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ON SOME GENERALIZATIONS OF THE REVERSIBILITY IN NONUNITAL RINGS

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Abstract. This paper is intended as a discussion of some generalizations of the notion of a reversible ring, which may be obtained by the restriction of the zero commutative property from the whole ring to some of its subsets. By the INCZ property we will mean the commutativity of idempotent elements of a ring with its nilpotent elements at zero, and by ICZ property we will mean the commutativity of idempotent elements of a ring at zero. We will prove that the INCZ property is equivalent to the abelianity even for nonunital rings. Thus the INCZ property implies the ICZ property. Under the assumption on the existence of unit, also the ICZ property implies the INCZ property. As we will see, in the case of nonunital rings, there are a few classes of rings separating the class of INCZ rings from the class of ICZ rings. We will prove that the classes of rings, that will be discussed in this note, are closed under extending to the rings of polynomials and formal power series.

1. Preliminaries

All rings considered in this paper are assumed to be associative but not necessarily with unit. The standard extension of a ring R to a unital ring with the help of the ring of integers is denoted by R^1. The sets of idempotent elements in R and nilpotent elements in R are denoted by $E(R)$ and $N(R)$ respectively.

J. Lambek in [13] introduced the notion of a symmetric ring understood as a unital ring R in which $rst = 0$ implies $rts = 0$ for any $r, s, t \in R$, and proved that an equivalent condition on a unital ring R to be symmetric is that $r_1 \cdot r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} \cdot r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any positive integer n, any elements $r_1, r_2, \ldots, r_n \in R$ and any permutation σ of the set $\{1, 2, \ldots, n\}$. D. D. Anderson and V. Camillo in [1] continued the study of rings whose zero

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products commute, defining the notion of a ring satisfying the ZC_n property as a not necessarily unital ring R in which $r_1 \cdot r_2 \cdots r_n = 0$ implies $r_{\sigma(1)} \cdot r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any elements $r_1, r_2, \ldots, r_n \in R$ and any permutation σ of the set $\{1, 2, \ldots, n\}$, and proving that the ZC_3 property implies the ZC_n property for any $n \geq 3$. B. H. Shafee and S. K. Nauman in [16] distinguished between the right and left symmetries, defining the notions of right and left symmetric rings as not necessarily unital rings R in which $rst = 0$ implies $rts = 0$ and $srt = 0$ respectively, for any $r, s, t \in R$. In this context *a symmetric ring* means a ring both right and left symmetric, or equivalently a ring satisfying the ZC_3 property. J. M. Habeb in [7] introduced the notion of a ZC ring understood as a ring R in which $rs = 0$ implies $sr = 0$ for any $r, s \in R$. P. M. Cohn in [6] was the first who used the term *a reversible ring* instead of a ZC ring. Finally, H. E. Bell in [3] defined the notion of a ring satisfying the IFP property as a ring R in which $rs = 0$ implies $rRs = 0$ for any $r, s \in R$ (in both the definitions, there is no reason to require R to be unital). J. M. Habeb in [7] referred to rings satisfying the IFP property as ZI rings. L. Motais de Narbonne in [15] was the first who used the term *a semicommutative ring* instead of a ring satisfying the IFP property.

Commutative rings, as well as reduced rings, are both symmetric and reversible. Symmetric rings with unit are obviously reversible. For nonunital rings this is no longer true, as shown by B. H. Shafee and S. K. Nauman in [16]. Right symmetric rings, as well as reversible rings, are semicommutative. The classes of right symmetric rings, reversible rings, and semicommutative rings are not closed under standard adjoining unit. For a deeper discussion of the above mentioned classes of rings under the assumption that these rings are unital, we refer the readers to [14].

Further generalizations of the commutative property may be obtained by the restriction of this property from the whole ring to some of its subsets. A not necessarily unital ring R in which $er = re$ holds for any $e \in E(R)$ and $r \in R$, according to the definition introduced by I. Kaplansky in [10], is said to be *abelian.* I. Kaplansky studied of the abelian property in the class of Baer rings. An equivalent condition on a unital ring R to be abelian is that $ere = er$ holds for any $e \in E(R)$ and $r \in R$. Another equivalent condition on a unital ring R to be abelian is that $er = 0$ implies $eRr = 0$ for any $e \in E(R)$ and $r \in R$. According to the definition introduced by G. F. Birkenmeier in [4], an idempotent e of a ring R is said to be *right semicentral* or *left semicentral* in R if $ere = er$ or $ere = re$, respectively, holds for any $r \in R$. W. Chen in [5] introduced the notion of *a semiabelian ring* understood as a ring R in which every idempotent is either right semicentral or left semicentral. J. Wei in [18] defined the notion of *a right almost abelian ring* as a ring R in which $er = 0$ implies $eRr = 0$ for any $e \in E(R)$ and $r \in N(R)$ (in both the definitions, there is no reason to require R to be unital).

Semicommutative rings with unit are abelian. As we will see in Theorem 2.1, for symmetric rings, as well as for reversible rings, this implication still holds if we drop the assumption on the existence of unit. An example of a right symmetric ring without unit, which is nonabelian, was given by B. H. Shafee and S. K. Nauman in [16]. This example confirms that semicommutative rings without unit are not abelian in general. As we will see in Corollary 2.2, abelian rings form a class closed under standard adjoining unit. Abelian rings are obviously both semiabelian and right almost abelian. J. Wei in [18] showed that neither semiabelian rings need not be right almost abelian nor right almost abelian rings need not be semiabelian even then these rings are unital.

This paper is intended as a discussion of some generalizations of the notion of a reversible ring, which may be obtained by the restriction of the zero commutative property from the whole ring to some of its subsets. The subsets of idempotent elements and nilpotent elements of this ring are natural subsets for considering such restrictions. For a ring R, we consider the following properties:

INCZ: idempotents of R commute with nilpotents of R at zero, which means that the equivalence $er = 0$ if and only if $re = 0$ holds for any $e \in E(R)$ and $r \in N(R)$.

ICZ: idempotents of R commute at zero, which means that $ef = 0$ implies $fe = 0$ for any $e, f \in E(R)$.

We can directly verify the following connections between the above properties:

$$\text{abelianity} \Rightarrow \text{INCZ} \Rightarrow \text{ICZ}.$$

To see the latter implication, we assume that $ef = 0$ where $e, f \in E(R)$. Then since $fe \in N(R)$ and $e(fe) = 0$, it follows that also $fe = (fe)e = 0$ by the assumption on the INCZ property. As we will see in Theorem 2.1, even for nonunital rings, the INCZ property implies the abelianity. As we will see in Theorem 2.3, under the assumption on the existence of unit, the ICZ property implies the abelianity. As we will see in Section 3, in the case of nonunital rings, there are a few classes of rings separating the class of abelian rings from the class of rings satisfying the ICZ property. In Section 4 we will prove that the classes of rings, that will be discussed in Section 3, are closed under extending to the rings of polynomials and formal power series.

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2. Generalizations of reversible rings without unit

Recall that unless otherwise stated we do not require rings to be unital. Theorems 2.1, 2.3 and 2.9 were partially noticed by V. K. Kharchenko et al. in [9], J. Han et al. in [8] and G. Shin in [17].

Theorem 2.1. *For every ring R, the following statements are equivalent*:

1. *R is abelian*;

2. *$et = te$ holds for any $e \in E(R)$ and $t \in N(R)$;*
3. *$ef = fe$ holds for any $e, f \in E(R)$;*
4. *$ere = e$ and $rer = r$ imply $e = r$ for any $e \in E(R)$ and $r \in R$;*
5. *$efe = e$ and $fef = f$ imply $e = f$ for any $e, f \in E(R)$;*
6. *R satisfies the INCZ property.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 6$, $1 \Rightarrow 3 \Rightarrow 5$ and $1 \Rightarrow 4 \Rightarrow 5$ are obvious. In the proofs of both the implications $5 \Rightarrow 1$ and $6 \Rightarrow 1$, we let $e \in E(R)$ and $r \in R$. Then also $e + er - ere, e + re - ere \in E(R)$ and $er - ere, re - ere \in N(R)$. In the case when the statement 5 holds, since $e(e + er - ere)e = e$ and $(e+er-ere)e(e+er-ere) = e+er-ere$, and simultaneously $e(e+re-ere)e = e$ and $(e + re - ere)e(e + re - ere) = e + re - ere$, from this it follows that $e = e+er-ere$ and $e = e+re-ere$, and thus $er = ere = re$. In the case when the statement 6 holds, since $(er - ere)e = 0$ and $e(re - ere) = 0$, from this it follows that $e(er - ere) = 0$ and $(re - ere)e = 0$, and thus $er = ere = re$. □

Since the early 50's, the notion of a inverse semigroup, understood as a semigoup S in which for every $s \in S$ there exists a unique $u \in S$ such that $sus = s$ and $usu = u$, is of fundamental importance in semigroup theory. As we see in Theorem 2.1, an equivalent condition on a ring R to be abelian is that idempotent elements in R form an inverse semigroup.

