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COMPARISON THEOREMS IN RIEMANN-FINSLER GEOMETRY WITH LINE RADIAL INTEGRAL CURVATURE BOUNDS AND RELATED RESULTS

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Abstract. We establish some Hessian comparison theorems and volume comparison theorems for Riemann-Finsler manifolds under various line radial integral curvature bounds. As their applications, we obtain some results on first eigenvalue, Gromov pre-compactness and generalized Myers theorem for Riemann-Finsler manifolds under suitable line radial integral curvature bounds. Our results are new even in the Riemannian case.

1. Introduction

Comparison technique is a powerful tool in global analysis in differential geometry, and it has been well developed in Riemannian geometry. Volume, as the important geometric invariant, plays a key role in comparison technique. Recently comparison technique has been developed for Finsler manifolds and the relationship between curvature and topology of Finsler manifolds has also been investigated [1, 3–6, 12]. It should be pointed out here that volume form is uniquely determined by the given Riemannian metric, while there are different choices of volume forms for Finsler metrics. As the result, we usually need to control the S-curvature in order to obtain volume comparison theorems as well as results on curvature and topology. This additional assumption on S-curvature has been removed by author recently by using the extreme volume forms (the maximal and minimal volume forms) [7, 10], and we may also consider different curvature bounds, such as integral curvature bounds [8, 11].

The main purpose of the present paper is to study comparison theorems under line radial integral curvature bounds. We establish some Hessian comparison theorems and volume comparison theorems for Riemann-Finsler manifolds under various line radial integral curvature bounds. As their applications, we

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obtain some results on first eigenvalue, Gromov pre-compactness and generalized Myers theorem for Riemann-Finsler manifolds under suitable line radial integral curvature bounds. Our results are new even in the Riemannian case.

2. Finsler geometry

Let (M,F) be a Finsler n-manifold with Finsler metric $F:TM\to[0,\infty)$. Let $(x,y)=(x^i,y^i)$ be local coordinates on TM, and $\pi:TM\backslash 0\to M$ the natural projection. Unlike in the Riemannian case, most Finsler quantities are functions of TM rather than M. The *fundamental tensor* g_{ij} and the *Cartan tensor* C_{ijk} are defined by

$$g_{ij}(x,y):=\frac{1}{2}\frac{\partial^2F^2(x,y)}{\partial y^i\partial y^j},\quad C_{ijk}(x,y):=\frac{1}{4}\frac{\partial^3F^2(x,y)}{\partial y^i\partial y^j\partial y^k}.$$

Let $V=V^i\partial/\partial x^i$ be a non-vanishing vector field on an open subset $\mathcal{U}\subset M$. One can introduce a Riemannian metric $\widetilde{g}=\mathbf{g}_V$ and a linear connection ∇^V on the tangent bundle over $\mathcal{U}$ as follows:

$$\mathbf{g}_V(X,Y):=X^iY^jg_{ij}(x,v),\quad \forall X=X^i\frac{\partial}{\partial x^i},Y=Y^i\frac{\partial}{\partial x^i};$$

$$\nabla^V_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}:=\Gamma^k_{ij}(x,v)\frac{\partial}{\partial x^k},$$

here Γ^k_{ij} are the Chern connection coefficients. From the torsion freeness and almost g-compatibility of Chern connection we have

$$\nabla^V_XY-\nabla^V_YX=[X,Y], \tag{2.1}$$

$$X\cdot\mathbf{g}_V(Y,Z)=\mathbf{g}_V(\nabla^V_XY,Z)+\mathbf{g}_V(Y,\nabla^V_XZ)+2\mathbf{C}_V(\nabla^V_XV,Y,Z), \tag{2.2}$$

here $\mathbf{C}_V$ is defined by

$$\mathbf{C}_V(X,Y,Z)=X^iY^jZ^kC_{ijk}(x,v),$$

and it satisfies

$$\mathbf{C}_V(V,X,Y)=0. \tag{2.3}$$

By (2.1)-(2.3) we see that the Chern connection ∇^V and the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ are related by

$$\begin{aligned}\mathbf{g}_V(\nabla^V_XY,Z)&=\mathbf{g}_V(\widetilde{\nabla}_XY,Z)-\mathbf{C}_V(\nabla^V_XV,Y,Z)\\&\quad-\mathbf{C}_V(\nabla^V_YV,X,Z)+\mathbf{C}_V(\nabla^V_ZV,X,Y).\end{aligned} \tag{2.4}$$

The *Chern curvature* $\mathbf{R}^V(X,Y)Z$ for vector fields X,Y,Z on $\mathcal{U}$ is defined by

$$\mathbf{R}^V(X,Y)Z:=\nabla^V_X\nabla^V_YZ-\nabla^V_Y\nabla^V_XZ-\nabla^V_{[X,Y]}Z.$$

In the Riemannian case this curvature does not depend on V and coincides with the Riemannian curvature tensor. For a *flag* $(V;P)$ (or $(V;W)$) consisting of a non-zero tangent vector $V \in T_xM$ and a 2-plane $P \subset T_xM$ with $V \in P$ the *flag curvature* $\mathbf{K}(V; P)$ is defined as follows:

$$\mathbf{K}(V; P) = \mathbf{K}(V; W) := \frac{\mathbf{g}_V(\mathbf{R}^V(V, W)W, V)}{\mathbf{g}_V(V, V)\mathbf{g}_V(W, W) - \mathbf{g}_V(V, W)^2}.$$

Here W is a tangent vector such that V, W span the 2-plane P and $V \in T_xM$ is extended to a geodesic field, i.e., $\nabla^V_V V = 0$ near x. In the Riemannian case the flag curvature is the sectional curvature of the 2-plane P and does dot depend on V. In the literature there are several connections used in Finsler geometry, but for the definition of the flag curvature it does not make a difference whether one uses the Chern, the Cartan or the Berwald connection. The *Ricci curvature* of V is defined as

$$\mathbf{Ric}(V) = \sum_i \mathbf{K}(V; E_i),$$

where $E_1, \ldots, E_n$ is the local $\mathbf{g}_V$-orthonormal frame over $\mathcal{U}$. By (2.4) it is easy to see that $\nabla^V_V V = \widetilde{\nabla}_V V$, and consequently, V is a geodesic field of F if and only if it is a geodesic field of $\widetilde{g}$, and when V is a geodesic field, then $\nabla^V_V = \widetilde{\nabla}_V$, and for any plane P containing V, the flag curvature $\mathbf{K}(P, V)$ is just the sectional curvature $\widetilde{\mathbf{K}}(P)$ of $\widetilde{g}$ (see [3,4]).

Given a Finsler manifold (M, F), the *dual Finsler metric* F^* on M is defined by

$$F^*(\xi_x) := \sup_{Y \in T_xM \backslash 0} \frac{\xi(Y)}{F(Y)}, \quad \forall \xi \in T^*M,$$

and the corresponding fundamental tensor is defined by

$$g^{*kl}(\xi) = \frac{1}{2} \frac{\partial^2 F^{*2}(\xi)}{\partial \xi_k \partial \xi_l}.$$

The *Legendre transformation* $l : TM \to T^*M$ is defined by

$$l(Y) = \begin{cases} \mathbf{g}_Y(Y, \cdot), & Y \neq 0, \\ 0, & Y = 0. \end{cases}$$

It is well-known that for any $x \in M$, the Legendre transformation is a smooth diffeomorphism from $T_xM \backslash 0$ onto $T^*_xM \backslash 0$, and it is norm-preserving, namely, $F(Y) = F^*(l(Y)), \forall Y \in TM$. Consequently, $g^{ij}(Y) = g^{*ij}(l(Y))$.

