

THE SECOND-ORDER STABILIZED GAUGE-UZAWA METHOD FOR INCOMPRESSIBLE FLOWS WITH VARIABLE DENSITY

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ABSTRACT. The Navier-Stokes equations with variable density are challenging problems in numerical analysis community. We recently built the 2nd order stabilized Gauge-Uzawa method [SGUM] to solve the Navier-Stokes equations with constant density and have estimated theoretically optimal accuracy. Also we proved that SGUM is unconditionally stable. In this paper, we apply SGUM to the Navier-Stokes equations with nonconstant variable density and find out the stability condition of the algorithms. Because the condition is rather strong to apply to real problems, we consider Allen-Cahn scheme to construct unconditionally stable scheme.

1. Introduction

The Navier-Stokes equations [NSE] with variable density describe fluid motion due to density differences in mixture of several fluid, like water, oil, and air. Given a bounded polygon Ω in \mathbb{R}^d with $d = 2$ or 3 ,

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we consider the NSE with variable density defined in $\Omega \times (0, T]$:

$$(1.1) \quad \begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho &= 0, \\ \rho(\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0, \end{aligned}$$

where the unknowns ρ , \mathbf{u} and p are the density, the velocity field and the pressure, respectively; μ is the dynamic viscosity, \mathbf{f} represents the external force and $T > 0$ is a fixed time. The initial and boundary conditions of the system (1.1) for \mathbf{u} and ρ are given as

$$(1.2) \quad \begin{cases} \rho(\mathbf{x}, 0) = \rho_0 & \text{in } \Omega & \text{and} & \rho(\mathbf{x}, t) = r(\mathbf{x}, t) & \text{on } \Gamma_{\mathbf{u}(\mathbf{x}, t)}, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0 & \text{in } \Omega & \text{and} & \mathbf{u}(\mathbf{x}, t) = \mathbf{u}_b(\mathbf{x}, t) & \text{on } \Gamma, \end{cases}$$

where Γ is boundary of Ω , and $\Gamma_{\mathbf{v}}$ is the inflow boundary defined by, for any velocity field \mathbf{v} ,

$$\Gamma_{\mathbf{v}} := \{\mathbf{x} \in \Gamma : \mathbf{v}(\mathbf{x}) \cdot \boldsymbol{\nu} < 0\}$$

with $\boldsymbol{\nu}$ being the outward unit normal vector. The forcing function \mathbf{f} is given, the nondimensional number $\mu = Re^{-1}$ is reciprocal of the Reynolds and $\int_{\Omega} p = 0$.

Since the saddle point approximation requires high computational cost, projection type methods are popularly used in real computation areas. But it is difficult to apply the methods to solve (1.1)-(1.2), because of the nonlinear terms constructed by the multiplication of non-constant density function. The well-known skew-symmetry equation

$$\int_{\Omega} (\rho_0 \mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} d\mathbf{x} = 0, \quad \text{for } \mathbf{u}, \mathbf{v} \text{ smooth enough and } \mathbf{u} \cdot \boldsymbol{\nu}|_{\Gamma} = 0,$$

for constant density ρ_0 , is a crucial tool to analyze NSE with constant density. But this property does not hold anymore because ρ is not a constant in (1.1). So we need the following equations in [3, 15] to treat the convection term: let $\nabla \cdot \mathbf{u} = 0$ in Ω and $\mathbf{u} \cdot \boldsymbol{\nu} = 0$ on Γ ,

$$(1.3) \quad \int_{\Omega} (\rho \mathbf{u} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} d\mathbf{x} + \frac{1}{2} \int_{\Omega} \nabla \cdot (\rho \mathbf{u}) \mathbf{v} \cdot \mathbf{v} d\mathbf{x} = 0,$$

$$(1.4) \quad \int_{\Omega} \mathbf{u} \cdot \nabla \rho \cdot \rho d\mathbf{x} = 0, \quad \text{and} \quad \int_{\Omega} \rho \nabla \cdot \mathbf{u} \rho d\mathbf{x} = 0.$$

