of real-valued sequences $y = (y_l)_{l\in\mathbb{N}}$ equipped with the inner product

$$\langle y, z\rangle := \sum_{l=1}^{\infty} \lambda_l y_l z_l.$$

We consider the H-valued random vectors $Y_n := (\phi_l(X_n))_{l\in\mathbb{N}}$. Then $\{Y_n, n \geq 1\}$ is a sequence of H-valued random vectors and

$$(3)\quad V_n = \frac{1}{b_n^2} \sum_{ij=1}^{n} \sum_{l=1}^{\infty} \lambda_l \phi_l(X_i)\phi_l(X_j) = \sum_{l=1}^{\infty} \lambda_l \Big(\frac{1}{b_n} \sum_{k=1}^{n} \phi_l(X_k)\Big)^2 = \Big\| \frac{1}{b_n} \sum_{k=1}^{n} Y_n \Big\|^2.$$

Now, let $\{X_n, n \geq 1\}$ be a sequence of real-valued pairwise independent random variables. Then $\{Y_n, n \geq 1\}$ is a sequence of H-valued coordinatewise pairwise NQD random vectors. Thus, since the limit result of H-valued coordinatewise pairwise NQD random vectors $\{Y_n, n \geq 1\}$, we obtain the convergence of V_n.

The organization of the paper is as follows. The concept of coordinatewise pairwise NQD random vectors in Hilbert spaces is defined in Section 2, where we also prove some useful lemmas. The results for weak laws of large numbers for weighted coordinatewise pairwise NQD random vectors in Hilbert spaces in the case that the decay order of tail probability is r for some $0 < r < 2$ are contained in Section 3. As applications, in Section 4, we give the extended Pareto-Zipf distribution, the generalized St. Petersburg game and present some results about weak laws of large numbers for each model, respectively.

2. Preliminaries

Let $\{a_n; n \geq 1\}$ and $\{b_n; n \geq 1\}$ be sequences of positive real numbers. We use notation $a_n \asymp b_n$ instead of $0 < \liminf a_n/b_n \leq \limsup a_n/b_n < \infty$;

$a_n = o(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = 0$; notation $a_n \sim b_n$ is used for $\lim_{n\to\infty} a_n/b_n = 1$. These notations are also used for positive real functions $f(x)$ and $g(x)$. The indicator function of A is denoted by $I(A)$. Throughout this paper, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

Let H be a real separable Hilbert space with the norm $\|\cdot\|$ generated by an inner product $\langle\cdot,\cdot\rangle$ and let $\{e_j, j \in B\}$ be an orthonormal basis in H. Huan et al. [8] introduced the concept of coordinatewise negatively association (CNA) random vectors in H and Mi-Hwa Ko [10] introduced the concept of coordinatewise asymptotical negatively association (CANA) random vectors in H. In this paper, we present the concept of coordinatewise pairwise NQD random vectors in H.

Definition. A sequence $\{X_n, n \geq 1\}$ of H-valued random vectors is said to be coordinatewise pairwise NQD if for any $j \in B$, the sequence of random variables $\{< X_n, e_j >, n \geq 1\}$ is pairwise NQD.

Remark 2.1. If a sequence of H-valued random vectors is NA [11] (or CNA, or pairwise independent), then it is coordinatewise pairwise NQD.

Example 2.2. Let $\{Y_n; n \geq 1\}$ be a sequence of pairwise NQD random variables. For each $n \geq 1, j \in B$, put $X_n^j = |c_j| Y_n$ where $\sum_{j\in B} c_j^2 < \infty$. We consider $X_n = \sum_{j\in B} X_n^j e_j$, $n \geq 1$, then $\{X_n, n \geq 1\}$ is a sequence of H-valued coordinatewise pairwise NQD random vectors.

The following example shows a sequence of H-valued coordinatewise pairwise NQD random vectors which is not pairwise independent.

Example 2.3. Let $\{Z_n, n \geq 1\}$ be i.i.d. $N(0,1)$ random variables. Then $\{Z_n - Z_{n+1}, n \geq 1\}$ are identically distributed $N(0,2)$ random variables. Let F be the $N(0,2)$ distribution function and $\{F_n, n \geq 1\}$ be a sequence of continuous distribution functions. For $n \geq 1$, put

$$F_n^{-1}(t) = \inf\{x : F_n(x) \geq t\} \text{ and } \ Y_n = F_n^{-1}(F(Z_n - Z_{n+1})).$$

Li et al. [13] showed that $\{Y_n, n \geq 1\}$ is a sequence of pairwise NQD random variables and for all $n \geq 1$, the distribution function of Y_n is F_n. For each $n \geq 1$, $j \in B$, put $X_n^j = |c_j| Y_n$ where $\sum_{j\in B} c_j^2 < \infty$. We consider $X_n = \sum_{j\in B} X_n^j e_j$, $n \geq 1$ (see Example 2.2), then $\{X_n, n \geq 1\}$ is a sequence of H-valued coordinatewise pairwise NQD random vectors.