Now let $f : M \to \mathbb{R}$ be a smooth function on M. The *gradient* of f is defined by $\nabla f = l^{-1}(df)$. Thus we have

$$df(X) = \mathbf{g}_{\nabla f}(\nabla f, X), \quad X \in TM.$$

Let $\mathcal{U} = \{x \in M : \nabla f \mid_x \neq 0\}$. We define the *Hessian* $H(f)$ of f on $\mathcal{U}$ as follows:

$$H(f)(X, Y) := XY(f) - \nabla^{\nabla f}_X Y(f), \quad \forall X, Y \in TM \mid_{\mathcal{U}}.$$

By (2.1)-(2.4) it is easy to know that $H(f)$ is symmetric, and it can be rewritten as (see [11,12])

$$H(f)(X,Y) = \mathbf{g}_{\nabla f}(\nabla_X^{\nabla f}\nabla f, Y). \tag{2.5}$$

A *volume form* $d\mu$ on Finsler manifold (M,F) is nothing but a global non-degenerate n-form on M. In local coordinates we can express $d\mu$ as $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. The frequently used volume forms in Finsler geometry are so-called Busemann-Hausdorff volume form and Holmes-Thompson volume form. In [7] we introduce the maximal and minimal volume forms for Finsler manifolds which play the important role in comparison technique in Finsler geometry. They are defined as following. Let

$$dV_{\max} = \sigma_{\max}(x)dx^1 \wedge \cdots \wedge dx^n$$

and

$$dV_{\min} = \sigma_{\min}(x)dx^1 \wedge \cdots \wedge dx^n$$

with

$$\sigma_{\max}(x) := \max_{y\in T_xM\setminus 0}\sqrt{\det(g_{ij}(x,y))}, \quad \sigma_{\min}(x) := \min_{y\in T_xM\setminus 0}\sqrt{\det(g_{ij}(x,y))}.$$

Then it is easy to check that the n-forms $dV_{\max}$ and $dV_{\min}$ are well-defined on M. $dV_{\max}$ and $dV_{\min}$ are called the *maximal volume form* and the *minimal volume form* of (M,F), respectively. Both maximal volume form and minimal volume form are called *extreme volume form*, and we shall denote by dV_{ext} the maximal or minimal volume form. The volume with respect to $dV_{\max}$ (resp. $dV_{\min}$) is called the *maximal volume* (resp. *minimal volume*). Maximal volume and minimal volume are both called *extreme volume*.

The *uniformity function* $\mu : M \to \mathbb{R}$ is defined by

$$\mu(x) = \max_{y,z,u\in T_xM\setminus 0}\frac{\mathbf{g}_y(u,u)}{\mathbf{g}_z(u,u)}.$$

$\mu_F = \max_{x\in M}\mu(x)$ is called the *uniformity constant*. It is clear that

$$\mu^{-1}F^2(u) \leqslant \mathbf{g}_y(u,u) \leqslant \mu F^2(u).$$

Similarly, the *reversible function* $\lambda : M \to \mathbb{R}$ is defined by

$$\lambda(x) = \max_{y\in T_xM\setminus 0}\frac{F(y)}{F(-y)}.$$

$\lambda_F = \max_{x\in M}\lambda(x)$ is called the *reversibility* of (M,F), and (M,F) is called *reversible* if $\lambda_F = 1$. It is clear that $\lambda(x)^2 \leqslant \mu(x)$.

Fix $x \in M$, let $I_x = \{v \in T_xM : F(v) = 1\}$ be the indicatrix at x. For $v \in I_x$, the *cut-value* $c(v)$ is defined by

$$c(v) := \sup\{t > 0 : d(x, \exp_x(tv)) = t\}.$$

Then, we can define the *tangential cut locus* $\mathbf{C}(x)$ of x by $\mathbf{C}(x) := \{c(v)v : c(v) < \infty, v \in I_x\}$, the *cut locus* $C(x)$ of x by $C(x) = \exp_x \mathbf{C}(x)$, and the *injectivity radius* i_x at x by $i_x = \inf\{c(v) : v \in I_x\}$, respectively. It is known that $C(x)$ has zero Hausdorff measure in M. Also, we set $\mathbf{D}_x = \{tv : 0 \leqslant t < c(v), v \in I_x\}$ and $D_x = \exp_x \mathbf{D}_x$. It is known that $\mathbf{D}_x$ is the largest domain, which is starlike with respect to the origin of T_xM for which $\exp_x$ restricted to that domain is a diffeomorphism, and $D_x = M \backslash C(x)$.

In the following we consider the polar coordinates on $D(x)$. For any $q \in D(x)$, the polar coordinates of q are defined by $(r, \theta) = (r(q), \theta^1(q), \dots, \theta^{n-1}(q))$, where $r(q) = F(v), \theta^\alpha(q) = \theta^\alpha(u)$, here $v = \exp_x^{-1}(q)$ and $u = v/F(v)$. It is clear that r is just the distance function with respect to x. Then by the Gauss lemma (see [1], page 140), the unit radial coordinate vector $T = d(\exp_x)\left(\frac{\partial}{\partial r}\right)$ is $\mathbf{g}_T$-orthogonal to coordinate vectors ∂_α which is defined by

$$\begin{aligned}\partial_\alpha|_{\exp_x(ru)} &= d(\exp_x)\left(\frac{\partial}{\partial \theta^\alpha}\right)\Big|_{\exp_x(ru)} \\ &= d(\exp_x)_{ru}\left(r\frac{\partial}{\partial \theta^\alpha}\right) = rd(\exp_x)_{ru}\left(\frac{\partial}{\partial \theta^\alpha}\right)\end{aligned}$$

for $\alpha = 1, \dots, n-1$, and consequently, $T = \nabla r$. Consider the singular Riemannian metric $\widetilde{g} = \mathbf{g}_{\nabla r}$ on $D(x)$, then it is clear that

$$\widetilde{g} = dr^2 + \widetilde{g}_{\alpha\beta} d\theta^\alpha d\theta^\beta, \quad \widetilde{g}_{\alpha\beta} = \mathbf{g}_{\nabla r}(\partial_\alpha, \partial_\beta).$$