If we denote $\sigma = \sqrt{\rho}$

$$\sigma(\sigma \mathbf{u})_t = \rho \mathbf{u}_t + \frac{1}{2} \rho_t \mathbf{u} = \rho \mathbf{u}_t - \frac{1}{2} \nabla \cdot (\rho \mathbf{u}) \mathbf{u},$$

then we can readily get the following system in conserved form in [3]:

$$(1.5) \quad \begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho + \frac{\rho}{2} \nabla \cdot \mathbf{u} &= 0, \\ \sigma(\sigma \mathbf{u})_t + \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{2} \nabla \cdot (\rho \mathbf{u}) \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} &= \mathbf{f}, \\ \nabla \cdot \mathbf{u} &= 0. \end{aligned}$$

We note here that the system (1.5) is equivalent to the original system (1.1). The main advantage of the conserved form (1.5) is, by (1.3),

$$\left\langle \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{2} \nabla \cdot (\rho \mathbf{u}) \mathbf{u}, \mathbf{u} \right\rangle = 0.$$

This vanishment is crucial to analyzing algorithms in §3.

the Gauge-Uzawa method which is introduced in [11, 12] is the first order unconditionally stable algorithm for NSE with constant density. The method is applied to solve the variable density problem (1.1) and proved that the method is unconditionally stable in [15]. Guermond and Salgado construct the second order fractional time-stepping method in [5] to solve (1.1). The method is based on the pressure correction method which is constructed in [18] and is estimated errors and stability condition for the Stokes equations in [6].

We recently construct the second order stabilized Gauge-Uzawa method (SGUM) in [13] to solve NSE with constant density. We proved that SGUM is unconditionally stable and is the optimal accuracy algorithm. The goal of this paper is to introduce new algorithms for solving (1.1) based on SGUM and to prove the stability without adjusting of the algorithm. Unfortunately, Algorithm 1 and 2 that will be introduced in Section 2 are not unconditionally stable, thus we construct a new unconditionally stable algorithm applying the Allen-Cahn scheme induced in [9]. In order to avoid mass conservation problem of Allen-Cahn scheme, we hire the Lagrange multiplier which is presented in [16].

The paper is composed of follows. We first introduce the algorithms that play with our paper in Section 2 and then prove their stability condition in Section 3. Next, we present the finite element discretization in Section 4. In Section 5, we show some numerical results which bring out the convergence rate of our schemes for (\mathbf{u}, p, ρ) and exhibit numerical simulations of the Rayleigh-Taylor instability compared with [3]. finally, we conclude some remarks in the last section.

2. The 2nd order SGUMs to solve NSE with variable density

We start this section with definition of notations: τ is the time marching size and δ is difference of 2 consecutive functions, for example, for any sequence functions z^{n+1} ,

$$\delta z^{n+1} = z^{n+1} - z^n, \quad \delta \delta z^{n+1} = \delta(\delta z^{n+1}) = z^{n+1} - 2z^n + z^{n-1}, \quad \dots$$

We now introduce assumptions for the initial conditions of (1.1) and the regularity of ρ :

ASSUMPTION 1. There exist positive constants χ and ϱ satisfying

$$\begin{aligned} \rho_0 &\in L^\infty(\Omega), \quad 0 < \chi \leq \rho_0 \leq \varrho, \\ \mathbf{u}_0 &\in \mathbf{L}^2(\Omega). \end{aligned}$$

In addition, we assume upper bounds of σ^n and ρ^n for numerical solution.

ASSUMPTION 2. There exists a positive constant M such that

$$\begin{aligned} \max_{0 \leq n \leq N-1} \left(\left\| \frac{\sigma^{n+1} - \sigma^n}{\sigma^n} \right\|_{L^\infty(\Omega)}^2, \left\| \frac{\rho^{n+1} - \rho^n}{\rho^n} \right\|_{L^\infty(\Omega)} \right) &\leq M\tau, \\ \max_{0 \leq n \leq N-1} \left\| \frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{\rho^{n+1}} \right\|_{L^\infty(\Omega)} &\leq M\tau, \\ \max_{0 \leq n \leq N-1} \left(\left\| \frac{\sigma^{n+1} - \sigma^n}{\sigma^{n+1}} \right\|_{L^\infty(\Omega)}^2, \left\| \frac{\rho^{n+1} - \rho^n}{\rho^{n+1}} \right\|_{L^\infty(\Omega)} \right) &\leq M\tau. \end{aligned}$$