But

$$\mathrm{Cov}(Z_n - Z_{n+1}, Z_{n+1} - Z_{n+2}) = -1$$

then X_n and X_{n+1} are not independent. Consequently, $\{X_n, n \geq 1\}$ is not a sequence of pairwise independent random vectors.

The following lemma plays an essential role in our main results.

Lemma 2.4 ([12])**.** *Let X and Y be $\mathbb{R}$-valued NQD random variables. Then,*

i) $cov(X,Y) \le 0$.

ii) *If f and g are Borel functions, both of which are monotone increasing (or both are monotone decreasing), then $f(X)$ and $g(X)$ are NQD.*

Lemma 2.5. *Let $(X_n, n \ge 1)$ be a sequence of H-valued coordinatewise pairwise NQD random vectors with mean 0 and finite second moments. Then,*

$$E\left\|\sum_{k=1}^{n} X_k\right\|^2 \le \sum_{k=1}^{n} E\|X_k\|^2.$$

Proof. For $n \ge 1$, we have by Lemma 2.4 that

$$\begin{aligned} E\left\|\sum_{k=1}^{n} X_k\right\|^2 &= E\sum_{j\in B}\Big(\langle\sum_{k=1}^{n} X_k, e_j\rangle\Big)^2 \\ &= \sum_{j\in B} E\Big(\sum_{k=1}^{n}\langle X_k, e_j\rangle\Big)^2 \\ &= \sum_{j\in B}\Big(\sum_{k=1}^{n} E\langle X_k, e_j\rangle^2 + \sum_{k\ne i} cov(\langle X_k, e_j\rangle, \langle X_i, e_j\rangle)\Big) \\ &\le \sum_{k=1}^{n}\sum_{j\in B} E\langle X_k, e_j\rangle^2 = \sum_{k=1}^{n} E\|X_k\|^2. \end{aligned}$$

□

Lemma 2.6. *Let X be an H-valued random vector. Suppose that*

$$\sum_{j\in B} P(|\langle X, e_j\rangle| > x) \asymp x^{-r} \textit{ for a fixed } 0 < r < 2.$$

Then

(a) $\sum_{j\in B} E(|\langle X, e_j\rangle| I(|\langle X, e_j\rangle| > x)) \asymp x^{1-r}$ *if* $1 < r < 2$.

(b) $\sum_{j\in B} E(|\langle X, e_j\rangle|^{\alpha} I(|\langle X, e_j\rangle| \le x)) \asymp x^{\alpha-r}$ *if* $\alpha > r > 0$.

Proof. To prove (a), we have

$$\begin{aligned} \sum_{j\in B} E(|\langle X, e_j\rangle| I(|\langle X, e_j\rangle| > x)) &\asymp \sum_{j\in B}\sum_{k\ge x} kP(k < |\langle X, e_j\rangle| \le k+1) \\ &= \sum_{j\in B}\sum_{k\ge x} P(|\langle X, e_j\rangle| > k) \\ &= \sum_{k\ge x}\sum_{j\in B} P(|\langle X, e_j\rangle| > k) \\ &\asymp \sum_{k\ge x} k^{-r} \asymp x^{1-r}. \end{aligned}$$

For the proof of (b), we see

$$
\begin{aligned}
&\sum_{j\in B} E(|\langle X, e_j\rangle|^{\alpha} I(|\langle X, e_j\rangle| \le x)) \\
\asymp &\sum_{j\in B}\sum_{0\le k\le x-1} (k+1)^{\alpha} P(k < |\langle X, e_j\rangle| \le k+1) \\
\asymp &\sum_{j\in B}\sum_{0\le k\le x-1} ((k+1)^{\alpha} - k^{\alpha}) P(|\langle X, e_j\rangle| > k) \\
\asymp &\sum_{j\in B}\sum_{1\le k\le x-1} k^{\alpha-1} P(|\langle X, e_j\rangle| > k) \\
= &\sum_{1\le k\le x-1} k^{\alpha-1} \sum_{j\in B} P(|\langle X, e_j\rangle| > k) \\
\asymp &\sum_{1\le k\le x-1} k^{\alpha-1-r} \asymp x^{\alpha-r}.
\end{aligned}
$$

□

3. The main results

We consider $\{X_n, n \ge 1\}$ is a sequence of coordinatewise pairwise NQD random vectors in H. For $k \ge 1, j \in B$, we set

$$X_k^j = \langle X_k, e_j\rangle.$$

Now, the main results can be stated and proved.