Corollary 2.2. *If in a ring R the equality $et = te$ holds for any $e \in E(R)$ and $t \in N(R)$, then the same equality holds also in the unital ring R^1 for any $e \in E(R^1)$ and $t \in N(R^1)$. In particular, abelian rings form a class closed under standard adjoining unit.*

Proof. The former of the statements follows immediately from the fact that $E(R^1) = E(R) \cup (1 - E(R))$ and $N(R^1) = N(R)$. The latter of the statements follows directly from Theorem 2.1. □

Theorem 2.3. *For every unital ring R, the following statements are equivalent*:

1. *R is abelian*;
2. *$ete = et$ holds for any $e \in E(R)$ and $t \in N(R)$;*
3. *$efe = ef$ holds for any $e, f \in E(R)$;*
4. *$te = 0$ implies $et = 0$ for any $e \in E(R)$ and $t \in N(R)$;*
5. *$ef \in E(R)$ holds for any $e, f \in E(R)$;*
6. *R satisfies the ICZ property.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 5 \Rightarrow 6$ are obvious. In the proofs of both the implications $4 \Rightarrow 1$ and $6 \Rightarrow 1$, we assume that $e, f \in E(R)$ are orthogonal and $r \in R$. Then also $e + erf \in E(R)$ and $erf \in N(R)$. In the case when the statement 4 holds, since $(erf)e = 0$, it follows that also $erf = e(erf) = 0$. In the case when the statement 6 holds, since $f(e + erf) = 0$, it follows that also $erf = (e+erf)f = 0$. In both the cases, $erf = 0$ holds for any orthogonal $e, f \in E(R)$ and any $r \in R$. In particular, $er(1-e) = 0 = (1-e)re$, and, in consequence, $er = ere = re$. □

Theorem 2.4. *If in a ring R the property $ef \in E(R)$ holds for any $e, f \in E(R)$, then $e_1 \cdot e_2 \cdots e_n = 0$ implies $e_{\sigma(1)} \cdot e_{\sigma(2)} \cdots e_{\sigma(n)} = 0$ for any positive integer n, any elements $e_1, e_2, \ldots, e_n \in E(R)$ and any permutation σ of the set $\{1, 2, \ldots, n\}$. In particular, the ring R satisfies the ICZ property.*

Proof. The proof is the simple adaptation of the proof of the theorem, according to which reduced rings satisfy the ZC_n property for any positive integer n, see for instance [1, Theorem 1.3]. □

In semigroup theory, the notion of an E-semigroup is defined as a semigroup whose idempotent elements form a subsemigroup. For this reason, a ring R, in which the property $ef \in E(R)$ holds for any $e, f \in E(R)$, might be called an E-ring.

Theorem 2.5. *For every ring R, the following statements are equivalent:*

1. *R satisfies the ICZ property;*
2. *$ef = 0$ implies $fRe = 0$ for any $e, f \in E(R)$;*
3. *$ef = 0$ implies $eRf = 0$ for any $e, f \in E(R)$.*

Proof. In the proofs of all three implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$, we let $e, f \in E(R)$ with $ef = 0$ and $r \in R$. In the case when the statement 1 holds, also $fe = 0$. Since $e + re - ere \in E(R)$ and $(e + re - ere)f = 0$, it follows that $fre = f(e+re-ere) = 0$. In the case when the statement 2 holds, also $fe \in fRe = 0$, and thus $eRf = 0$. In the case when the statement 3 holds, since $f - fe \in E(R)$ and $(f - fe)e = 0$, it follows that $fe = (f - fe)fe \in (f - fe)Re = 0$. □

Corollary 2.6. *Every semicommutative ring satisfies the ICZ property.*

Proof. The corollary is a simple consequence of Theorem 2.5. □

Theorem 2.7. *For every ring R, the following statements are equivalent:*

1. *$erf = efr$ holds for any $e, f \in E(R)$ and $r \in R$;*
2. *$etf = eft$ holds for any $e, f \in E(R)$ and $t \in N(R)$;*
3. *$efg = egf$ holds for any $e, f, g \in E(R)$;*
4. *$ere = er$ holds for any $e \in E(R)$ and $r \in R$;*
5. *$ete = et$ holds for any $e \in E(R)$ and $t \in N(R)$;*
6. *$efe = ef$ holds for any $e, f \in E(R)$;*
7. *$re = 0$ implies $er = 0$ for any $e \in E(R)$ and $r \in R$;*
8. *$te = 0$ implies $et = 0$ for any $e \in E(R)$ and $t \in N(R)$;*
9. *$re = 0$ implies $eRr = 0$ for any $e \in E(R)$ and $r \in R$;*
10. *$te = 0$ implies $eRt = 0$ for any $e \in E(R)$ and $t \in N(R)$.*

Proof. The implications $1 \Rightarrow 4 \Rightarrow 9 \Rightarrow 7 \Rightarrow 8$, $1 \Rightarrow 2 \Rightarrow 5 \Rightarrow 10 \Rightarrow 8$ and $1 \Rightarrow 3 \Rightarrow 6$ are obvious. In the proofs of all three implications $6 \Rightarrow 4$, $8 \Rightarrow 4$ and $4 \Rightarrow 1$, we let $e, f \in E(R)$ and $r \in R$. Then, as we know, $e+er-ere \in E(R)$ and $er-ere \in N(R)$. In the case when the statement 6 holds, since $e(e+er-ere)e = e(e + er - ere)$, it follows that $ere = er$. In the case when the statement 8 holds, since $(er - ere)e = 0$, from this it follows that $e(er - ere) = 0$, and thus $ere = er$. We suppose now that the statement 4 holds. Then idempotent elements of the ring R form a semigroup, and hence the ring R satisfies the ICZ property by Theorem 2.4. Since $(e-ef)^2 = e-ef-efe+(ef)^2 = e-ef$, which means that $e - ef \in E(R)$, and since $(e - ef)f = 0$, from this it follows that $(e-ef)rf \in (e-ef)Rf = 0$ by Theorem 2.5, and thus $erf = e(frf) = efr$. $\square$

N. K. Kim et al. in [12] defined the notions of rings satisfying *the right* and *left IIP properties* as rings R in which $rse = 0$ implies $res = 0$ and $ers = 0$ implies $res = 0$ respectively, for any $e \in E(R)$ and $r, s \in R$. These authors also defined the notions of rings satisfying *the right* and *left IR properties* as rings R in which $re = 0$ implies $er = 0$ and $er = 0$ implies $re = 0$ respectively, for any $e \in E(R)$ and $r \in R$, and proved that an equivalent condition on a ring R to satisfy the right IR property is that $ere = er$ holds for any $e \in E(R)$ and $r \in R$. Note that in Theorem 2.7 we gave a deeper characterization of rings satisfying the right IR property. For every ring R, the following statements are equivalent: (1) R is abelian; (2) R satisfies both the right and left IIP properties; (3) R satisfies both the right and left IR properties. Moreover, the following connections between the properties defined above hold:

$$\text{abelianity} \;\Rightarrow\; \text{right IIP} \;\Rightarrow\; \text{right IR}.$$

N. K. Kim et al. in [12, Examples 2.3 and 2.6] showed that both the converse implications need not be true in general. As we saw in Corollary 2.6, semicommutative rings satisfy the ICZ property. N. K. Kim et al. in [12, Examples 2.11] showed that semicommutative rings need not satisfy the IR property.

Theorem 2.8.