3. Hessian comparison theorems

Notations as above. Notice that $T = \nabla r$ is a geodesic field, by (2.5) it follows that $H(r)(\nabla r, \cdot) = 0$, thus we need only to consider $H(r)$ on the normal space with respect to the radial geodesic field ∇r. Let $E_1, \dots, E_{n-1}, E_n = \nabla r$ be the local $\mathbf{g}_{\nabla r}$-orthonormal frame along geodesic rays, by (2.1)-(2.5) we get

$$\begin{aligned}&\frac{\partial}{\partial r}\left(H(r)(E_i, E_j)\right) \\ &= \nabla r \cdot \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_i} \nabla r, E_j\right) \\ &= \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{\nabla r}\nabla^{\nabla r}_{E_i} \nabla r, E_j\right) + \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_i}\nabla r, \nabla^{\nabla r}_{\nabla r} E_j\right) \\ &= \mathbf{g}_{\nabla r}\left(\mathbf{R}^{\nabla r}(\nabla r, E_i)\nabla r, E_j\right) + \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{[\nabla r, E_i]}\nabla r, E_j\right) \\ &\quad + \sum_k \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_i}\nabla r, E_k\right)\mathbf{g}_{\nabla r}\left(E_k, \nabla^{\nabla r}_{\nabla r} E_j\right) \\ &= -\mathbf{g}_{\nabla r}\left(\mathbf{R}^{\nabla r}(E_i, \nabla r)\nabla r, E_j\right) + \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{\nabla^{\nabla r}_{\nabla r} E_i}\nabla r, E_j\right) \\ &\quad - \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{\nabla^{\nabla r}_{E_i}\nabla r}\nabla r, E_j\right) + \sum_k \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_i}\nabla r, E_k\right)\mathbf{g}_{\nabla r}\left(E_k, \nabla^{\nabla r}_{\nabla r} E_j\right) \\ &= -\mathbf{g}_{\nabla r}\left(\mathbf{R}^{\nabla r}(E_i, \nabla r)\nabla r, E_j\right) + \sum_k \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{\nabla r} E_i, E_k\right)\mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_k}\nabla r, E_j\right) \\ &\quad - \sum_k \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_i}\nabla r, E_k\right)\mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_k}\nabla r, E_j\right)\end{aligned}$$

$$+\sum_k \mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{E_i}\nabla r, E_k\right)\mathbf{g}_{\nabla r}\left(E_k, \nabla^{\nabla r}_{\nabla r}E_j\right),$$

and consequently,

$$\begin{aligned}\frac{\partial}{\partial r}(H(r)(E_i,E_j)) = & -\mathbf{g}_{\nabla r}\left(\mathbf{R}^{\nabla r}(E_i,\nabla r)\nabla r, E_j\right)\\ & -\sum_k H(r)(E_i,E_k)\cdot H(r)(E_k,E_j)\\ & +\sum_k H(r)(E_k,E_j)\mathbf{g}_{\nabla r}\left(\nabla^{\nabla r}_{\nabla r}E_i, E_k\right)\\ & +\sum_k H(r)(E_i,E_k)\mathbf{g}_{\nabla r}\left(E_k, \nabla^{\nabla r}_{\nabla r}E_j\right).\end{aligned} \tag{3.1}$$

We note here that the similar formula was obtained when $E_1,\dots,E_{n-1}$ are parallel along geodesic rays, here we need this general formula for later use. The volume form of $\widetilde{g}$ is $dV_{\widetilde{g}} = \widetilde{\sigma}(r,\theta)dr\wedge d\theta^1\wedge\cdots\wedge\theta^{n-1} := \widetilde{\sigma}(r,\theta)dr\wedge d\theta$, here $\widetilde{\sigma}(r,\theta)=\sqrt{\det(\widetilde{g}_{\alpha\beta})}$. Let $h(r)=\mathrm{trace}_{\mathbf{g}_{\nabla r}}H(r)$, by (2.1)-(2.5) and (3.1) we have (see also [11, 12])

$$\frac{\partial}{\partial r}(\log\widetilde{\sigma}) = h,\quad \frac{\partial h}{\partial r}+\frac{h^2}{n-1}\leqslant -\mathbf{Ric}(\nabla r). \tag{3.2}$$

Let

$$\sigma_c(r)=\mathfrak{s}_c(r)^{n-1},\quad h_c(r)=(n-1)\mathfrak{ct}_c(r),$$

where

$$\mathfrak{s}_c(r)=\begin{cases}\frac{\sin(\sqrt{c}r)}{\sqrt{c}}, & c>0,\\ r, & c=0,\\ \frac{\sinh(\sqrt{-c}r)}{\sqrt{-c}}, & c<0,\end{cases}\qquad \mathfrak{ct}_c(r)=\begin{cases}\sqrt{c}\cot(\sqrt{c}r), & c>0,\\ \frac{1}{r}, & c=0,\\ \sqrt{-c}\coth(\sqrt{-c}r), & c<0.\end{cases}$$

Then

$$(\log\sigma_c)'=h_c,\quad h_c'+\frac{h_c^2}{n-1}=-(n-1)c. \tag{3.3}$$

Let us first consider line integrate Ricci curvature bounds. In polar coordinates we write $\rho_c=\rho_c(r,\theta)=\max\{(n-1)c-\mathbf{Ric}(\nabla r),0\}$, and define $\psi_c=\psi_c(r,\theta)=\max\{0,h(r,\theta)-h_c(r)\}$. It is clear that ψ_c is defined on $D_x\backslash\{x\}$ (when $c>0$, we require that $r<\pi/\sqrt{c}$). For sufficiently small ϵ let a and b be the lower and upper bounds of flag curvature on $B_x(\epsilon)$, then by Hessian comparison theorem [12] it follows that

$$h_b(r)\leqslant h(r,\theta)\leqslant h_a(r),\quad \forall r<\epsilon,$$

which together with the fact that

$$\lim_{r\to 0}(h_a(r)-h_b(r))=0,\quad \forall a\neq b,$$

we have

$$\lim_{r\to 0+}\psi_c(r,\theta)=0. \tag{3.4}$$

On the other hand, by (3.2) and (3.3) we have that ψ_c is absolutely continuous on $D_x\backslash\{x\}$ and satisfies

$$\frac{\partial \psi_c}{\partial r} + \frac{\psi_c^2}{n-1} + 2\frac{\psi_c \cdot h_c}{n-1} \leqslant \rho_c. \tag{3.5}$$

For any $p \geqslant 1$, by multiplying (3.5) by $(2p-1)\psi_c^{2p-2}$ and simplifying we get

$$\frac{\partial}{\partial r}\psi_c^{2p-1} + \frac{2p-1}{n-1}\psi_c^{2p} + \frac{4p-2}{n-1}\psi_c^{2p-1}h_c \leqslant (2p-1)\rho_c\psi_c^{2p-2}. \tag{3.6}$$

We assume that $r \leqslant \frac{\pi}{2\sqrt{c}}$ when $c > 0$, then $h_c \geqslant 0$. Integrating (3.6) from 0 to r and using (3.4) and the Hölder's inequality we have

$$\begin{aligned}
&\psi_c^{2p-1}(r,\theta) + \frac{2p-1}{n-1}\int_0^r \psi_c^{2p}(t,\theta)dt \\
&\leqslant (2p-1)\int_0^r \rho_c(t,\theta)\psi_c^{2p-2}(t,\theta)dt \\
&\leqslant (2p-1)\left(\int_0^r \rho_c^p(t,\theta)dt\right)^{\frac{1}{p}}\left(\int_0^r \psi_c^{2p}(t,\theta)dt\right)^{1-\frac{1}{p}}.
\end{aligned} \tag{3.7}$$