Theorem 3.1. *Let $\{X_n; n \ge 1\}$ be a sequence of H-valued coordinatewise pairwise NQD random vectors with zero mean and infinite rth moments for some $1 < r < 2$ whose distributions satisfy that $\sum_{j\in B} P(|X_n^j| > x) \asymp x^{-r}$ for all $n \ge 1$ and $\limsup_{x\to\infty} \sup_{n\ge 1} x^r \sum_{j\in B} P(|X_n^j| > x) < \infty$. Let $(a_{nk}; 1 \le k \le m_n, n \ge 1)$ be an array of positive real numbers such that*

$$\sum_{k=1}^{m_n} a_{nk}^r = o(1). \tag{4}$$

Then

$$\sum_{k=1}^{m_n} a_{nk} X_k \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Proof. For each $j \in B, n \ge 1$ and $1 \le k \le m_n$. Put

$$Y_{nk}^j = X_k^j I(|X_k^j| \le a_{kn}^{-1}) + a_{nk} I(X_k^j > a_{nk}^{-1}) - a_{nk} I(X_k^j < -a_{nk}^{-1}),$$

$$Y_{nk} = \sum_{j\in B} Y_{nk}^j e_j,\ U_n = \sum_{k=1}^{m_n} a_{nk}[Y_{nk} - EY_{nk}].$$

By using Lemma 2.4, it is easy to see that $\{Y_{nk} - EY_{nk},\ 1 \le k \le m_n\}$ is a sequence of H-valued coordinatewise pairwise NQD random vectors with mean 0. For an arbitrary $\varepsilon > 0$, by Lemma 2.5 and Lemma 2.6, we have

$$\begin{aligned}
P(\|U_n\| > \varepsilon) &\le \frac{1}{\varepsilon^2} E(\|U_n\|^2) \\
&\le \frac{1}{\varepsilon^2} \sum_{k=1}^{m_n} a_{nk}^2 E\|Y_{nk} - EY_{nk}\|^2 \\
&= \frac{1}{\varepsilon^2} \sum_{k=1}^{m_n} a_{nk}^2 \sum_{j \in B} E(Y_{nk}^j - EY_{nk}^j)^2 \\
&\le \frac{1}{\varepsilon^2} \sum_{k=1}^{m_n} a_{nk}^2 \sum_{j \in B} E(Y_{nk}^j)^2 \\
&\le \frac{1}{\varepsilon^2} \sum_{k=1}^{m_n} a_{nk}^2 \sum_{j \in B} E(|X_k^j|^2 I(|a_{nk} X_k^j| \le 1)) \\
&\quad + \frac{1}{\varepsilon^2} \sum_{k=1}^{m_n} \sum_{j \in B} P(|X_k^j| > a_{nk}^{-1}) \\
&\le \frac{C}{\varepsilon^2} \sum_{k=1}^{m_n} a_{nk}^r \to 0 \text{ as } n \to \infty. \qquad (5)
\end{aligned}$$

On the other hand, we also have by Lemma 2.6 that

$$\begin{aligned}
\sum_{k=1}^{m_n} a_{nk} \|EY_{nk}\| &= \sum_{k=1}^{m_n} a_{nk} \left\| \sum_{j \in B} EY_{nk}^j e_j \right\| \\
&\le \sum_{k=1}^{m_n} a_{nk} \left\| \sum_{j \in B} E\left(X_k^j I(|X_k^j| \le a_{nk}^{-1}) \right) e_j \right\| \\
&\quad + \sum_{k=1}^{m_n} \sum_{j \in B} P(|X_k^j| > a_{nk}^{-1}) \\
&= \sum_{k=1}^{m_n} a_{nk} \left\| \sum_{j \in B} E\left(X_k^j I(|X_k^j| > a_{nk}^{-1}) \right) e_j \right\| \\
&\quad + \sum_{k=1}^{m_n} \sum_{j \in B} P(|X_k^j| > a_{nk}^{-1}) \\
&\le \sum_{k=1}^{m_n} a_{nk} \sum_{j \in B} E\left(|X_k^j| I(|X_k^j| > a_{nk}^{-1}) \right)
\end{aligned}$$