1. *If a ring R satisfies the right IR property, then $es = 0$ implies $eRs = 0$ for any $e \in E(R)$ and $s \in R$. In particular, the ring R is right almost abelian.*
2. *If a ring R is right almost abelian, then $erese = erse$ holds for any $e \in E(R)$ and $r, s \in R$. In particular, in the ring R the equality $(ef)^3 = (ef)^2$ holds for any $e, f \in E(R)$.*
3. *For every ring R, the following statements are equivalent*:
   a. *R is abelian*;
   b. *R satisfies the following conditions*:
      i. *R is right almost abelian*;
      ii. *$eRt = 0$ implies $te = 0$, and simultaneously $tRe = 0$ implies $et = 0$, both the implications hold for any $e \in E(R)$ and $t \in N(R)$.*

Proof. In the proofs of all three statements, we let $e \in E(R)$ and $r, s \in R$. If R satisfies the right IR property, then $es = 0$ implies $eRs = eRes = 0$ by Theorem 2.7. If R is right almost abelian, then since $se - ese \in N(R)$ and $e(se - ese) = 0$, from this it follows that $eR(se - ese) = 0$, and thus

$erse = erese$. The implication a⇒b is obvious. In the proof of the converse implication b⇒a, we additionally let $t \in N(R)$. If $et = 0$, then $eRt = 0$, and from this it follows that also $te = 0$. If $te = 0$, then since $et \in N(R)$ and $etRe = eteRe$ by statement 2, from this it follows that also $et = e(et) = 0$. In consequence, R is abelian by Theorem 2.1. □

To summarize, the following connections between the properties discussed in this paper hold:

$$\begin{array}{ccccc} & \nearrow & \text{right IIP} \Rightarrow \text{right IR} & \searrow & \\ \text{abelianity} \Leftrightarrow \text{INCZ} & & & & \begin{array}{l}\text{idempotent elements}\\ \text{in a ring}\\ \text{form a semigroup}\end{array} \Rightarrow \text{ICZ} \\ & \searrow & \text{left IIP} \Rightarrow \text{left IR} & \nearrow & \end{array}$$

In the case of unital rings, the converse implications also hold by Theorem 2.3. Example 3.1 shows that there exist nonabelian rings satisfying the right IIP property. N. K. Kim et al. in [12, Example 2.6] showed that the right IR property need not imply the right IIP property. Examples 3.2–3.4 show that there exist rings whose idempotent elements form a semigroup, and which need not satisfy the right IR property. Finally, Example 3.5 shows that there exist rings satisfying the ICZ property, and whose idempotent elements need not form a semigroup.

Theorem 2.9. *For every ring R, the following statements are equivalent*:

1. *R is abelian*;
2. *R satisfies the following conditions*:
   a. *$ef \in E(R)$ holds for any $e, f \in E(R)$*;
   b. *$eRt = 0$ implies $te = 0$, and simultaneously $tRe = 0$ implies $et = 0$, both the implications hold for any $e \in E(R)$ and $t \in N(R)$.*

Proof. The implication $1 \Rightarrow 2$ is obvious. In the proof of the converse implication $2 \Rightarrow 1$, we let $e, f \in E(R)$ and $r \in R$. Since $f + fr - frf, f + rf - frf \in E(R)$, it follows that also $e(f + fr - frf), (f + rf - frf)e \in E(R)$. Right multiplying $e(f + fr - frf)e(f + fr - frf) = e(f + fr - frf)$ by f, and then applying the assumption, we obtain $efr(ef - fef) = 0$, which means that $efR(ef - fef) = 0$. Since $ef \in E(R)$ and $fe - fef \in N(R)$, from this it follows that $(ef - fef)ef = 0$, and thus $ef = fef$. Similarly, left multiplying $(f + rf - frf)e(f + rf - frf)e = (f + rf - frf)e$ by f we obtain $(fe-fef)Rfe = 0$. Since $fe \in E(R)$ and $fe-fef \in N(R)$, from this it follows that $fe(fe - fef) = 0$, and thus $fe = fef$. In consequence, $ef = fef = fe$ holds for any $e, f \in E(R)$, which forces R to be abelian by Theorem 2.1. □

Corollary 2.10. *For every semiprime ring R, the following statements are equivalent*:

1. *R is abelian*;
2. *R is right almost abelian*;
3. *$ef \in E(R)$ holds for any $e, f \in E(R)$.*

Proof. The corollary is a simple consequence of Theorems 2.8 and 2.9. □

Example 3.7 shows that even for prime rings, the ICZ property need not imply the abelianity.

Theorem 2.11. *For every von Neumann regular ring R, the following statements are equivalent*:

1. *R is reduced*;
2. *R is abelian*;
3. *R satisfies the ICZ property.*

Proof. Both the implications $1 \Rightarrow 2 \Rightarrow 3$ are obvious. In the proof of the implication $3 \Rightarrow 1$, for any $t \in N(R)$ with $t^2 = 0$ we let $x \in R$ such that $t = txt$. Since $xt, tx \in E(R)$ and $(xt)(tx) = 0$, from this it follows that $(xt)R(tx) = 0$ by Theorem 2.5, and thus $tRt = txtRtxt = 0$. In consequence, $t = 0$. □

3. Examples of rings satisfying ICZ property

For a ring R, we denote by $R[X]$ and $R\langle X\rangle$ the rings of polynomials in commuting and noncommuting variables $\{x \mid x \in X\}$ respectively, both with coefficients from R. The polynomial rings in commuting and noncommuting variables $\{x \mid x \in X\}$ with zero constant term are denoted by $\sum_{x\in X} xR[X]$ and $\sum_{x\in X} xR\langle X\rangle$ respectively. The formal power series ring with coefficients from R is denoted by $R[[x]]$. We denote by $M_n(R)$ and $U_n(R)$ the rings of $n\times n$ matrices and upper triangular $n \times n$ matrices respectively, both with entries from R. The subring of $U_n(R)$ of $n \times n$ matrices with fixed element on the main diagonal is denoted by $D_n(R)$.

Example 3.1. Let P be a commutative ring with unit, and let

$$R = \sum_{x\in X} xP\langle X\rangle/(xy - x \mid x, y \in X)$$

be a homomorphic image of the polynomial ring in noncommuting variables with zero constant term. Every element of the ring R is expressed uniquely as $\sum_{x\in X} \alpha_x\overline{x}$ where $\alpha_x \in P$ equals zero for almost every $x \in X$. For simplicity of notation, we will write α instead of $\sum_{x\in X} \alpha_x\overline{x}$. In the ring R,

$$\alpha\beta\gamma = \sum_{x\in X}\alpha_x\big(\sum_{y\in X}\beta_y\big)\big(\sum_{z\in X}\gamma_z\big)\overline{x} = \sum_{x\in X}\alpha_x\big(\sum_{y\in X}\gamma_y\big)\big(\sum_{z\in X}\beta_z\big)\overline{x} = \alpha\gamma\beta$$

holds for any $\alpha, \beta, \gamma \in R$. This evidently forces R to be right symmetric, and hence to satisfy the right IIP property.

Simultaneously, $\alpha\overline{y} = \alpha$ holds for any $\alpha \in R$ and $y \in X$, in spite of that $\overline{y} \in E(R)$ and if $\sum_{x\in X} \alpha_x = 0$, then $\overline{y}\alpha = 0$. This obviously means that the ring R does not satisfy the left IR property.

Example 3.2. Under the notation used in Example 3.1, let $S = \left(\begin{smallmatrix} R & R \\ 0 & 0 \end{smallmatrix}\right)$ be a subring in the matrix ring $M_2(R)$. Since every idempotent matrix in the ring S is expressed uniquely as $\left(\begin{smallmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{smallmatrix}\right)$ where $\varepsilon \in E(R)$ and $\alpha \in R$, from this it follows that $\mathscr{E}\mathscr{F} \in E(S)$ holds for any $\mathscr{E}, \mathscr{F} \in E(S)$.

Simultaneously, for idempotent matrices $\mathscr{E} = \left(\begin{smallmatrix} \overline{x} & \overline{x} \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathscr{F} = \left(\begin{smallmatrix} \overline{y} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in the ring S we have $\mathscr{E}\mathscr{F} \neq \mathscr{E}\mathscr{F}\mathscr{E} \neq \mathscr{F}\mathscr{E}$. This means that the ring S satisfies neither the right nor left IR property.