The Assumption 2 is rather strong condition, but it is still an open problem to construct unconditionally stable second order methods to solve (1.1) and all other papers on this topic hire similar assumptions. In [5], they use two hypotheses that they restrict the bound of ρ^n :

$$(2.1) \quad \chi \leq \min_{\mathbf{x} \in \Omega} \rho^n(\mathbf{x}), \quad \max_{\mathbf{x} \in \Omega} \rho^n(\mathbf{x}) \leq \varrho,$$

and there is a uniform constant M so that

$$(2.2) \quad \max_{0 \leq n \leq N-1} \left\| \frac{\rho^{n+1} - \rho^n}{\tau} \right\|_{L^\infty} \leq M\chi.$$

They, in [10], proved that for the staggered discretization mesh if $\chi \leq \rho^n \leq \varrho$ holds, then

$$\chi \leq \rho^{n+1} \leq \varrho, \quad n = 1, \dots, N.$$

So we assert that Assumption 2 is reasonable and similar to (2.2) if ρ^n is bounded and τ is small enough.

From now, we construct two SGUM algorithms to solve (1.1), so called the convective form in §2.1 and the conserved form in §2.2. We also introduce the fractional time-stepping method constructed in [5] to compare numerical results with new algorithms. In order to remove Assumption 2, we consider Allen-Cahn method which is performed unconditionally stable in §2.4.

2.1. Convective Form. We impose the 2nd order backward Euler formula [BDF2] for density equation and SGUM for NSE in [14] to obtain the following convective form of SGUM:

ALGORITHM 1 (*SGUM in convective form*). For $\rho^0 = \rho_0$, $\mathbf{u}^0 = \mathbf{u}_0$ and $s^0 = 0$, compute \mathbf{u}^1 , ρ^1 and p^1 via any first order projection method. Set $\psi^1 = -\frac{2\tau}{3}p^1$ and $s^1 = 0$ and then repeat for $2 \leq n \leq N \leq T/\tau - 1$:

Step 1: Set $\bar{\mathbf{u}}^{n+1} = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ and find ρ^{n+1} as the solution of

$$(2.3) \quad \frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{2\tau} + \bar{\mathbf{u}}^{n+1} \cdot \nabla \rho^{n+1} = 0, \\ \rho^{n+1}|_{\Gamma_{\mathbf{u}^n}} = r^{n+1}.$$

Step 2: Find $\hat{\mathbf{u}}^{n+1}$ as the solution of

$$(2.4) \quad \rho^{n+1} \frac{3\hat{\mathbf{u}}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\tau} + \rho^{n+1}(\bar{\mathbf{u}}^{n+1} \cdot \nabla)\hat{\mathbf{u}}^{n+1} + \nabla p^n - \mu \Delta \hat{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1}, \\ \hat{\mathbf{u}}^{n+1}|_{\Gamma} = \mathbf{u}_b.$$

Step 3: Find ψ^{n+1} as the solution of

$$(2.5) \quad -\nabla \cdot \left(\frac{1}{\rho^{n+1}} \nabla (\psi^{n+1} - \psi^n) \right) = \nabla \cdot \hat{\mathbf{u}}^{n+1}, \\ \partial_\nu \psi^{n+1}|_{\Gamma} = 0.$$

Step 4: Update \mathbf{u}^{n+1} and s^{n+1} by

$$(2.6) \quad \mathbf{u}^{n+1} = \hat{\mathbf{u}}^{n+1} + \frac{1}{\rho^{n+1}} \nabla (\psi^{n+1} - \psi^n), \\ s^{n+1} = s^n - \nabla \cdot \hat{\mathbf{u}}^{n+1}.$$

Step 5: Update p^{n+1} by

$$(2.7) \quad p^{n+1} = -\frac{3}{2\tau} \psi^{n+1} + \mu s^{n+1}.$$

We will prove the following stability result of Algorithm 1 in §3. For the sake of simplicity, we consider only homogeneous Dirichlet boundary conditions for the velocity, i.e., $\mathbf{u}|_\Gamma = \mathbf{0}$.