$$+\sum_{k=1}^{m_n}\sum_{j\in B}P(|X_k^j|>a_{nk}^{-1})$$

$$\leq C\sum_{k=1}^{m_n}a_{nk}^r \to 0 \text{ as } n\to\infty. \tag{6}$$

Consequently, (5) and (6) yield that $\sum_{k=1}^{m_n} a_{nk}Y_{nk}$ converges in probability to 0. Finally, put $V_n = \sum_{k=1}^{n} a_{nk}Y_{nk}$ and $S_n = \sum_{k=1}^{m_n} a_{nk}X_k$, we have that for arbitrary $\epsilon > 0$,

$$P(\|V_n - S_n\| > \epsilon) \leq \sum_{k\leq m_n} P(Y_{nk}\neq X_k) \leq \sum_{k\leq m_n}\sum_{j\in B}P(Y_{nk}^j\neq X_k^j)$$

$$= \sum_{k\leq m_n}\sum_{j\in B}P(|X_k^j|>a_{nk}^{-1}) \leq C\sum_{k=1}^{m_n}a_{nk}^r \to 0 \text{ as } n\to\infty.$$

Therefore, $S_n \xrightarrow{P} 0$ as $n\to\infty$. □

When $m_n = n$ and $a_{nk} = \frac{1}{b_n}(1\leq k\leq n)$, we obtain the following corollary.

Corollary 3.2. *Let $\{X_n; n\geq 1\}$ be a sequence of H-valued coordinatewise pairwise NQD random vectors with zero mean and infinite rth moments for some $1<r<2$ whose distributions satisfy that $\sum_{j\in B}P(|X_n^j|>x)\asymp x^{-r}$ for $n\geq 1$ and $\limsup_{x\to\infty}\sup_{n\geq 1}x^r\sum_{j\in B}P(|X_n^j|>x)<\infty$. Let $\{b_n; n\geq 1\}$ be a sequence of positive real numbers such that*

$$\lim_{n\to\infty}\frac{n^{1/r}}{b_n}=0.$$

Then,

$$\frac{1}{b_n}\sum_{k=1}^{n}X_k \xrightarrow{P} 0 \text{ as } n\to\infty.$$

Example 3.3. Let $\{X_n, n\geq 1\}$ be the sequence of random vectors defined in Example 2.3 with (c_j) satisfying the condition $\sum_{j\in B}|c_j|^r<\infty$ for some $1<r<2$ and $\{F_n, n\geq 1\}$ be the distribution functions of a common density function

$$f(x)=\begin{cases}\dfrac{r}{2x^{r+1}} & \text{for } |x|>1,\\ 0 & \text{otherwise.}\end{cases}$$

One readily checks that, $EX_1 = 0, E\|X_1\|^r=\infty$ so that the Marcinkiewicz-Zygmund weak law does not hold. However, it is easy to see that $\sum_{j\in B}P(|X_1^j| > x)\asymp x^{-r}$ then applying Corollary 3.2 we get

$$\frac{1}{n^{1/r}l(n)}\sum_{k=1}^{n}X_k \xrightarrow{P} 0 \text{ as } n\to\infty,$$

where $\lim_{n\to\infty}l(n)=\infty$.

The obvious question that comes to mind is whether or not there is almost sure convergence in Corollary 3.2. The following example shows that the weak law established in Corollary 3.2 cannot be extended to a strong law.

Example 3.4. Consider the real Hilbert space ℓ_2 of all square summable real sequences with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \text{ for } x, y \in \ell_2.$$

The standard orthonormal basis of ℓ_2 is $\{e_n, n \leq 1\}$ where e_n denote the element of ℓ_2 having 1 in its nth position and 0 elsewhere. For each $1 < r \leq 2$, let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. symmetric random vectors in ℓ_2 space such that

$$X_n = (X_n^1, X_n^2, \dots, X_n^j, \dots),$$

where

$$P(|X_n^j| > x) = c_j x^{-r} \ (c_j > 0, \sum_{j=1}^{\infty} c_j < \infty) \text{ for all } x > x_0 > 0, \text{ and } j = 1, 2, \dots.$$

Then the hypotheses of Corollary 3.2 are met. Now, let $b_n = (n \log n)^{1/r}, n \geq 1$. For each $j > 0$, we obtain

$$\sum_{n=1}^{\infty} P(|X_n^j| > cb_n) \asymp c \sum_{n=1}^{\infty} \frac{1}{b_n^r} = c \sum_{n=1}^{\infty} \frac{1}{n \log n} = \infty \text{ for any } c > 0,$$

so that, by the Borel-Cantelli lemma,

$$P(|X_n^j| > cb_n \text{ infinitely often}) = 1 \quad \text{for any } c > 0.$$

Thus

$$\limsup_{n\to\infty} \frac{X_n^j}{b_n} = \infty \quad \text{almost surely.}$$

On the other hand,

$$\frac{X_n^j}{b_n} = \frac{\sum_{k=1}^n X_k^j}{b_n} - \frac{b_{n-1}}{b_n} \frac{\sum_{k=1}^{n-1} X_k^j}{b_{n-1}}.$$