Example 3.3. Let S be the same as in Example 3.2. A matrix $\left(\begin{smallmatrix} \mathscr{E} & \mathscr{A} \\ 0 & \mathscr{E} \end{smallmatrix}\right)$ is idempotent in the ring $D_2(S)$ if and only if $\mathscr{E} \in E(S)$ and $\mathscr{E}\mathscr{A} + \mathscr{A}\mathscr{E} = \mathscr{A}$, and from this $\mathscr{E}\mathscr{A}\mathscr{E} = 0$. According to Example 3.2, $\mathscr{E} = \left(\begin{smallmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{smallmatrix}\right)$ where $\varepsilon \in E(R)$ and $\alpha \in R$. Let $\mathscr{A} = \left(\begin{smallmatrix} \beta & \gamma \\ 0 & 0 \end{smallmatrix}\right)$ where $\beta, \gamma \in R$. We conclude from

$$\begin{pmatrix} \varepsilon\beta & \varepsilon\beta\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \beta & \gamma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

that $\varepsilon\beta = 0$, hence that

$$\begin{pmatrix} \beta\varepsilon & \beta\varepsilon\alpha + \varepsilon\gamma \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \beta & \gamma \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \beta & \gamma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \beta & \gamma \\ 0 & 0 \end{pmatrix},$$

and thus that $\beta = \beta\varepsilon$ and $\gamma = \beta\varepsilon\alpha + \varepsilon\gamma$. Needless to say, every matrix of the form $\left(\begin{smallmatrix} \mathscr{E} & \mathscr{A} \\ 0 & \mathscr{E} \end{smallmatrix}\right)$ where $\mathscr{E} = \left(\begin{smallmatrix} \varepsilon & \varepsilon\alpha \\ 0 & 0 \end{smallmatrix}\right)$, $\mathscr{A} = \left(\begin{smallmatrix} \beta\varepsilon & \beta\varepsilon\alpha+\varepsilon\gamma \\ 0 & 0 \end{smallmatrix}\right)$, $\varepsilon \in E(R)$ and $\alpha, \beta, \gamma \in R$ with $\varepsilon\beta = 0$, is idempotent in the ring $D_2(S)$. We consider another idempotent matrix of the form $\left(\begin{smallmatrix} \mathscr{F} & \mathscr{B} \\ 0 & \mathscr{F} \end{smallmatrix}\right)$ where $\mathscr{F} = \left(\begin{smallmatrix} \phi & \phi\delta \\ 0 & 0 \end{smallmatrix}\right)$, $\mathscr{B} = \left(\begin{smallmatrix} \eta\phi & \eta\phi\delta+\phi\mu \\ 0 & 0 \end{smallmatrix}\right)$, $\phi \in E(R)$ and $\delta, \eta, \mu \in R$ with $\phi\eta = 0$. Since $\mathscr{E}\mathscr{F} = \left(\begin{smallmatrix} \varepsilon\phi & \varepsilon\phi\delta \\ 0 & 0 \end{smallmatrix}\right)$, $\varepsilon\phi \in E(R)$,

$$\mathscr{E}\mathscr{B} + \mathscr{A}\mathscr{F} = \begin{pmatrix} (\varepsilon\eta + \beta)\varepsilon\phi & (\varepsilon\eta + \beta)\varepsilon\phi\delta + \varepsilon\phi\mu \\ 0 & 0 \end{pmatrix}$$

and $\varepsilon\phi(\varepsilon\eta + \beta) = \varepsilon\phi\eta + \varepsilon\beta\phi = 0$, from this it follows that

$$\begin{pmatrix} \mathscr{E} & \mathscr{A} \\ 0 & \mathscr{E} \end{pmatrix}\begin{pmatrix} \mathscr{F} & \mathscr{B} \\ 0 & \mathscr{F} \end{pmatrix} = \begin{pmatrix} \mathscr{E}\mathscr{F} & \mathscr{E}\mathscr{B} + \mathscr{A}\mathscr{F} \\ 0 & \mathscr{E}\mathscr{F} \end{pmatrix} \in E(D_2(S)).$$

Simultaneously, for idempotent matrices $E = \left(\begin{smallmatrix} \mathscr{E} & 0 \\ 0 & \mathscr{E} \end{smallmatrix}\right)$ and $F = \left(\begin{smallmatrix} \mathscr{F} & 0 \\ 0 & \mathscr{F} \end{smallmatrix}\right)$ in the ring $D_2(S)$ where $\mathscr{E}$ and $\mathscr{F}$ are the same as in Example 3.2, we have $EF \neq EFE \neq FE$. This means that the ring $D_2(S)$ satisfies neither the right nor left IR property.

Example 3.4. Let P be a commutative ring with unit, and let

$$\begin{aligned} R = (x_1P\langle X\rangle + y_1P\langle X\rangle + x_2P\langle X\rangle + y_2P\langle X\rangle)/(x_ix_j - x_j, x_iy_j - x_j,& \\ y_ix_j - y_j, y_iy_j - y_j \mid i, j \in \{1, 2\})& \end{aligned}$$

be a homomorphic image of the polynomial ring in noncommuting variables $X = \{x_1, y_1, x_2, y_2\}$ with zero constant term. Every element of the ring R is expressed uniquely as $\alpha_1\overline{x_1} + \beta_1\overline{y_1} + \alpha_2\overline{x_2} + \beta_2\overline{y_2}$ where $\alpha_1, \beta_1, \alpha_2, \beta_2 \in P$. Since

$$\begin{aligned} &(\alpha_1\overline{x_1} + \beta_1\overline{y_1} + \alpha_2\overline{x_2} + \beta_2\overline{y_2})(\gamma_1\overline{x_1} + \delta_1\overline{y_1} + \gamma_2\overline{x_2} + \delta_2\overline{y_2}) \\ (1)\qquad &= (\alpha_1 + \alpha_2)(\gamma_1 + \delta_1)\overline{x_1} + (\beta_1 + \beta_2)(\gamma_1 + \delta_1)\overline{y_1} \end{aligned}$$

$$+ (\alpha_1 + \alpha_2)(\gamma_2 + \delta_2)\overline{x_2} + (\beta_1 + \beta_2)(\gamma_2 + \delta_2)\overline{y_2},$$

from this it follows that an element $\alpha_1\overline{x_1} + \beta_1\overline{y_1} + \alpha_2\overline{x_2} + \beta_2\overline{y_2}$ is idempotent in the ring R if and only if

$$\begin{aligned}(\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) &= \alpha_1, \quad (\beta_1 + \beta_2)(\alpha_1 + \beta_1) = \beta_1,\\ (\alpha_1 + \alpha_2)(\alpha_2 + \beta_2) &= \alpha_2, \quad (\beta_1 + \beta_2)(\alpha_2 + \beta_2) = \beta_2,\end{aligned} \tag{2}$$

and thus

$$\varepsilon = \alpha_1 + \beta_1 + \alpha_2 + \beta_2 \in E(P).$$

Substituting $\beta_2 = \varepsilon - \alpha_1 - \beta_1 - \alpha_2$ into (2) we obtain

$$\begin{aligned}&(\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) = \alpha_1,\\ &\varepsilon(\alpha_1 + \beta_1) - (\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) = \beta_1,\\ &\varepsilon(\alpha_1 + \alpha_2) - (\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) = \alpha_2,\\ &\varepsilon(2\alpha_1 + \beta_1 + \alpha_2) - (\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) = \alpha_1 + \beta_1 + \alpha_2,\end{aligned} \tag{3}$$

and then substituting $(\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) = \alpha_1$ into (3) we obtain $(1 - \varepsilon)(\alpha_1 + \beta_1) = 0$ and $(1-\varepsilon)(\alpha_1+\alpha_2) = 0$. But $(1-\varepsilon)\alpha_1 = (1-\varepsilon)(\alpha_1+\alpha_2)(\alpha_1+\beta_1) = 0$. From this we obtain

$$\begin{aligned}&(1 - \varepsilon)\alpha_1 = (1 - \varepsilon)\beta_1 = (1 - \varepsilon)\alpha_2 = (1 - \varepsilon)\beta_2 = 0,\\ &(\alpha_1 + \alpha_2)(\alpha_1 + \beta_1) = \alpha_1, \quad \alpha_1 + \beta_1 + \alpha_2 + \beta_2 = \varepsilon.\end{aligned} \tag{4}$$

Needless to say, every element $\alpha_1\overline{x_1} + \beta_1\overline{y_1} + \alpha_2\overline{x_2} + \beta_2\overline{y_2}$ satisfying (4) where $\varepsilon \in E(P)$, is idempotent in the ring R. Assuming (4) and additionally

$$\begin{aligned}&(1 - \phi)\gamma_1 = (1 - \phi)\delta_1 = (1 - \phi)\gamma_2 = (1 - \phi)\delta_2 = 0,\\ &(\gamma_1 + \gamma_2)(\gamma_1 + \delta_1) = \gamma_1, \quad \gamma_1 + \delta_1 + \gamma_2 + \delta_2 = \phi,\end{aligned}$$

where $\phi \in E(P)$, we can check that the element (1) is idempotent in the ring R. This means that $\mathscr{E}\mathscr{F} \in E(R)$ holds for any $\mathscr{E}, \mathscr{F} \in E(R)$.