THEOREM 2.1 (Stability of SGUM in convective form). *If Assumption 1 is satisfied, then the Algorithm 1 holds the following a priori bounds, for all $\tau > 0$ and $1 \leq N \leq T/\tau - 1$,*

$$(2.8) \quad \|\rho^{N+1}\|_0^2 + \|2\rho^{N+1} - \rho^N\|_0^2 + \sum_{n=1}^N \|\delta\delta\rho^{n+1}\|_0^2 = \|\rho^1\|_0^2 + \|2\rho^1 - \rho^0\|_0^2.$$

If Assumption 2 holds, then we have

$$(2.9) \quad \begin{aligned} & \|\sigma^{N+1}\mathbf{u}^{N+1}\|_0^2 + \|\sigma^{N+1}(2\mathbf{u}^{N+1} - \mathbf{u}^N)\|_0^2 + 3\left\|\frac{1}{\sigma^{N+1}}\nabla\psi^{N+1}\right\|_0^2 \\ & + \sum_{n=1}^N \left(\|\sigma^{n+1}\delta\delta\mathbf{u}^{n+1}\|_0^2 + \left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2 \right) \\ & + 2\mu\tau\|s^{N+1}\|_0^2 + \sum_{n=1}^N \mu\tau\|\nabla\hat{\mathbf{u}}^{n+1}\|_0^2 \leq \mathbf{e}^{CMT} \frac{C\tau}{\mu} \sum_{n=1}^N \|\mathbf{f}^{n+1}\|_{-1}^2 \\ & + \mathbf{e}^{CMT} \left(\|\sigma^1\mathbf{u}^1\|_0^2 + \|\sigma^1(2\mathbf{u}^1 - \mathbf{u}^0)\|_0^2 + 3\left\|\frac{1}{\sigma^1}\nabla\psi^1\right\|_0^2 + 2\mu\tau\|s^1\|_0^2 \right). \end{aligned}$$

2.2. Conserved Form. We impose time discretization on equations (1.5) to get the conserved form of SGUM:

ALGORITHM 2 (SGUM in conserved form). For $\rho^0 = \rho_0$, $\mathbf{u}^0 = \mathbf{u}_0$ and $s^0 = 0$, compute \mathbf{u}^1 , ρ^1 and p^1 via any first order projection method. Set $\psi^1 = -\frac{2\tau}{3}p^1$ and $s^1 = 0$ and then repeat for $2 \leq n \leq N \leq T/\tau - 1$:

Step 1: Set $\bar{\mathbf{u}}^{n+1} = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ and find ρ^{n+1} as the solution of (2.3)

Step 2: Find $\hat{\mathbf{u}}^{n+1}$ as the solution of

$$(2.10) \quad \begin{aligned} & \sigma^{n+1} \frac{3\sigma^{n+1}\hat{\mathbf{u}}^{n+1} - 4\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1}}{2\tau} + \rho^{n+1}(\bar{\mathbf{u}}^{n+1} \cdot \nabla)\hat{\mathbf{u}}^{n+1} \\ & + \frac{1}{2}\nabla \cdot (\rho^{n+1}\bar{\mathbf{u}}^{n+1})\hat{\mathbf{u}}^{n+1} + \nabla p^n - \mu\Delta\hat{\mathbf{u}}^{n+1} = \mathbf{f}^{n+1}, \\ & \hat{\mathbf{u}}^{n+1}|_\Gamma = \mathbf{u}_b. \end{aligned}$$

Step 3: Find ψ^{n+1} as the solution of (2.5)

Step 4: Update \mathbf{u}^{n+1} and s^{n+1} by (2.6)

Step 5: Update p^{n+1} by (2.7)

The following theorem is the stability result of Algorithm 2 and is proved in §3.

THEOREM 2.2 (Stability of SGUM in conserved form). *If Assumption 1 is satisfied, then the Algorithm 2 holds the following a priori bounds, for all $\tau > 0$ and $1 \leq N \leq T/\tau - 1$,*
(2.11)

$$\|\rho^{N+1}\|_0^2 + \|2\rho^{N+1} - \rho^N\|_0^2 + \sum_{n=1}^N \|\delta\delta\rho^{n+1}\|_0^2 = \|\rho^1\|_0^2 + \|2\rho^1 - \rho^0\|_0^2.$$