This implies for each $j = 1, 2, \dots,$

$$\frac{\sum_{k=1}^n X_k^j}{b_n} \not\to 0 \quad \text{almost surely.}$$

Therefore, SLLN

$$\frac{\sum_{k=1}^n X_k}{b_n} \to 0 \quad \text{almost surely}$$

fails.

Theorem 3.5. *Let* $\{X_n; n \geq 1\}$ *be a sequence of* H*-valued coordinatewise pairwise NQD random vectors whose distributions satisfying* $\sum_{j\in B} P(|X_n^j| > x) \asymp x^{-r}$ *for* $n \geq 1$ *and* $\limsup_{x\to\infty} \sup_{n\geq 1} x^r \sum_{j\in B} P(|X_n^j| > x) < \infty$ $(0 < r \leq 1)$*. Let* $(a_{nk}; 1 \leq k \leq m_n, n \geq 1)$ *be an array of positive real numbers such that*

$$\sum_{k=1}^{m_n} a_{nk}^r = o(1). \tag{7}$$

Then

$$\sum_{k=1}^{m_n} a_{nk}\Big(X_k - EY_{nk}\Big) \xrightarrow{P} 0 \text{ as } n \to \infty, \tag{8}$$

where Y_{nk} *are defined as in proof of Theorem 3.1.*

Proof. It is well known that $\{Y_{nk} - EY_{nk}, 1 \leq k \leq m_n\}$ is a sequence of coordinatewise pairwise NQD random vectors in H. Let ε be an arbitrary positive number. We have, by Lemma 2.6,

$$\begin{aligned}
&P\Big(\Big\|\sum_{k=1}^{m_n} a_{nk}\Big(Y_{nk} - EY_{nk}\Big)\Big\| > \varepsilon\Big) \\
&\leq \frac{1}{\varepsilon^2} E\Big\|\sum_{k=1}^{m_n} a_{nk}\Big(Y_{nk} - EY_{nk}\Big)\Big\|^2 \\
&\leq \frac{1}{\epsilon^2}\sum_{k=1}^{m_n} a_{nk}^2 E\|Y_{nk} - EY_{nk}\|^2 = \frac{1}{\epsilon^2}\sum_{k=1}^{m_n} a_{nk}^2 \sum_{j\in B} E(Y_{nk}^j - EY_{nk}^j)^2 \\
&\leq \frac{1}{\epsilon^2}\sum_{k=1}^{m_n} a_{nk}^2 \sum_{j\in B} E(Y_{nk}^j)^2 \\
&\leq \frac{1}{\epsilon^2}\sum_{k=1}^{m_n} a_{nk}^2 \sum_{j\in B} E(|X_k^j|^2 I(|a_{nk}X_k^j| \leq 1)) + \frac{1}{\epsilon^2}\sum_{k=1}^{m_n}\sum_{j\in B} P(|X_k^j| > a_{nk}^{-1}) \\
&\leq C\sum_{k=1}^{m_n} a_{nk}^r \to 0 \text{ as } n \to \infty.
\end{aligned}$$

This implies that

$$\sum_{k=1}^{m_n} a_{nk}\Big(Y_{nk} - EY_{nk}\Big) \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{9}$$

On the other hand, we see

$$P(\Big\|\sum_{k=1}^{m_n} a_{nk}(X_k - Y_{nk})\Big\| > \epsilon) \leq \sum_{k\leq m_n} P(Y_{nk} \neq X_k) \leq \sum_{k\leq m_n}\sum_{j\in B} P(Y_{nk}^j \neq X_k^j)$$

$$= \sum_{k \le m_n} \sum_{j \in B} P(|X_k^j| > a_{nk}^{-1})$$

$$\le C \sum_{k=1}^{m_n} a_{nk}^r \to 0 \text{ as } n \to \infty. \tag{10}$$

(9) and (10) yield the conclusion (8). $\square$

Corollary 3.6. *Under the assumptions of Theorem 3.5, if* $0 < r < 1$*, then we have*

$$\sum_{k=1}^{m_n} a_{nk} X_k \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{11}$$