Simultaneously, for $\overline{x_1}, \overline{y_2} \in E(R)$ we have $\overline{x_1} \cdot \overline{y_2} \neq \overline{x_1} \cdot \overline{y_2} \cdot \overline{x_1} \neq \overline{y_2} \cdot \overline{x_1}$. This means that the ring R satisfies neither the right nor left IR property.

Example 3.5. Let P be a reduced ring, and let

$$R = \left\{ \begin{pmatrix} 0 & 0 & 0 & b\\ 0 & c & 0 & 0\\ a & 0 & c & d\\ 0 & 0 & 0 & c \end{pmatrix} \mid a, b, c, d \in P \right\}$$

be a subring in the matrix ring $M_4(P)$. A matrix

$$\begin{pmatrix} 0 & 0 & 0 & b\\ 0 & e & 0 & 0\\ a & 0 & e & d\\ 0 & 0 & 0 & e \end{pmatrix}$$

is idempotent in the ring R if and only if $e \in E(P)$, $a = ea$, $b = be$ and $d = ab + ed + de$, and hence $ab + ede = 0$. From this it follows that $d =$

$-ede+ed+de$, which means that $(d-ed)^2 = 0 = (d-de)^2$, and, in consequence, that $d - ed = 0 = d - de$ by the assumption. Thus $d = -ab = -eabe$. Needless to say, every matrix of the form

$$E = \begin{pmatrix} 0 & 0 & 0 & be \\ 0 & e & 0 & 0 \\ ea & 0 & e & -eabe \\ 0 & 0 & 0 & e \end{pmatrix},$$

where $e \in E(P)$ and $a, b \in P$, is idempotent in the ring R. We consider another idempotent matrix of the form

$$F = \begin{pmatrix} 0 & 0 & 0 & df \\ 0 & f & 0 & 0 \\ fc & 0 & f & -fcdf \\ 0 & 0 & 0 & f \end{pmatrix},$$

where $f \in E(P)$ and $c, d \in P$. If $EF = 0$, then $ef = 0$, from this it follows that $fPe = 0$ by the assumption, and, in consequence, $FE = 0$. This confirms that the ring R satisfies the ICZ property.

Simultaneously,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ e & 0 & e & 0 \\ 0 & 0 & 0 & e \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & e \\ 0 & 0 & 0 & e \end{pmatrix} \notin E(R)$$

for every nonzero $e \in E(P)$, in sprite of that both the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ e & 0 & e & 0 \\ 0 & 0 & 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & e \end{pmatrix} \in E(R).$$

Example 3.6. Let P be a ring, and let

$$R = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & c & 0 \\ b & 0 & 0 & c \end{pmatrix} \mid a, b, c \in P \right\}$$

be a subring in the matrix ring $M_4(P)$. Since every idempotent matrix in the ring R is expressed uniquely as

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & ae & 0 \\ 0 & 0 & e & 0 \\ eb & 0 & 0 & e \end{pmatrix},$$

where $e \in E(P)$ and $a, b \in P$, from this it follows that if in the ring P the property $ef \in E(P)$ holds for any $e, f \in E(P)$ (respectively, the ring P satisfies the ICZ property), then the same is true for the ring R.

Simultaneously, for matrices

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & e & 0 \\ e & 0 & 0 & e \end{pmatrix} \in E(R)$$

and

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in N(R),$$

where $e \in E(P)$ is nonzero, we have $EA = A$ and $BE = B$, in spite of that $AE = 0$ and $EB = 0$. This means that the ring R satisfies neither the right nor left IR property.

Example 3.7. Let P be a domain with unit, and let

$$R = (xP\langle x, y\rangle + yP\langle x, y\rangle)/(x^2 - x)$$

be a homomorphic image of the polynomial ring in noncommuting variables with zero constant term. Every element of the ring R is expressed uniquely as $\sum_i \alpha_i \overline{u_i}$, where $\alpha_i \in P$ equals zero for almost every i, $\overline{u_i} \in \{\overline{x}, \overline{y}\} \cup \mathscr{M}$ and $\mathscr{M} = \{\overline{y}^{\eta},\ \overline{x}\cdot\overline{y}^{\eta_1}\cdot\overline{x}\cdot\overline{y}^{\eta_2}\cdots\overline{x}\cdot\overline{y}^{\eta_k}\cdot\overline{x},\ \overline{x}\cdot\overline{y}^{\eta_1}\cdot\overline{x}\cdot\overline{y}^{\eta_2}\cdots\overline{x}\cdot\overline{y}^{\eta_k},\ \overline{y}^{\eta_1}\cdot\overline{x}\cdot\overline{y}^{\eta_2}\cdots\overline{x}\cdot\overline{y}^{\eta_k}\cdot\overline{x},\ \overline{y}^{\eta_1}\cdot\overline{x}\cdot\overline{y}^{\eta_2}\cdots\overline{x}\cdot\overline{y}^{\eta_{k+1}} \mid k \geq 1, \eta \geq 2, \eta_1, \eta_2, \ldots, \eta_{k+1} \geq 1\}$. For the proof of the primeness in the ring R, we let $\overline{a} = \sum_i \alpha_i \overline{u_i}$ and $\overline{b} = \sum_i \beta_i \overline{u_i}$ from R, both nonzero. Then we denote by $\alpha_j \overline{u_j}$ and $\beta_k \overline{u_k}$ monomials of the lowest degrees in the polynomials $\overline{a}$ and $\overline{b}$ with $\alpha_j \neq 0$ and $\beta_k \neq 0$ respectively. Since the coefficient of $\overline{u_j}\cdot\overline{y}\cdot\overline{u_k}$ in the polynomial $\overline{a}\cdot\overline{y}\cdot\overline{b}$ equals $\alpha_j\beta_k \neq 0$, from this it follows that $\overline{a}\cdot\overline{y}\cdot\overline{b} \neq 0$, and thus $\overline{a}R\overline{b} \neq 0$. This confirms that the ring R is prime.

If a polynomial $\overline{e} = \varepsilon\overline{x} + \alpha\overline{y} + \sum_i \alpha_i \overline{u_i}$ where $\overline{u_i} \in \mathscr{M}$, is idempotent in the ring R, then since $\overline{e}^2 = \varepsilon^2\overline{x} + \sum_i \beta_i \overline{u_i}$ for some $\beta_i \in P$, thus $\varepsilon \in E(P) = \{0, 1\}$ and $\alpha = 0$. We now assume that the polynomial $\overline{e}$ is nonzero with $\varepsilon = 0$, and that the lowest degree of the monomials $\alpha_i\overline{u_i}$ with $\alpha_i \neq 0$ is equal to n, obviously $n \geq 2$. But $\overline{e} = \overline{e}^2 = \sum_{i,j} \alpha_i\alpha_j \overline{u_i}\cdot\overline{u_j}$ and the lowest degree of the monomials $\alpha_i\alpha_j\overline{u_i}\cdot\overline{u_j}$ with $\alpha_i\alpha_j \neq 0$, is no smaller than $2n - 1 \geq n + 1$. This contradiction means that every nonzero idempotent polynomial in the ring R is expressed uniquely as $\overline{x} + \sum_i \alpha_i\overline{u_i}$ for some $\alpha_i \in P$ and some $\overline{u_i} \in \mathscr{M}$. In consequence, $\overline{e}\cdot\overline{f} = 0$ implies $\overline{e} = 0$ or $\overline{f} = 0$ for any $\overline{e}, \overline{f} \in E(R)$. This finally confirms that the ring R satisfies the ICZ property.