And if Assumption 2 holds, then we have
(2.12)

$$\begin{aligned} & \|\sigma^{N+1}\mathbf{u}^{N+1}\|_0^2 + \|2\sigma^{N+1}\mathbf{u}^{N+1} - \sigma^N\mathbf{u}^N\|_0^2 + 3\left\|\frac{1}{\sigma^{N+1}}\nabla\psi^{N+1}\right\|_0^2 \\ & + \sum_{n=1}^N \left(\|\sigma^{n+1}\hat{\mathbf{u}}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1}\|_0^2 + \left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2 \right) \\ & + 2\mu\tau\|s^{N+1}\|_0^2 + \mu\tau \sum_{n=1}^n \|\nabla\hat{\mathbf{u}}^{n+1}\|_0^2 \leq e^{CMT} \frac{\tau}{\mu} \sum_{n=1}^N \|\mathbf{f}^{n+1}\|_{-1}^2 \\ & + e^{CMT} \left(\|\sigma^1\mathbf{u}^1\|_0^2 + \|2\sigma^1\mathbf{u}^1 - \sigma^0\mathbf{u}^0\|_0^2 + 3\left\|\frac{1}{\sigma^1}\nabla\psi^1\right\|_0^2 + 2\mu\tau\|s^1\|_0^2 \right). \end{aligned}$$

2.3. The fractional time-stepping method. Guermond and Salgado established the second order method called the fractional time-stepping method in order to solve (1.1) in [5]. They assumed the stability hypothesis (2.1). In order to use (1.3), they treat (1.1) with the following process. Since density equation can be changed by $\rho_t + \nabla \cdot (\rho\mathbf{u}) = 0$, we get

$$(2.13) \quad \frac{1}{2}\rho_t\mathbf{u} + \frac{1}{2}\nabla \cdot (\rho\mathbf{u}) \cdot \mathbf{u} = 0$$

by multiplying $\frac{\mathbf{u}}{2}$. Then, we have the following equation by adding (2.13) to a momentum equation of (1.1):

$$(2.14) \quad \rho\mathbf{u}_t + \frac{1}{2}\rho_t\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{2}(\nabla \cdot (\rho\mathbf{u}))\mathbf{u} + \nabla p - \mu\Delta\mathbf{u} = \mathbf{f}.$$

By applying the splitting method and BDF2 idea to (2.14), they construct the following algorithm:

ALGORITHM 3 (*the fractional time-stepping method [5]*). For $\rho^0 = \rho_0$, $\mathbf{u}^0 = \mathbf{u}_0$ and $s^0 = 0$, compute \mathbf{u}^1 , ρ^1 and p^1 via any first order projection method. Set $\psi^1 = -\frac{2\tau}{3}p^1$ and $s^1 = 0$ and then repeat for $2 \leq n \leq N \leq T/\tau - 1$:

Step 1: Set $\bar{\mathbf{u}}^{n+1} = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ and compute ρ^{n+1} using any order 2 algorithm satisfying (2.1) and (2.2).

Step 2: Define

$$\begin{aligned}\rho^* &:= \frac{3}{2}\rho^{n+1} - \frac{2}{3}\rho^n + \frac{1}{6}\rho^{n-1} = \rho^{n+1} + \frac{1}{6}(3\rho^{n+1} - 4\rho^n + \rho^{n-1}), \\ p^\# &:= p^n + \frac{4}{3}\psi^n - \frac{1}{3}\psi^{n-1}.\end{aligned}$$

Then find \mathbf{u}^{n+1} as the solution of

$$\begin{aligned}\frac{3\rho^*\mathbf{u}^{n+1} - 4\rho^{n+1}\mathbf{u}^n + \rho^{n+1}\mathbf{u}^{n-1}}{2\tau} + \rho^{n+1}(\bar{\mathbf{u}}^{n+1} \cdot \nabla)\mathbf{u}^{n+1} \\ + \frac{1}{2}\nabla \cdot (\rho^{n+1}\bar{\mathbf{u}}^{n+1})\mathbf{u}^{n+1} + \nabla p^\# - \mu\Delta\mathbf{u}^{n+1} = \mathbf{f}^{n+1}, \\ \mathbf{u}^{n+1}|_\Gamma = \mathbf{u}_b.\end{aligned}$$