Proof. By Lemma 2.6, we have

$$\begin{aligned}
\Big\| \sum_{k=1}^{m_n} a_{nk} EY_{nk} \Big\| &\le \sum_{k=1}^{m_n} a_{nk} \|EY_{nk}\| \le \sum_{k=1}^{m_n} a_{nk} \sum_{j \in B} E|Y_{nk}^j| \\
&\le \sum_{k=1}^{m_n} a_{nk} \sum_{j \in B} E\left(|X_k^j| I(|X_k^j| \le a_{nk}^{-1})\right) \\
&\quad + \sum_{k=1}^{m_n} \sum_{j \in B} P(|X_k^j| > a_{nk}^{-1}) \\
&\le C \sum_{k=1}^{m_n} a_{nk}^r \to 0 \text{ as } n \to \infty. \qquad (12)
\end{aligned}$$

$\square$

Example 3.7. Let $\{X_n, n \ge 1\}$ be the sequence of random vectors defined in Example 2.3 with (c_j) such that $\sum_{j \in B} |c_j|^r < \infty$ for some $0 < r < 1$ and $\{F_n, n \ge 1\}$ be the distribution functions of common density function

$$f(x) = \begin{cases} \dfrac{r}{2x^{r+1}} & \text{for } |x| > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{j \in B} P(|X_1^j| > x) \asymp x^{-r}$. Using Corollary 3.6 with $a_{nk} = \frac{1}{n^{1/r} l(n)}$, $1 \le k \le n$, where $\lim_{n \to \infty} l(n) = \infty$, we get

$$\frac{1}{n^{1/r} l(n)} \sum_{k=1}^{n} X_k \xrightarrow{P} 0 \text{ as } n \to \infty.$$

In the case $l(n) = (\log n)^{1/r}$, we get the same result as in Example 6.4.4 (p. 282) of [7].

Corollary 3.8. *Under the assumptions of Theorem 3.5, if* $r = 1$ *and there exists* $A \in H$ *such that*

$$\lim_{n \to \infty} \sum_{k=1}^{m_n} a_{nk} \sum_{j \in B} EX_k^j I(|X_k^j| \le a_{nk}^{-1}) e_j = A, \tag{13}$$

then we have

$$\sum_{k=1}^{m_n} a_{nk} X_k \xrightarrow{P} A \text{ as } n \to \infty. \tag{14}$$

Proof. We have

$$\begin{aligned}
&\Big\| \sum_{k=1}^{m_n} a_{nk} EY_{nk} - \sum_{k=1}^{m_n} a_{nk} \sum_{j\in B} EX_k^j I(|X_k^j| \le a_{nk}^{-1}) e_j \Big\| \\
&\sum_{k=1}^{m_n} a_{nk} \sum_{j\in B} E|Y_{nk}^j - X_k^j I(|X_k^j| \le a_{nk}^{-1})| \\
&= \sum_{k=1}^{m_n} \sum_{j\in B} P(|X_k^j| > a_{nk}^{-1}) \le C \sum_{k=1}^{m_n} a_{nk}^r \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Then

$$\lim_{n\to\infty} \sum_{k=1}^{m_n} a_{nk} EY_{kn} = A,$$

by Theorem 3.5, we obtain (14). □

In the following corollary, we restrict to the identically distributed coordinatewise pairwise NQD random vector case. We recall the concept of slowly varying function at infinity as follows: Let $a \ge 0$, a positive measurable function $f(x)$ on $[a;\infty)$ is said to be *slowly varying at infinity* if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = 1 \text{ for all } t > 0.$$

Clearly, $\log x$, $\log\log x$ are slowly varying functions at infinity (the readers may refer to [5, 7]).

Corollary 3.9. *Let $\{X_n; n \ge 1\}$ be a sequence of H-valued identically distributed coordinatewise pairwise NQD random vectors whose common distributions satisfy $\sum_{j\in B} P(|X^j| > x) \asymp x^{-1}$. In addition, we suppose that there exists a sequence of real numbers $(\alpha_j)_{j\in B}$ such that $\sum_{j\in B} \alpha_j^2 < \infty$ and*

$$\lim_{x\to\infty} \frac{EX^j I(|X^j| \le x)}{\log x} = \alpha_j \quad (j \in B) \tag{15}$$

then for each real number $\beta > -1$ and a slowly varying sequence $l(n)$ we have

$$\frac{1}{n^{\beta+1} l(n) \log n} \sum_{k=1}^{n} k^\beta l(k) X_k \xrightarrow{P} \frac{\alpha}{1+\beta} \text{ as } n \to \infty, \tag{16}$$

where $\alpha = \sum_{j\in B} \alpha_j e_j$.