By (3.7) we easily obtain

$$\int_0^r \psi_c^{2p}(t,\theta)dt \leqslant (n-1)^p \int_0^r \rho_c^p(t,\theta)dt,$$

which together with (3.7) yields

$$\psi_c^{2p-1}(r,\theta) \leqslant (2p-1)(n-1)^{p-1}\int_0^r \rho_c^p(t,\theta)dt. \tag{3.8}$$

By the definition of ψ_c it is clear that $h \leqslant h_c + \psi_c$, thus we have the following:

Theorem 3.1. *Let (M,F) be a forward complete Finsler n-manifold. Suppose that $r = d(x,\cdot)$ is smooth at $y \in M$, and γ be the unique minimal normal geodesic from x to y. For $c \in \mathbb{R}$, $p \geqslant 1$ (we require that $r(y) \leqslant \frac{\pi}{2\sqrt{c}}$ when $c > 0$), we have*

$$\begin{aligned}
&\operatorname{trace}_{\mathbf{g}_{\nabla r}} H(r)(y) \\
&\leqslant (n-1)\mathfrak{ct}_c(r(y)) \\
&\quad + \left[(2p-1)(n-1)^{p-1}\int_\gamma \left(\max\{(n-1)c - \mathbf{Ric}(\gamma'(t)), 0\}\right)^p dt\right]^{\frac{1}{2p-1}}.
\end{aligned}$$

In the following we consider the case when $c > 0$. In this situation $h_c(r) = \sqrt{c}\cot\sqrt{c}r$, and (3.6) can be written as

$$\frac{\partial}{\partial r}\psi_c^{2p-1} + \frac{4p-2}{n-1}\psi_c^{2p-1}\cdot\sqrt{c}\cot(\sqrt{c}r) + \frac{2p-1}{n-1}\psi_c^{2p} \leqslant (2p-1)\rho_c\psi_c^{2p-2}.$$

The integrating factor of the first terms is $\sin^{\frac{4p-2}{n-1}}(\sqrt{c}r)$. Multiplying by the integrating factor and integrating from 0 to r we get

(3.9)
$$\begin{aligned}
&\sin^{\frac{4p-2}{n-1}}(\sqrt{c}r)\psi_c^{2p-1}(r,\theta)+\frac{2p-1}{n-1}\int_0^r\sin^{\frac{4p-2}{n-1}}(\sqrt{c}t)\psi_c^{2p}(t,\theta)dt\\
&\leqslant(2p-1)\int_0^r\sin^{\frac{4p-2}{n-1}}(\sqrt{c}t)\rho_c(t,\theta)\psi_c^{2p-2}(t,\theta)dt\\
&\leqslant(2p-1)\left(\int_0^r\sin^{\frac{4p-2}{n-1}}(\sqrt{c}t)\rho_c^p(t,\theta)dt\right)^{\frac{1}{p}}\left(\int_0^r\sin^{\frac{4p-2}{n-1}}(\sqrt{c}t)\psi_c^{2p}(t,\theta)dt\right)^{1-\frac{1}{p}}.
\end{aligned}$$

By (3.9) we easily obtain

$$\sin^{\frac{4p-2}{n-1}}(\sqrt{c}r)\psi_c^{2p-1}(r,\theta)\leqslant(2p-1)(n-1)^{p-1}\int_0^r\rho_c^p(t,\theta)dt. \tag{3.10}$$

Thus we have:

Theorem 3.2. *Let (M,F) be a forward complete Finsler n-manifold. Suppose that $r=d(x,\cdot)$ is smooth at $y\in M$, γ is the unique minimal normal geodesic from x to y, and $r(y)<\frac{\pi}{\sqrt{c}}$ for some $c>0$. Then for any $p\geqslant 1$ we have*

$$\begin{aligned}
&\mathrm{trace}_{\mathbf{g}_{\nabla r}}H(r)(y)\\
&\leqslant(n-1)\sqrt{c}\cot(\sqrt{c}r(y))\\
&\quad+\left[\frac{(2p-1)(n-1)^{p-1}}{\sin^{\frac{4p-2}{n-1}}(\sqrt{c}r(y))}\int_\gamma(\max\{(n-1)c-\mathbf{Ric}(\gamma'(t)),0\})^p\,dt\right]^{\frac{1}{2p-1}}.
\end{aligned}$$

Now let $c<0$, we define $\varrho_c=\varrho_c(r,\theta)=\max\{\mathbf{Ric}(\nabla r)-c,0\}$ and $\varphi_c=\varphi_c(r,\theta)=\max\{0,\lambda_c(r)-h(r,\theta)\}$ with $\lambda_c(r)=\frac{h_c(r)}{n-1}=\sqrt{-c}\coth(\sqrt{-c}r)$. We assume that M has nonpositive flag curvature, then by Hessian comparison theorem [12] it follows that the eigenvalues of $H(r)$ are all nonnegative. Thus we have

$$\sum_{i,j}(H(r)(E_i,E_j))^2\leqslant h(r)^2,$$

which together with (3.1) yields

$$\frac{\partial h}{\partial r}+h^2\geqslant-\mathbf{Ric}(\nabla r). \tag{3.11}$$

Since $\lambda_c(r)=\sqrt{-c}\coth(\sqrt{-c}r)$, it satisfies

$$\lambda_c'+\lambda_c^2=-c. \tag{3.12}$$

(3.11) and (3.12) implies that

$$\frac{\partial}{\partial r}\varphi_c+\varphi_c^2+2\varphi_c h\leqslant\varrho_c. \tag{3.13}$$

With (3.13) at hand, we can prove the following result with the same method as in Theorem 3.1.

Theorem 3.3. *Let (M, F) be a forward complete Finsler n-manifold with nonpositive flag curvature. Suppose that $r = d(x, \cdot)$ is smooth at $y \in M$, and γ be the unique minimal normal geodesic from x to y. Then for any $c < 0$, $p \geqslant 1$ we have*

$$\begin{aligned}\mathrm{trace}_{\mathbf{g}_{\nabla r}} H(r)(y) \geqslant{}& \sqrt{-c}\coth(\sqrt{-c}r(y)) \\ &- \left[(2p-1)\int_\gamma (\max\{\mathbf{Ric}(\gamma'(t)) - c, 0\})^p \, dt\right]^{\frac{1}{2p-1}}.\end{aligned}$$

In the following we consider the line integrate flag curvature bounds. Notice that $H(r)$ is a symmetric bilinear form, we may choose the local frame $E_1, \ldots, E_{n-1}, E_n = \nabla r$ such that $E_i, 1 \leqslant i \leqslant n-1$ are eigenvectors of $H(r)$ with eigenvalues λ_i, and by (2.2), (2.3) and (3.1) it follows that

$$\frac{\partial}{\partial r}\lambda_i + \lambda_i^2 = \mathbf{K}(\nabla r; E_i). \tag{3.14}$$