Simultaneously, $\overline{x}\cdot(\overline{x}+\overline{x}\cdot\overline{y}-\overline{x}\cdot\overline{y}\cdot\overline{x}) = \overline{x}+\overline{x}\cdot\overline{y}-\overline{x}\cdot\overline{y}\cdot\overline{x} \neq \overline{x} = (\overline{x}+\overline{x}\cdot\overline{y}-\overline{x}\cdot\overline{y}\cdot\overline{x})\cdot\overline{x}$, in spite of that $\overline{x} \in E(R)$. This means that the ring R is not abelian.

4. Extending to the formal power series ring

N. K. Kim and Y. Lee in [11, Lamma 8] proved that under the assumption on the abelianity of a ring R, every idempotent element in the ring $R[[x]]$ belongs in fact to the ring R (the proof of this lemma does not require R to be unital). In consequence, also the rings $R[[x]]$ and $R[x]$ are abelian, the latter as a subring in $R[[x]]$. The problem becomes more complicated when we leave the class of abelian rings. For instance, under the notation used in Example 3.6, the element $E + \sum_{n\geq 1}(A+B)x^n$ is idempotent in the ring $R[[x]]$.

Theorem 4.1 was in fact proved by N. K. Kim et al. in [12, Theorem 2.14]. Methods used by these authors are different from those will be used by us.

Theorem 4.1. *If in a ring R the equality $ere = er$ holds for any $e \in E(R)$ and $r \in R$, then the same equality holds also in the ring $R[[x]]$. If particular, if a ring R satisfies the right IR property, then also the rings $R[[x]]$ and $R[x]$ satisfy the right IR property.*

Proof. An element $\sum_{n\geq 0} e_n x^n$ is idempotent in the ring $R[[x]]$ if and only if

$$\sum_{i+j=n} e_i e_j = e_n \tag{5}$$

holds for every $n \geq 0$. Applying the mathematical induction on $n \geq 0$ we will prove that

$$e_i e_j = \begin{cases} e_i & \text{if } j = 0, \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

holds for any $i, j \in \{0, 1, \ldots, n\}$. The case when $n = 0$ follows immediately from the fact that $e_0 \in E(R)$ by (5). Suppose now that (6) holds for a fixed $n \geq 0$. Substituting (6) into (5) we obtain

$$e_0e_{n+1} + e_{n+1}e_0 = e_0e_{n+1} + \sum_{i=1}^{n} e_i e_{n+1-i} + e_{n+1}e_0 = \sum_{i=0}^{n+1} e_i e_{n+1-i} = e_{n+1}, \tag{7}$$

then right multiplying (7) by e_0 we obtain $e_0e_{n+1}e_0 = 0$, and hence

$$e_0 e_{n+1} = 0 \tag{8}$$

by the assumption. Substituting (8) into (7) we obtain

$$e_{n+1}e_0 = e_{n+1}. \tag{9}$$

From (6), (8) and (9) we now conclude that

$$e_i e_{n+1} = e_i e_0 e_{n+1} = 0$$

for every $i \in \{0, 1, \ldots, n+1\}$, and that

$$e_{n+1} e_j = e_{n+1} e_0 e_j = 0$$

for every $j \in \{1, 2, \ldots, n+1\}$.

For any $\overline{e} = \sum_{n\geq 0} e_n x^n \in E(R[[x]])$ and $\overline{r} = \sum_{n\geq 0} r_n x^n \in R[[x]]$ we now obtain

$$\begin{aligned}\overline{e}\cdot\overline{r}\cdot\overline{e} &= \sum_{n\geq 0}\Big(\sum_{i+j+k=n} e_i r_j e_k\Big)x^n = \sum_{n\geq 0}\Big(\sum_{i+j+k=n} e_i e_0 r_j e_k\Big)x^n\\ &= \sum_{n\geq 0}\Big(\sum_{i+j+k=n} e_i e_0 r_j e_0 e_k\Big)x^n = \sum_{n\geq 0}\Big(\sum_{i+j=n} e_i e_0 r_j e_0\Big)x^n\\ &= \sum_{n\geq 0}\Big(\sum_{i+j=n} e_i e_0 r_j\Big)x^n = \sum_{n\geq 0}\Big(\sum_{i+j=n} e_i r_j\Big)x^n = \overline{e}\cdot\overline{r}.\end{aligned}$$
□

Theorem 4.2. *Assume that in a ring R the property $ef \in E(R)$ holds for any $e, f \in E(R)$. Then the following statements are equivalent*:

1. $\overline{e}\cdot\overline{f} \in E(R[[x]])$ *holds for any* $\overline{e}, \overline{f} \in E(R[[x]])$;
2. $e\cdot\overline{f} \in E(R[[x]])$ *holds for any* $e \in E(R)$ *and* $\overline{f} \in E(R[[x]])$;
3. $ef_m e = 0$ *holds for any* $e \in E(R)$, $\sum_{n\geq 0} f_n x^n \in E(R[[x]])$ *and* $m \geq 1$.

Proof. The implication $1 \Rightarrow 2$ is obvious. In the proofs of both the implications $2 \Rightarrow 3$ and $3 \Rightarrow 1$, we let $e \in E(R)$ and $\sum_{n\geq 0} f_n x^n \in E(R[[x]])$. Then, as we know,

$$\sum_{i=0}^{n} f_i f_{n-i} = f_n$$

holds for every $n \geq 0$. In particular, $f_0 \in E(R)$, also $ef_0, f_0e \in E(R)$ by the assumption, and hence $e - ef_0e, f_0 - f_0ef_0 \in E(R)$. Since $(e - ef_0e)f_0 = 0$ and $(f_0 - f_0ef_0)e = 0$, from this it follows that

$$\begin{gathered}(e - ef_0e)Rf_0 = 0 \quad\text{and}\quad f_0R(e - ef_0e) = 0,\\ (f_0 - f_0ef_0)Re = 0 \quad\text{and}\quad eR(f_0 - f_0ef_0) = 0\end{gathered}\tag{10}$$

by Theorems 2.4 and 2.5. Applying the mathematical induction on $n \geq 0$ we will prove that

$$ef_n = ef_0ef_n \quad\text{and}\quad f_ne = f_nef_0e.\tag{11}$$

In the case when $n = 0$, the conclusion is evident. Suppose now that $(e - ef_0e)f_i = 0$ and $f_i(e - ef_0e) = 0$ hold for a fixed $n \geq 0$ and every $i \in \{0, 1, \ldots, n\}$. Then applying the induction hypothesis and (10) we conclude that

$$\begin{aligned}(e - ef_0e)f_{n+1} &= (e - ef_0e)\Big(\sum_{i=0}^{n} f_i f_{n+1-i} + f_{n+1}f_0\Big)\\ &= \sum_{i=0}^{n}(e - ef_0e)f_i f_{n+1-i} + (e - ef_0e)f_{n+1}f_0 = 0\end{aligned}$$

and

$$f_{n+1}(e - ef_0e) = \Big(f_0 f_{n+1} + \sum_{i=1}^{n+1} f_i f_{n+1-i}\Big)(e - ef_0e)$$

$$= f_0 f_{n+1}\big(e - ef_0 e\big) + \sum_{i=1}^{n+1} f_i f_{n+1-i}\big(e - ef_0 e\big) = 0.$$

In the proof of the implication $2 \Rightarrow 3$, according to the assumption, we have

$$e \cdot \sum_{n \geq 0} f_n x^n = e \cdot \sum_{n \geq 0} f_n x^n \cdot e \cdot \sum_{n \geq 0} f_n x^n = \sum_{n \geq 0} \big(\sum_{i=0}^{n} ef_i ef_{n-i}\big) x^n.$$

In combination with (11), this gives

$$ef_m = \sum_{i=0}^{m} ef_i ef_{m-i} = ef_0 ef_m + \sum_{i=1}^{m} ef_i ef_{m-i} = ef_m + \sum_{i=1}^{m} ef_i ef_{m-i}$$

for every $m \geq 1$. Thus

$$\sum_{i=1}^{m} ef_i ef_{m-i} = 0 \tag{12}$$

holds for every $m \geq 1$. Applying the mathematical induction on $m \geq 1$ we will prove that

$$ef_m e = 0.$$

In the case when $m = 1$, from (12) it follows that $ef_1 ef_0 = 0$, and thus $ef_1 e = ef_1 ef_0 e = 0$ by (11). Suppose now that $ef_i e = 0$ holds for a fixed $m \geq 1$ and every $i \in \{1, 2, \ldots, m\}$. Then applying the induction hypothesis and (12) we conclude that

$$ef_{m+1} ef_0 = \sum_{i=1}^{m} ef_i ef_{m+1-i} + ef_{m+1} ef_0 = \sum_{i=1}^{m+1} ef_i ef_{m+1-i} = 0,$$

and, in consequence, that

$$ef_{m+1} e = ef_{m+1} ef_0 e = 0$$

by (11).