Proof. Using Corollary 3.8, with $m_n = n$, $a_{nk} = \frac{b_k}{c_n}$, $b_k = k^\beta l(k)$ and $c_n = n^{\beta+1} l(n) \log n$. We note

$$\sum_{j=1}^n b_j \sim \frac{1}{1+\beta} n^{\beta+1} l(n),$$
$$\sum_{j=1}^n b_j \log b_j \sim \frac{\beta}{1+\beta} n^{\beta+1} l(n) \log n \quad \text{and} \quad \log c_n \sim (\beta+1) \log n,$$

$$\begin{aligned}
\sum_{j\in B} E(|X^j| I(|X^j| \le x)) &\asymp \sum_{j\in B} \sum_{0\le k\le x-1} (k+1) P(k < |X^j| \le k+1) \\
&= \sum_{j\in B} \sum_{1\le k\le x-1} P(|X^j| > k) \\
&= \sum_{1\le k\le x-1} \sum_{j\in B} P(|X^j| > k) \\
&\asymp \sum_{1\le k\le x-1} k^{-1} \asymp \log x.
\end{aligned}$$

This implies that

$$\begin{aligned}
\sum_{j\in B} \Big| \sum_{k=1}^n a_{nk} E X^j I(|X^j| \le a_{nk}^{-1}) \Big| &\le \frac{1}{c_n} \sum_{k=1}^n b_k \sum_{j\in B} E|X^j| I(|X^j| \le \frac{c_n}{b_k}) \\
&\asymp \frac{1}{c_n} \sum_{k=1}^n b_k \log(\frac{c_n}{b_k}) \to \frac{1}{1+\beta} \quad \text{as } n \to \infty,
\end{aligned}$$

so

$$\sup_n \sum_{j\in B} \Big| \sum_{k=1}^n a_{nk} E X^j I(|X^j| \le a_{nk}^{-1}) \Big| < \infty. \tag{17}$$

Moreover, for each $j \in B$, by (15) we have

$$\sum_{k=1}^n a_{nk} E X^j I\{|X^j| \le a_{nk}^{-1}\} \sim \alpha_j \frac{\sum_{k=1}^n b_k \log(\frac{c_n}{b_k})}{c_n} \to \frac{\alpha_j}{1+\beta} \quad \text{as } n \to \infty.$$

Using (17),

$$A = \lim_{n\to\infty} \sum_{k=1}^n a_{nk} \sum_{j\in B} E X_k^j I(|X_k^j| \le a_{nk}^{-1}) e_j \to \frac{\alpha}{1+\beta} \text{ as } n \to \infty.$$

This completes the proof. $\square$

4. Applications

4.1. Extended Pareto-Zipf distribution

In this subsection, we extend Theorem 3.1 of Nakata [16] as follows.

We consider $\{X_n, n \geq 1\}$ as a sequence of coordinatewise pairwise NQD random vectors in H whose distributions are defined by $P(X_n^j = 0) = 1 - \frac{\alpha_j}{c_n^r}$, $(j \in B)$ and the tail probability

$$P(X_n^j > x) = \frac{\alpha_j}{(x + c_n)^r} \quad \text{for } x > 0, j \in B,$$

where $r > 0$, $\alpha_j \geq 0$, $\sum_{j \in B} \alpha_j < \infty$ and $\{c_n; n \geq 1\}$ is a sequence of positive numbers such that

$$c_n^r \geq \max\{\alpha_j\} \quad \text{for all} \quad n.$$

Clearly,

$$\sum_{j \in B} P(|X_n^j| > x) \asymp x^{-r}$$

for $n \geq 1$ and

$$\limsup_{x \to \infty} \sup_{n \geq 1} x^r \sum_{j \in B} P(|X_n^j| > x) < \infty.$$

We have the following theorem.

Theorem 4.1. *Let $0 < r < 2; r \neq 1$. Suppose that*

$$C_n = \sum_{k=1}^{n} c_k^{-r} \to \infty \textit{ as } n \to \infty,$$

and

$$C_n = o(b_n^r).$$

If $0 < r < 1$, then

$$\frac{1}{b_n} \sum_{k=1}^{n} \frac{X_k}{c_k} \xrightarrow{P} 0 \textit{ as } n \to \infty. \tag{18}$$

If $1 < r < 2$, then

$$\frac{1}{b_n} \Big(\sum_{k=1}^{n} \frac{X_k}{c_k} - \frac{C_n \alpha}{r - 1} \Big) \xrightarrow{P} 0 \textit{ as } n \to \infty, \tag{19}$$

where $\alpha = \sum_j \alpha_j e_j$.