Let

$$\overline{\mathbf{K}}(\nabla r) = \max_{\mathbf{g}_{\nabla r}(\nabla r, E)=0} \mathbf{K}(\nabla r; E),$$

and $\phi_i = \phi_i(r, \theta) = \max\{0, \lambda_c(r) - \lambda_i(r, \theta)\}$ for $1 \leqslant i \leqslant n-1$. Then by (3.12) and (3.14) we get

$$\frac{\partial}{\partial r}\phi_i + \phi_i^2 + 2\phi_i\lambda_i \leqslant \max\{\overline{\mathbf{K}}(\nabla r) - c, 0\}, \quad 1 \leqslant i \leqslant n-1. \tag{3.15}$$

Now we assume that $c < 0$, and M has nonpositive flag curvature, then by (3.15) and the similar argument as in Theorem 3.1 we obtain:

Theorem 3.4. *Let (M, F) be a forward complete Finsler n-manifold with nonpositive flag curvature. Suppose that $r = d(x, \cdot)$ is smooth at $y \in M$, and γ be the unique minimal normal geodesic from x to y. Then for any $c < 0$, $p \geqslant 1$, and $X \in T_yM$ with $\mathbf{g}_{\nabla r}(\nabla r, X) = 0$ and $\mathbf{g}_{\nabla r}(X, X) = 1$ we have*

$$\begin{aligned}H(r)(X, X) \geqslant{}& \sqrt{-c}\coth(\sqrt{-c}r(y)) \\ &- \left[(2p-1)\int_\gamma \left(\max\{\overline{\mathbf{K}}(\nabla r) - c, 0\}\right)^p dt\right]^{\frac{1}{2p-1}}.\end{aligned}$$

4. Volume comparison theorems

To study the volume comparison theorem, we shall use polar coordinates described in §2. Fix $x \in M$. For $r > 0$, let $\mathbf{D}_x(r) \subset I_x$ be defined by $\mathbf{D}_x(r) = \{v \in I_x : rv \in \mathbf{D}_x\}$. It is easy to see that $\mathbf{D}_x(r_1) \subset \mathbf{D}_x(r_2)$ for $r_1 > r_2$ and

$\mathbf{D}_x(r) = I_x$ for $r < i_x$. Since $C(x)$ has zero Hausdorff measure in M, we have

$$\begin{aligned} \operatorname{vol}_{\widetilde{g}}(B_x(R)) &= \int_{B_x(R)} dV_{\widetilde{g}} = \int_{B_x(R)\cap D_x} dV_{\widetilde{g}} \\ &= \int_{\exp_x^{-1}(B_x(R))\cap\mathbf{D}_x} \exp_x^*(dV_{\widetilde{g}}) = \int_0^R dr \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(r,\theta)d\theta. \end{aligned} \tag{4.1}$$

Let

$$V_{c,\Lambda,n}(R) = \operatorname{vol}(\mathbb{S}^{n-1}(1)) \int_0^R e^{\Lambda t}\mathfrak{s}_c(t)^{n-1}dt.$$

When $\Lambda = 0, V_{c,0,n}(R)$ is equal to $\operatorname{vol}(\mathbb{B}_c^n(R))$ when $R \leqslant i_c$, here $\mathbb{B}_c^n(R)$ denotes the geodesic ball of radius R in space form of constant c, and i_c the corresponding injectivity radius. The following lemma is crucial to prove volume comparison theorem.

Lemma 4.1. *Suppose that f, g are two positive integrable functions of t, and $\frac{f}{g}$ is monotone increasing (resp. decreasing). Then the function*

$$\frac{\displaystyle\int_0^r f(t)dt}{\displaystyle\int_0^r g(t)dt}$$

is also monotone increasing (resp. decreasing).

Now we are ready to prove the following relative volume comparison theorem with upper line integrate radial curvature bounds.

Theorem 4.2. *Let (M,F) be a forward complete Finsler n-manifold with non-positive flag curvature, and $c < 0$, $p \geqslant 1$.*

(1) *Suppose that there is $C > 0$ such that the radial flag curvature at $x \in M$ satisfies*

$$\int_\gamma \left(\max\{\overline{\mathbf{K}}(\nabla r) - c, 0\}\right)^p dt \leqslant C$$

for any minimal normal geodesic γ issuing from x, then

$$\frac{\operatorname{vol}_{\mathrm{ext}}(B_x(r))}{V_{c,\Lambda,n}(r)} \leqslant \max_{x\in B_x(R)} \mu(x)^{\frac{n}{2}} \cdot \frac{\operatorname{vol}_{\mathrm{ext}}(B_x(R))}{V_{c,\Lambda,n}(R)}$$

holds for any $r < R \leqslant i_x$, here $\operatorname{vol}_{\mathrm{ext}}$ denotes the extreme volume (i.e., the maximal volume $\operatorname{vol}_{\max}$ or minimal volume $\operatorname{vol}_{\min}$), $\Lambda = -(n-1)\left[(2p-1)C\right]^{\frac{1}{2p-1}}$, and i_x the injectivity radius of x.

(2) *Suppose that there is $C > 0$ such that the radial Ricci curvature at $x \in M$ satisfies*

$$\int_\gamma (\max\{\mathbf{Ric}(\nabla r) - c, 0\})^p\, dt \leqslant C$$

for any minimal normal geodesic γ issuing from x, then

$$\frac{\mathrm{vol}_{\mathrm{ext}}(B_x(r))}{V_{c,\Lambda,2}(r)} \leqslant \max_{x\in B_x(R)} \mu(x)^{\frac{n}{2}} \cdot \frac{\mathrm{vol}_{\mathrm{ext}}(B_x(R))}{V_{c,\Lambda,2}(R)}$$

holds for any $r < R \leqslant i_x$, here $\Lambda = -\left[(2p-1)C\right]^{\frac{1}{2p-1}}$.

Proof. Here we only prove (1), (2) may be verified similarly. By (3.2) and Theorem 3.4 we have

$$\begin{aligned} \frac{\partial}{\partial r}\log\widetilde{\sigma} = h &\geqslant (n-1)\left[\sqrt{-c}\coth(\sqrt{-c}r) - ((2p-1)C)^{\frac{1}{2p-1}}\right] \\ &= \frac{d}{dr}\log\left(e^{\Lambda r}\sinh^{n-1}(\sqrt{-c}r)\right), \quad \Lambda = -(n-1)\left[(2p-1)C\right]^{\frac{1}{2p-1}}. \end{aligned} \tag{4.2}$$