In the proof of the implication $3 \Rightarrow 1$, since $ef_0, f_0 e \in E(R)$, according to the assumption, we have $ef_0 f_m ef_0 = 0$ and $f_0 ef_m f_0 e = 0$ for every $m \geq 1$. On right multiplying the former and left multiplying the latter of the equalities by e, and next applying (11) we obtain $ef_0 f_m e = 0$ and $ef_m f_0 e = 0$ respectively. From (10) we now see that

$$f_0 f_m e = f_0 f_m e - f_0 ef_0 f_m e = (f_0 - f_0 ef_0) f_m e = 0$$

and

$$ef_m f_0 = ef_m f_0 - ef_m f_0 ef_0 = ef_m (f_0 - f_0 ef_0) = 0.$$

Thus

$$ef_m e = 0, \quad f_0 f_m e = 0 \quad \text{and} \quad ef_m f_0 = 0 \tag{13}$$

hold for any $e \in E(R)$, $\sum_{n \geq 0} f_n x^n \in E(R[[x]])$ and $m \geq 1$. For every $r \in R$ with $ere = 0$, since $e + er \in E(R)$, we have $(e + er) f_m (e + er) = 0$ and $(e + er) f_m f_0 = 0$ by (13). On right multiplying the former of the equalities by

e, and next applying (13) we obtain $erf_me = 0$. On applying (13) in the latter of the equalities we obtain $erf_mf_0 = 0$. Thus

$$\text{if } dere = 0, \text{ then } erf_me = 0 \text{ and } erf_mf_0 = 0 \tag{14}$$
$$\text{hold for any } e \in E(R), \sum_{n\geq 0} f_nx^n \in E(R[[x]]) \text{ and } m \geq 1.$$

Finally, let $\overline{e} = \sum_{n\geq 0} e_nx^n$, $\overline{f} = \sum_{n\geq 0} f_nx^n \in E(R[[x]])$. Since $e_0, f_0 \in E(R)$, from this it follows that $e_0e_me_0 = 0$ and $f_0e_mf_0 = 0$ hold for every $m \geq 1$ by the assumption, and thus $e_0e_mf_nf_0 = 0$ and $f_0e_mf_nf_0 = 0$ hold for any $m, n \geq 1$ by (14). Simultaneously, $e_0e_mf_0 = 0$ and $f_0e_mf_0 = 0$ hold for every $m \geq 1$ by (13). This confirms that

$$e_0e_mf_nf_0 = 0 \quad \text{and} \quad f_0e_mf_nf_0 = 0 \tag{15}$$

hold for any $m \geq 1$ and $n \geq 0$. Applying the mathematical induction on $n \geq 0$ we will prove that

$$e_0e_mf_n = 0, \tag{16}$$

where $m \geq 1$. The case when $n = 0$ follows immediately from (15). Suppose now that $e_0e_mf_i = 0$ holds for a fixed $n \geq 0$ and every $i \in \{0, 1, \ldots, n\}$. Then applying the induction hypothesis and (15) we conclude that

$$\begin{aligned} e_0e_mf_{n+1} &= e_0e_m\Big(\sum_{i=0}^{n} f_if_{n+1-i} + f_{n+1}f_0\Big) \\ &= \sum_{i=0}^{n} e_0e_mf_if_{n+1-i} + e_0e_mf_{n+1}f_0 = 0. \end{aligned}$$

In the same way we may prove that also

$$f_0e_mf_n = 0 \quad \text{and} \quad e_0e_me_n = 0 \tag{17}$$

hold for any $m \geq 1$ and $n \geq 0$, the latter of the equalities being a consequence of the former with $\overline{e} = \overline{f}$. Once more applying the mathematical induction on $k \geq 0$ we will prove that

$$e_ke_mf_n = 0, \tag{18}$$

where $m \geq 1$ and $n \geq 0$. The case when $k = 0$ follows immediately from (16). Suppose now that $e_ie_mf_n = 0$ holds for a fixed $k \geq 0$ and every $i \in \{0, 1, \ldots, k\}$. Then applying the induction hypothesis and (17) we conclude that

$$\begin{aligned} e_{k+1}e_mf_n &= \Big(e_0e_{k+1} + \sum_{i=1}^{k+1} e_ie_{k+1-i}\Big)e_mf_n \\ &= e_0e_{k+1}e_mf_n + \sum_{i=1}^{k+1} e_ie_{k+1-i}e_mf_n = 0. \end{aligned}$$

In the same way we may prove that also

$$f_k e_m f_n = 0 \quad \text{and} \quad e_k f_m e_n = 0 \tag{19}$$

hold for any $m \geq 1$ and $k, n \geq 0$, the latter of the equalities being a consequence of the former with $\overline{e}$ and $\overline{f}$, which have swapped places with each other. From what has already been proved, it follows that

$$e_k f_l e_m f_n = 0 \tag{20}$$

holds for any $k, l, m, n \geq 0$ with $l + m \geq 1$.

In order to prove that $\overline{e} \cdot \overline{f} = \overline{e} \cdot \overline{f} \cdot \overline{e} \cdot \overline{f}$, it remains to proved that

$$\sum_{i+j=n} e_i f_j = \sum_{i+j+k+l=n} e_i f_j e_k f_l$$

holds for every $n \geq 0$. In the case when $n = 0$, the conclusion is evident. The case when $n = 1$ follows immediately from (11) and (20). Suppose now that $n \geq 2$ is fixed. Then applying (11), (18) and (20) we deduce that

$$\begin{aligned}
&\sum_{i+j+k+l=n} e_i f_j e_k f_l \\
&= \sum_{i+j+k+l=n,\ j+k\geq 1} e_i f_j e_k f_l + e_0 f_0 e_0 f_n + \sum_{m=1}^{n-1} e_m f_0 e_0 f_{n-m} + e_n f_0 e_0 f_0 \\
&= e_0 f_n + \sum_{m=1}^{n-1} \Big(\sum_{i=0}^{m-1} e_i e_{m-i} + e_m e_0 \Big) f_0 e_0 f_{n-m} + e_n f_0 \\
&= e_0 f_n + \sum_{m=1}^{n-1} \sum_{i=0}^{m-1} e_i e_{m-i} f_0 e_0 f_{n-m} + \sum_{m=1}^{n-1} e_m e_0 f_0 e_0 f_{n-m} + e_n f_0 \\
&= e_0 f_n + \sum_{m=1}^{n-1} e_m e_0 f_{n-m} + e_n f_0 \\
&= e_0 f_n + \sum_{m=1}^{n-1} \Big(e_m - \sum_{i=0}^{m-1} e_i e_{m-i} \Big) f_{n-m} + e_n f_0 \\
&= e_0 f_n + \sum_{m=1}^{n-1} e_m f_{n-m} - \sum_{m=1}^{n-1} \sum_{i=0}^{m-1} e_i e_{m-i} f_{n-m} + e_n f_0 \\
&= e_0 f_n + \sum_{m=1}^{n-1} e_m f_{n-m} + e_n f_0 = \sum_{i+j=n} e_i f_j.
\end{aligned}$$