Proof. If $0 < r < 1$, it is easy to obtain (18) by Corollary 3.6.

In the case $1 < r < 2$, we have by Theorem 3.1 that

$$\frac{1}{b_n} \sum_{k=1}^{n} \frac{1}{c_k} (X_k - E(X_k)) \xrightarrow{P} 0 \text{ as } n \to \infty.$$

Moreover,

$$\begin{aligned}\frac{1}{b_n}\sum_{k=1}^{n}\frac{1}{c_k}EX_k &= \frac{1}{b_n}\sum_{k=1}^{n}\frac{1}{c_k}\sum_{j\in B}EX_k^j e^j\\ &= \frac{1}{b_n}\sum_{k=1}^{n}\frac{1}{c_k}\sum_{j\in B}(\int_0^\infty P(X_k^j > x)dx)e^j\\ &= \frac{1}{b_n}\sum_{k=1}^{n}\frac{1}{c_k}\sum_{j\in B}(\int_0^\infty \frac{\alpha_j}{(x+c_k)^r}dx)e^j\\ &= \frac{\alpha}{b_n}\sum_{k=1}^{n}\frac{1}{(r-1)c_k^r} = \frac{C_n\alpha}{(r-1)b_n}.\end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let $r = 1$. Suppose that*

$$C_n = \sum_{k=1}^{n} c_k^{-1} \to \infty \text{ as } n \to \infty.$$

Then,

$$\frac{1}{C_n\log C_n}\sum_{k=1}^{n}\frac{X_k}{c_k} \xrightarrow{P} \alpha \text{ as } n\to\infty. \tag{20}$$

Proof. From Corollary 3.8, with $a_{nk} = \frac{1}{c_k C_n \log C_n}(1 \le k \le n)$, it is sufficient to show $A = \alpha$. It follows that for large $x > 0$,

$$\sum_{j\in B}EX_k^j I(X_k^j \le x)e_j = \sum_{j\in B}\Big(xP(X_k^j \le x) - \int_0^x P(X_k^j \le t)dt\Big)e_j \sim \alpha\log\frac{x}{c_k},$$

then

$$\begin{aligned}A &= \lim_{n\to\infty}\frac{1}{C_n\log C_n}\sum_{k=1}^{n}\frac{1}{c_k}\sum_{j\in B}EX_k^j I(X_k^j \le C_n\log C_n c_k)e_j\\ &= \alpha\lim_{n\to\infty}\frac{C_n\log(C_n\log C_n)}{C_n\log C_n} = \alpha.\end{aligned}$$

□

4.2. A generalized St. Petersburg Game

We consider $\{X_n, n \ge 1\}$ as a sequence of identically distributed coordinatewise pairwise NQD random vectors in H with

$$P(X_n^j = 2^{k/r}) = \alpha_j 2^{-k} \text{ for } k \ge 1, n \ge 1, j \in B, \text{ and } P(X_n^j = 0) = 1 - \alpha_j,$$

where $0 < r < 2$, $0 \le \alpha_j \le 1$, $\sum_{j\in B}\alpha_j < \infty$. Put $\alpha = \sum_{j\in B}\alpha_j e_j$.

If $1 < r < 2$, then $m_j = EX_n^j = \frac{\alpha_j}{1-2^{1/r-1}}$, $m = \sum_j m_j e_j = E(X_n) = \frac{\alpha}{1-2^{1/r-1}}$. Thus, we also have $\sum_{j\in B} P(|X_n^j - m_j| > x) \asymp x^{-r}$. Therefore, applying Corollary 3.2 with $b_n = n^{1/r}\log\log n$, we get

$$\frac{1}{n^{1/r}\log\log n}\sum_{k=1}^{n}(X_k - m) \xrightarrow{P} 0 \text{ as } n\to\infty.$$

If $0 < r < 1$, we also have $\sum_{j\in B} P(|X_n^j| > x) \asymp x^{-r}$. Therefore, applying Corollary 3.6 with $a_{nk} = \frac{1}{n^{1/r}\log\log n}(1\le k\le n)$, we get

$$\frac{1}{n^{1/r}\log\log n}\sum_{k=1}^{n} X_k \xrightarrow{P} 0 \text{ as } n\to\infty.$$

If $r = 1$, note that $EX^j I_{\{|X^j|\le x\}} \sim \alpha_j \log x/\log 2$ for all $j\in B$. By Corollary 3.9, for $\beta > -1$ and a slowly varying sequence $l(n)$, we have

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{n} j^\beta l(j) X_j}{n^{\beta+1} l(n)\log n} = \frac{1}{(1+\beta)\log 2} \text{ in probability.}$$

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