From (4.2) we see that the function

$$\frac{\displaystyle\int_{I_x}\widetilde{\sigma}(r,\theta)d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})e^{\Lambda r}\sinh^{n-1}(\sqrt{-c}r)}$$

is monotone increasing about $r(\leqslant i_x)$, and thus by Lemma 4.1 and (4.1) the function

$$\frac{\displaystyle\int_0^R\int_{I_x}\widetilde{\sigma}(r,\theta)drd\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\displaystyle\int_0^R e^{\Lambda r}\sinh^{n-1}(\sqrt{-c}r)dr} = \frac{\mathrm{vol}_{\widetilde{g}}(B_x(R))}{V_{c,\Lambda,n}(R)}$$

is also monotone increasing for $R \leqslant i_x$. Notice that $dV_{\min} \leqslant dV_{\widetilde{g}} \leqslant dV_{\max} \leqslant \mu(x)^{\frac{n}{2}} \cdot dV_{\min}$ (see e.g., [7,11]), it follows that

$$\frac{\mathrm{vol}_{\min}(B_x(r))}{V_{c,\Lambda,n}(r)} \leqslant \frac{\mathrm{vol}_{\widetilde{g}}(B_x(r))}{V_{c,\Lambda,n}(r)} \leqslant \frac{\mathrm{vol}_{\widetilde{g}}(B_x(R))}{V_{c,\Lambda,n}(R)} \leqslant \max_{x\in B_x(R)}\mu(x)^{\frac{n}{2}} \cdot \frac{\mathrm{vol}_{\min}(B_x(R))}{V_{c,\Lambda,n}(R)}$$

holds for any $r < R \leqslant i_x$. Similarly,

$$\frac{\mathrm{vol}_{\max}(B_x(r))}{V_{c,\Lambda,n}(r)} \leqslant \max_{x\in B_x(R)}\mu(x)^{\frac{n}{2}} \cdot \frac{\mathrm{vol}_{\max}(B_x(R))}{V_{c,\Lambda,n}(R)}$$

for any $r < R \leqslant i_x$, and (1) is proved. ☐

We also have the following relative volume comparison theorem with line integrate radial Ricci curvature bound.

Theorem 4.3. *Let (M,F) be a forward complete Finsler n-manifold. For $c \in \mathbb{R}$, $p \geqslant 1$, if there exists $C > 0$ such that the radial Ricci curvature satisfies*

$$\int_\gamma \left(\max\{(n-1)c - \mathbf{Ric}(\nabla r), 0\}\right)^p dt \leqslant C$$

for any minimal normal geodesic γ issuing from x, then for any $0 < r < R$ (we require $R < \frac{\pi}{\sqrt{c}}$ when $c > 0$),

$$\frac{\mathrm{vol}_{\mathrm{ext}}(B_x(r))}{V_{c,\Lambda,n}(r)} \geqslant \max_{x\in B_x(R)} \mu(x)^{-\frac{n}{2}} \cdot \frac{\mathrm{vol}_{\mathrm{ext}}(B_x(R))}{V_{c,\Lambda,n}(R)},$$

here

$$\Lambda = \Lambda(n,p,C,c,R) = \begin{cases} \left[(2p-1)(n-1)^{p-1}C\right]^{\frac{1}{2p-1}}, & c \leqslant 0, \\ \left[\dfrac{(2p-1)(n-1)^{p-1}C}{\sin^{\frac{4p-2}{n-1}}(\sqrt{c}R)}\right]^{\frac{1}{2p-1}}, & c > 0. \end{cases} \tag{4.3}$$

Proof. From (3.2), Theorems 3.1 and 3.2 it is clear that

$$\frac{\partial}{\partial r}\log\widetilde{\sigma} = \mathrm{trace}_{\mathbf{g}_{\nabla r}} H(r) \leqslant (n-1)\mathfrak{ct}_c(r) + \Lambda = \frac{d}{dr}\log\left(e^{\Lambda r}\mathfrak{s}_c(r)^{n-1}\right), \tag{4.4}$$

here Λ is given by (4.3). (4.4) means that the function

$$\frac{\widetilde{\sigma}(r,\theta)}{e^{\Lambda r}\mathfrak{s}_c(r)^{n-1}}$$

is monotone decreasing for r where it is smooth. Noting that $\mathbf{D}_x(R) \subset \mathbf{D}_x(r)$ for $R > r > 0$, we have for $R > r > 0$,

$$\begin{aligned} \frac{\int_{\mathbf{D}_x(r)} \widetilde{\sigma}(r,\theta)d\theta}{e^{\Lambda r}\mathfrak{s}_c(r)^{n-1}} &= \int_{\mathbf{D}_x(r)} \frac{\widetilde{\sigma}(r,\theta)}{e^{\Lambda r}\mathfrak{s}_c(r)^{n-1}} d\theta \geqslant \int_{\mathbf{D}_x(R)} \frac{\widetilde{\sigma}(r,\theta)}{e^{\Lambda r}\mathfrak{s}_c(r)^{n-1}} d\theta \\ &\geqslant \int_{\mathbf{D}_x(R)} \frac{\widetilde{\sigma}(R,\theta)}{e^{\Lambda R}\mathfrak{s}_c(R)^{n-1}} d\theta = \frac{\int_{\mathbf{D}_x(R)} \widetilde{\sigma}(R,\theta)d\theta}{e^{\Lambda R}\mathfrak{s}_c(R)^{n-1}}, \end{aligned}$$

which together with (4.1) and Lemma 4.1 implies that

$$\frac{\mathrm{vol}_{\widetilde{g}}(B_x(R))}{V_{c,\Lambda,n}(R)} = \frac{\int_0^R dr \int_{\mathbf{D}_x(r)} \widetilde{\sigma}(r,\theta)d\theta}{\mathrm{vol}(\mathbb{S}^{n-1})\int_0^R e^{\Lambda r}\mathfrak{s}_c(r)^{n-1}dr}$$

is monotone decreasing for any $R > 0$ (we require $R < \frac{\pi}{\sqrt{c}}$ when $c > 0$). Now the theorem follows similarly as Theorem 4.2. □

5. Gromov pre-compactness theorem

The notion of Hausdorff distance between metrics spaces was generalized by M. Gromov, and the corresponding pre-compactness theorem for Riemannian manifolds was proved in [2]. Gromov pre-compactness property has been generalized to Finsler manifolds by Shen [3] in reversible case and by Shen and Zhao [5] in non-reversible case. To state our result let us first recall some notations related to Gromov pre-compactness, for details one is referred to see

[5]. As we have seen before, any Finsler manifold (M,F) induces a general metric space (M,d). Let $(\mathcal{M}^\delta, d^\delta_{GH})$ denote the collection of compact general metric space with δ-Gromov-Hausdorff distance d^δ_{GH} whose reversibilities are not large than $\delta<\infty$, and $\mathrm{Cap}_M(\epsilon)$ be the maximal number of disjoint forward geodesic ball of radius ϵ in M. Also, let $(\mathcal{M}^\delta_*, d^\delta_{GH})$ be the collection of proper pointed general metric space whose reversibilities are not large than $\delta<\infty$.

Lemma 5.1 ([5])**.** (1) *Let $\mathcal{C}\subset(\mathcal{M}^\delta, d^\delta_{GH})$ be a class satisfying the following conditions*:

(a) *There is a constant D such that* $\mathrm{diam}M\leqslant D$ *for all $M\in\mathcal{C}$.*

(b) *For each $\varepsilon>0$ there exists $N=N(\varepsilon)<\infty$ such that* $\mathrm{Cap}_M(\varepsilon)\leqslant N(\varepsilon)$ *for all $M\in\mathcal{C}$.*

Then $\mathcal{C}$ is pre-compact in the δ-Gromov-Hausdorff topology.

(2) $(\mathcal{M}^\delta_*, d^\delta_{GH})$ *is pre-compact if for each $r>0$ and $\varepsilon>0$, there exists a number $N=N(r,\varepsilon)<\infty$ such that for every $\overline{\mathcal{B}_x(r)}\subset(M,x)\in\mathcal{C}$, one has* $\mathrm{Cap}_{\overline{\mathcal{B}_x(r)}}(\varepsilon)\leqslant N(r,\varepsilon)$.