□

Corollary 4.3. *If in a ring R the property $ef \in E(R)$ holds for any $e, f \in E(R)$, then the same property holds also in the rings $R[[x]]$ and $R[x]$.*

Proof. In the proof of the corollary, we let $e \in E(R)$ and $\sum_{n\geq 0} f_n x^n \in E(R[[x]])$. Then, as we know,

$$\sum_{i=0}^{n} f_i f_{n-i} = f_n$$

holds for every $n \geq 0$, and, in particular, $f_0 \in E(R)$. Applying the mathematical induction on $m \geq 1$ we will prove that

$$ef_0 f_m e = 0, \quad ef_m f_0 e = 0, \quad ef_m e = 0 \quad \text{and} \quad ef_i f_j e = 0, \tag{21}$$

where $e \in E(R)$, $\sum_{n\geq 0} f_n x^n \in E(R[[x]])$ and $i, j \in \{1, 2, \ldots, m\}$. In the case when $m = 1$, right multiplying $f_0 f_1 + f_1 f_0 = f_1$ by f_0 we have

$$f_0 f_1 f_0 = 0. \tag{22}$$

From this it follows that $f_0 + f_0 f_1, f_0 + f_1 f_0 \in E(R)$, and also $e(f_0 + f_0 f_1), (f_0 + f_1 f_0)e \in E(R)$ by the assumption. On right multiplying $e(f_0 + f_0 f_1)e(f_0 + f_0 f_1) = e(f_0 + f_0 f_1)$ by $f_0 e$, left multiplying $(f_0 + f_1 f_0)e(f_0 + f_1 f_0)e = (f_0 + f_1 f_0)e$ by ef_0, and next applying (11) and (22) we obtain

$$ef_0 f_1 e = 0 \quad \text{and} \quad ef_1 f_0 e = 0$$

respectively. From this it follows that

$$ef_1 e = e(f_0 f_1 + f_1 f_0)e = 0, \tag{23}$$

and thus $e + ef_1 \in E(R)$. But (23) holds for every $e \in E(R)$. On replacing e by $e + ef_1$ in (23), next right multiplying $(e + ef_1)f_1(e + ef_1) = 0$ by e, and finally applying (23) we obtain

$$ef_1 f_1 e = 0.$$

Suppose now that (21) holds for a fixed $m \geq 1$. Then both right and left multiplying $f_0 f_{m+1} + \sum_{i=1}^{m} f_i f_{m+1-i} + f_{m+1} f_0 = f_{m+1}$ by f_0, and next applying the induction hypothesis we have

$$f_0 f_{m+1} f_0 = 0. \tag{24}$$

In the same way as above we may prove that

$$ef_0 f_{m+1} e = 0 \quad \text{and} \quad ef_{m+1} f_0 e = 0.$$

From this and the induction hypothesis it follows that

$$ef_{m+1} e = e\big(f_0 f_{m+1} + \sum_{i=1}^{m} f_i f_{m+1-i} + f_{m+1} f_0\big)e = 0.$$

In this way

$$ef_j e = 0 \tag{25}$$

holds for any $e \in E(R)$ and $j \in \{1, 2, \ldots, m+1\}$, and thus $e + ef_i \in E(R)$ holds for every $i \in \{1, 2, \ldots, m+1\}$. On replacing e by $e + ef_i$ in (25), next right multiplying $(e + ef_i)f_j(e + ef_i) = 0$ by e, and finally applying (25) we obtain

$$ef_if_je = 0$$

for any $i, j \in \{1, 2, \dots, m+1\}$.

The corollary now follows immediately from Theorem 4.2. $\square$

Theorem 4.4. *Assume that a ring R satisfies the ICZ property. Then for any $\overline{e} = \sum_{n\geq 0} e_nx^n, \overline{f} = \sum_{n\geq 0} f_nx^n \in E(R[[x]])$, the following statements are equivalent*:

1. $\overline{e} \cdot \overline{f} = 0$;
2. $e_0f_0 = 0$;
3. $e_if_j = 0$ *holds for any* $i, j \geq 0$.

Proof. The implications $1 \Rightarrow 2$ and $3 \Rightarrow 1$ are obvious. In the proof of the implication $2 \Rightarrow 3$, we let $\sum_{n\geq 0} e_nx^n, \sum_{n\geq 0} f_nx^n \in E(R[[x]])$ with $e_0f_0 = 0$. Then, as we know,

$$\sum_{i=0}^{n} e_ie_{n-i} = e_n \quad \text{and} \quad \sum_{j=0}^{n} f_jf_{n-j} = f_n$$

hold for every $n \geq 0$. Applying the mathematical induction on $n \geq 0$ we will prove that

$$e_if_j = 0 \tag{26}$$

holds for any $i, j \in \{0, 1, \dots, n\}$. The case when $n = 0$ follows immediately from the assumption. Suppose now that (26) holds for a fixed $n \geq 0$. Applying the mathematical induction on $k \in \{0, 1, \dots, n\}$ we first will prove that

$$e_kf_{n+1} = 0.$$

In the case when $k = 0$, left multiplying $\sum_{j=0}^{n} f_jf_{n+1-j} + f_{n+1}f_0 = f_{n+1}$ by e_0, and next applying the induction hypothesis and Theorem 2.5 we have $e_0f_{n+1} = 0$. Suppose now that $e_if_{n+1} = 0$ holds for a fixed $k \in \{0, 1, \dots, n-1\}$ and every $i \in \{0, 1, \dots, k\}$. Then left multiplying $\sum_{j=0}^{n} f_jf_{n+1-j} + f_{n+1}f_0 = f_{n+1}$ by e_{k+1}, right multiplying $e_0e_{k+1} + \sum_{i=1}^{k+1} e_ie_{k+1-i} = e_{k+1}$ by f_{n+1}, and then applying the induction hypothesis we obtain

$$e_{k+1}f_{n+1}f_0 = e_{k+1}f_{n+1} \quad \text{and} \quad e_0e_{k+1}f_{n+1} = e_{k+1}f_{n+1}$$

respectively. From this and Theorem 2.5 we now conclude that

$$e_{k+1}f_{n+1} = e_{k+1}f_{n+1}f_0 = e_0e_{k+1}f_{n+1}f_0 = 0.$$

In the same way we may now prove that $e_{n+1}f_m = 0$ holds for every $m \in \{0, 1, \dots, n\}$, and also $e_{n+1}f_{n+1} = 0$. $\square$

Corollary 4.5. *If a ring R satisfies the ICZ property, then also the rings $R[[x]]$ and $R[x]$ satisfy the ICZ property.*

Proof. The corollary is a simple consequence of Theorem 4.4. $\square$

A ring R is said to be *Armendariz* if whenever polynomials $f(x) = f_0 + f_1x + \cdots + f_mx^m, g(x) = g_0 + g_1x + \cdots + g_nx^n \in R[x]$ satisfy $f(x) \cdot g(x) = 0$, then $f_ig_j = 0$ for any $i \in \{0, 1, \ldots, m\}$ and $j \in \{0, 1, \ldots, n\}$. E. P. Armendariz in [2, Lemma 1] proved that reduced rings with unit are Armendariz.

Theorem 4.6. *For every ring R, the following statements are equivalent*:

1. *R satisfies the ICZ property*;
2. *if polynomials $e(x) = e_0 + e_1x + \cdots + e_mx^m, f(x) = f_0 + f_1x + \cdots + f_nx^n \in E(R[x])$ satisfy $e(x) \cdot f(x) = 0$, then $e_if_j = 0$ holds for any $i \in \{0, 1, \ldots, m\}$ and $j \in \{0, 1, \ldots, n\}$*;
3. *if polynomials $e(x) = e_0 + e_1x, f(x) = f_0 + f_1x \in E(R[x])$ satisfy $e(x) \cdot f(x) = 0$, then $e_if_j = 0$ holds for any $i, j \in \{0, 1\}$.*

Proof. The implication $1 \Rightarrow 2$ is a simple consequence of Theorem 4.4. The implication $2 \Rightarrow 3$ is obvious. In the proof of the implication $3 \Rightarrow 1$, we let $e, f \in E(R)$ with $ef = 0$. Then $f + fex, e - fe - fex \in E(R[x])$, and since $(f + fex)(e - fe - fex) = 0$, from this it follows that also $fe = 0$ by the assumption. □

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