By Theorem 4.3 and Lemma 5.1 we can prove:

Theorem 5.2. *For any integer $n\geqslant 2$, $c\in\mathbb{R}$, $p\geqslant 1$ and $C,D>0(D<\pi/\sqrt{c}$ when $c>0)$, the following classes are pre-compact in the* (*pointed*) *δ-Gromov-Hausdorff topology*:

(1) *The collection $\{(M_i,F_i)\}$ of compact Finsler n-manifolds satisfying conditions*

$$\mathrm{diam}(M_i)\leqslant D,$$
$$\int_\gamma(\max\{(n-1)c-\mathbf{Ric}(\nabla r),0\})^p\,dt\leqslant C$$

for any minimal normal geodesic γ in M, and uniformity constant $\mu_{F_i}\leqslant\delta^2<\infty$ for all i.

(2) *The collection $\{(M_i,x_i,F_i)\}$ of compact Finsler n-manifolds* ($\mathrm{diam}(M_i)$ $<\pi/\sqrt{c}$ *when* $c>0$) *satisfying conditions*

$$\int_{\gamma_i}(\max\{(n-1)c-\mathbf{Ric}(\nabla r),0\})^p\,dt\leqslant C$$

for any minimal normal geodesic γ_i in M_i, and uniformity constant $\mu_{F_i}\leqslant\delta^2<\infty$ for all i.

Proof. We only prove (1), (2) may be verified by the same way. Note that $\lambda^2_{F_i}\leqslant\mu_{F_i}$, one has $\{(M_i,F_i)\}\subset(\mathcal{M}^\delta, d^\delta_{GH})$. For each (M_i,F_i), note that $\mathrm{diam}(M_i)\leqslant D$, one has $M_i=\overline{B_{x_i}(D)}$ for any $x_i\in M_i$. Since M_i is compact, there are finite disjoint forward geodesic balls $B_{x_1}(\varepsilon),\dots,B_{x_l}(\varepsilon)$ of radius ε in M_i. Let $B_{x_{l_0}}(\varepsilon)$ be the forward geodesic ball with the smallest minimal volume. Then by Theorem 4.3 we have

$$l\leqslant\frac{\mathrm{vol}_{\min}(M_i)}{\mathrm{vol}_{\min}(B_{x_{l_0}}(\varepsilon))}=\frac{\mathrm{vol}_{\min}(B_{x_{l_0}}(D))}{\mathrm{vol}_{\min}(B_{x_{l_0}}(\varepsilon))}\leqslant\frac{V_{c,\Lambda,n}(D)}{V_{c,\Lambda,n}(\varepsilon)}\cdot\delta^n,$$

here Λ is given by (4.3). Now (1) is easily followed by (1) of Lemma 5.1. $\square$

6. The Mckean type inequalities

In this section we shall study the first eigenvalue on Finsler manifolds and prove some McKean type theorems under the line integrate curvature bounds. Let us first recall the definition of the first eigenvalue for non-compact Finsler manifolds. Let $(M, F, d\mu)$ be a Finsler n-manifold with volume form $d\mu$, $\Omega \subset M$ a domain with compact closure and nonempty boundary $\partial\Omega$. The first eigenvalue $\lambda_1(\Omega)$ of Ω with respect to $d\mu$ is defined by (see [4], page 176)

$$\lambda_1(\Omega) = \inf_{f \in L^2_{1,0}(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega (F^*(df))^2 \, d\mu}{\int_\Omega f^2 d\mu} \right\},$$

where $L^2_{1,0}(\Omega)$ is the completion of C_0^∞ with respect to the norm

$$\| \varphi \|^2_\Omega = \int_\Omega \varphi^2 d\mu + \int_\Omega (F^*(d\varphi))^2 \, d\mu.$$

If $\Omega_1 \subset \Omega_2$ are bounded domains, then $\lambda_1(\Omega_1) \geqslant \lambda_1(\Omega_2) \geqslant 0$. Thus, if $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$ be bounded domains so that $\bigcup \Omega_i = M$, then the following limit

$$\lambda_1(M) = \lim_{i\to\infty} \lambda_1(\Omega_i) \geqslant 0$$

exists, and it is independent of the choice of $\{\Omega_i\}$.

Now let $B_p(R)$ be the forward geodesic ball of M with radius R centered at p, and $R < i_p$, where i_p denotes the injectivity radius about p. For $R > \varepsilon > 0$, let $\Omega_\varepsilon(R) = B_p(R)\setminus\overline{B_p(\varepsilon)}$. Then $r = d_F(p, \cdot)$ is smooth on $\Omega_\varepsilon(R)$, and thus $V = \nabla r$ is a unit geodesic vector field on $\Omega_\varepsilon(R)$, and we can consider the Riemannian metric $\widetilde{g} = \mathbf{g}_V$ on $\Omega_\varepsilon(R)$. Since the Legendre transformation $l : TM \to T^*M$ is norm-preserving, and thus it also preserves the uniformity constant. Hence, for any $f \in C_0^\infty(\Omega_\varepsilon(R))$,

$$\begin{aligned} (F^*(df))^2(x) &= g^{*ij}(x, df)\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \geqslant \frac{1}{\mu^*(x)} g^{*ij}(x, l(V(x)))\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} \\ &= \frac{1}{\mu(x)} g^{ij}(x, V(x))\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} = \frac{1}{\mu(x)}\|df\|^2_{\widetilde{g}}(x). \end{aligned} \tag{6.1}$$

Using (6.1), we get, for $d\mu = dV_{\min}$,

$$\frac{\int_{\Omega_\varepsilon(R)} (F^*(df))^2 \, dV_{\min}}{\int_{\Omega_\varepsilon(R)} f^2 dV_{\min}} \geqslant \frac{\int_{\Omega_\varepsilon(R)} (F^*(df))^2 dV_{\widetilde{g}}}{\Theta^{\frac{n}{2}} \int_{\Omega_\varepsilon(R)} f^2 dV_{\widetilde{g}}} \geqslant \frac{1}{\Theta^{1+\frac{n}{2}}} \frac{\int_{\Omega_\varepsilon(R)} \|df\|^2_{\widetilde{g}} dV_{\widetilde{g}}}{\int_{\Omega_\varepsilon(R)} f^2 dV_{\widetilde{g}}},$$

here

$$\Theta = \max_{x \in B_p(R)} \mu(x).$$

As the result, we have for $d\mu = dV_{\min}$,

$$\lambda_1(\Omega_\varepsilon(R)) \geqslant \frac{1}{\Theta^{1+\frac{n}{2}}}\widetilde{\lambda}_1(\Omega_\varepsilon(R)), \tag{6.2}$$

where $\widetilde{\lambda}_1(\Omega_\varepsilon(R))$ is the first eigenvalue of $\Omega_\varepsilon(R)$ with respect to $\widetilde{g}$. It is not difficult to see that (6.2) still holds for $d\mu = dV_{\max}$. In other words, (6.2) holds for $d\mu = dV_{\text{ext}}$. Now we are able to prove:

Theorem 6.1. *Let (M, F) be a forward complete noncompact and simply connected Finsler n-manifold with nonpositive flag curvature and finite uniformity constant $\mu_F, x \in M$, and $c < 0$, $p \geqslant 1$.*

(1) *Suppose that there is $C > 0$ with $[(2p-1)C]^{\frac{1}{2p-1}} < \sqrt{-c}$ such that the radial flag curvature at $x \in M$ satisfies*

$$\int_\gamma \left(\max\{\overline{\mathbf{K}}(\nabla r) - c, 0\}\right)^p dt \leqslant C$$

for any minimal normal geodesic γ issuing from x, then we have the following estimation for $\lambda_1(M)$ with respect to $d\mu = dV_{\text{ext}}$:

$$\lambda_1(M) \geqslant \frac{(n-1)^2\left[\sqrt{-c} - [(2p-1)C]^{\frac{1}{2p-1}}\right]^2}{4\mu_F^{1+\frac{n}{2}}}.$$

(2) *Suppose that there is $C > 0$ with $[(2p-1)C]^{\frac{1}{2p-1}} < \sqrt{-c}$ such that the radial Ricci curvature at $x \in M$ satisfies*

$$\int_\gamma (\max\{\mathbf{Ric}(\nabla r) - c, 0\})^p \, dt \leqslant C$$

for any minimal normal geodesic γ issuing from x, then we have the following estimation for $\lambda_1(M)$ with respect to $d\mu = dV_{\text{ext}}$:

$$\lambda_1(M) \geqslant \frac{\left[\sqrt{-c} - [(2p-1)C]^{\frac{1}{2p-1}}\right]^2}{4\mu_F^{1+\frac{n}{2}}}.$$

Proof. We only prove (1), (2) may be verified by the same way. Since (M, F) is a forward complete noncompact and simply connected Finsler manifold with nonpositive flag curvature, by Cartan-Hadamard theorem $r = d_F(x, \cdot)$ is smooth on $M\backslash\{x\}$. First we recall that $V = \nabla r$ is also a unit geodesic vector field on M with respect to $\widetilde{g}$. From the definition of gradient,

$$dr(X) = \mathbf{g}_V(V, X) = \widetilde{g}(V, X) = \widetilde{g}(\widetilde{\nabla} r, X),$$

namely, $\nabla r = \widetilde{\nabla} r$, here $\widetilde{\nabla} r$ is the gradient of r with respect to $\widetilde{g}$. Furthermore, by (2.3) and (2.4) we see that $\nabla^V_X V = \widetilde{\nabla}_X V$ for any $X \in TM$, and thus

$$\widetilde{H}(r)(X, Y) = \mathbf{g}_V(\widetilde{\nabla}_X V, Y) = \mathbf{g}_V(\nabla^V_X V, Y) = H(r)(X, Y),$$

here $\widetilde{H}$ is the Hessian of $\widetilde{g}$. Let $\widetilde{\Delta}$ and $\widetilde{\mathrm{div}}$ be the Laplacian and divergence with respect to $\widetilde{g}$, respectively. Then by Theorem 3.4 we get

$$\widetilde{\Delta}r = \widetilde{\mathrm{div}}\widetilde{\nabla}r = \mathrm{tr}_{\widetilde{g}}\widetilde{H}(r) = \mathrm{tr}_{\widetilde{g}}H(r) \geqslant (n-1)\left[\sqrt{-c} - [(2p-1)C]^{\frac{1}{2p-1}}\right].$$

By applying Lemma 3.7 in [11] to vector field V on $\Omega_\varepsilon(R)$ with respect to $\widetilde{g}$ and noticing (6.2) we have

$$\lambda_1(\Omega_\varepsilon(R)) \geqslant \frac{1}{\mu_F^{1+\frac{n}{2}}}\widetilde{\lambda}_1(\Omega_\varepsilon(R)) \geqslant \frac{(n-1)^2\left[\sqrt{-c} - [(2p-1)C]^{\frac{1}{2p-1}}\right]^2}{4\mu_F^{1+\frac{n}{2}}}.$$

Now letting $\varepsilon \to 0$ and $R \to \infty$ we obtain the desired result. □

7. Generalized Myers theorem

The celebrated Myers theorem in global Riemannian geometry says that if a Riemannian manifold M satisfies $\mathbf{Ric}(v) \geqslant (n-1)c$ for all unit vector v and some $c > 0$, then M is compact with $\mathrm{diam}(M) \leqslant \frac{\pi}{\sqrt{c}}$. Myers theorem has also been generalized to Finsler manifolds [1], and recently we establish a generalized Myers theorem under the line integral curvature bound for Finsler manifolds [9]. In this last section we shall prove another version of generalized Myers theorem for Finsler manifolds as follows.

Theorem 7.1. *Let (M, F) be an n-dimensional forward complete Finsler manifold, and $c > 0, p > 1$. If there is $\Lambda > 0$ with $\Lambda < (n-1)c$ such that for any $x \in M$ and each minimal normal geodesic γ emanating from x, the Ricci curvature satisfies*

$$\left[\frac{1}{L(\gamma)}\int_\gamma [\max\{(n-1)c - \mathbf{Ric}(\gamma'(t)), 0\}]^p\, dt\right]^{\frac{1}{p}} \leqslant \Lambda,$$

then M is compact with

$$\mathrm{diam}(M) \leqslant \frac{\pi}{\sqrt{c} - \dfrac{\Lambda}{(n-1)\sqrt{c}}}.$$

Proof. For any fixed $x, y \in M$ let $\gamma : [0, L(\gamma)] \to M$ be the minimal normal geodesic from x to y. Then by (3.11) we have the following inequality (see [9], page 836 or [11], page 99):

$$\pi \geqslant -\frac{1}{(n-1)\sqrt{c}}\int_\gamma \max\{(n-1)a - \mathbf{Ric}(\gamma'(t)), 0\}dt + L(\gamma)\sqrt{c},$$

which together with the Hölder inequality yields

$$\begin{aligned}\pi \geqslant\ & -\frac{1}{(n-1)\sqrt{c}}\left[\int_\gamma [\max\{(n-1)a - \mathbf{Ric}(\gamma'(t)), 0\}]^p dt\right]^{\frac{1}{p}}\left[\int_\gamma dt\right]^{1-\frac{1}{p}} \\ & + L(\gamma)\sqrt{c}\end{aligned}$$

$$= L(\gamma)\left(\sqrt{c} - \frac{1}{(n-1)\sqrt{c}}\left[\frac{1}{L(\gamma)}\int_\gamma [\max\{(n-1)a - \mathbf{Ric}(\gamma'(t)), 0\}]^p dt\right]^{\frac{1}{p}}\right)$$
$$\geqslant L(\gamma)\left(\sqrt{c} - \frac{\Lambda}{(n-1)\sqrt{c}}\right).$$

Consequently,

$$L(\gamma) \leqslant \frac{\pi}{\sqrt{c} - \frac{\Lambda}{(n-1)\sqrt{c}}},$$

and since x, y are arbitrary, we clearly have the desired result. □

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