Step 3: Find ψ^{n+1} as the solution of

$$-\Delta\psi^{n+1} = -\frac{3\chi}{2\tau}\nabla \cdot \mathbf{u}^{n+1}.$$

Step 4: Update p^{n+1} by

$$p^{n+1} = p^n + \psi^{n+1}.$$

2.4. Allen-Cahn Form. All algorithms mentioned above need rather strong stability condition. In order to overcome the weak stability, we consider the Allen-Cahn scheme which is introduced in [1]. To separate the areas of two fluids, they introduce a phase function ϕ such that

$$\phi(\mathbf{x}, t) = \begin{cases} 1 & \text{fluid 1,} \\ -1 & \text{fluid 2,} \end{cases}$$

with a smooth layer η connecting the two fluids. The set $\{\mathbf{x}|\phi(\mathbf{x}, t) = 0\}$ represents the interface. Let us define the Ginzburg-Landau energy functional

$$W(\phi) = \int_{\Omega} \left(\frac{1}{2}|\nabla\phi|^2 + F(\phi) \right) dx,$$

where $F(\phi) = \frac{1}{4\varepsilon^2}(\phi^2 - 1)^2$. The parameter ε is the gradient energy coefficient related to the interfacial width. We can construct the Allen-Cahn phase equation which is the dynamics of phase function:

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = \gamma \frac{\delta W}{\delta \phi} = -\gamma(\Delta\phi - f(\phi)),$$

where $f(\phi) = F'(\phi)$ and γ is a constant mobility. We take $\gamma \equiv 1$ for convenience.

In [9], they suggested the following splitting scheme to construct unconditionally stable algorithm for solving $\phi_t = -(\Delta\phi - f(\phi))$:

$$(2.15) \quad \frac{\phi^* - \phi^n}{\Delta t} = \frac{1}{2}(\Delta\phi^* + \Delta\phi^n),$$

$$(2.16) \quad \frac{\phi^{n+1} - \phi^*}{\Delta t} = \frac{\phi^{n+1} - (\phi^{n+1})^3}{\varepsilon^2}.$$

The equation (2.16) is a numerical time discretization of the ordinary differential equation:

$$(2.17) \quad \phi_t = \frac{\phi - \phi^3}{\varepsilon^2}$$

with the initial condition ϕ^* . The equation (2.17) can be solved analytically by separation of variables. Therefore the system (2.15) and (2.16) can be rewritten by the following system which is called the second-order hybrid numerical method:

$$(2.18) \quad \begin{aligned} \frac{\phi^* - \phi^n}{\Delta t} &= \frac{1}{2}(\Delta\phi^* + \Delta\phi^n), \\ \phi^{n+1} &= \frac{\phi^*}{\sqrt{e^{-\frac{2\Delta t}{\varepsilon^2}} + (\phi^*)^2(1 - e^{-\frac{2\Delta t}{\varepsilon^2}})}}. \end{aligned}$$

It is well known that the system (2.18) does not hold mass conservation. So, we hire a Lagrange multiplier ξ introduced in [16] to make the phase function satisfy

$$\frac{d}{dt} \int_{\Omega} \phi(\mathbf{x}, t) dx = 0$$

or

$$\int_{\Omega} \phi(\mathbf{x}, t) dx = \xi.$$

Since these idea can be readily imposed all above algorithms, we apply only SGUM in convective form as follows:

ALGORITHM 4 (*SGUM in convective form applying the Allen-Cahn scheme*). For $\phi^0 = \phi_0$, $\mathbf{u}^0 = \mathbf{u}_0$ and $s^0 = 0$, compute \mathbf{u}^1 , ϕ^1 and p^1 . Set $\psi^1 = -\frac{2\tau}{3}p^1$ and $s^1 = 0$ and then repeat for $2 \leq n \leq N \leq T/\tau - 1$:

Step 1: Set $\bar{\mathbf{u}}^{n+1} = 2\mathbf{u}^n - \mathbf{u}^{n-1}$ and $\xi^n = \frac{1}{\tau|\Omega|} \int_{\Omega} \phi^n d\mathbf{x}$, find ϕ^* as the solution of

$$(2.19) \quad \frac{\phi^* - \phi^n}{\tau} + \frac{1}{2}(\bar{\mathbf{u}}^{n+1} \cdot \nabla \phi^* + \bar{\mathbf{u}}^{n+1} \cdot \nabla \phi^n) = \frac{1}{2}(\Delta \phi^* + \Delta \phi^n) - \xi^n.$$

Then, update ϕ^{n+1} by

$$\phi^{n+1} = \frac{\phi^*}{\sqrt{e^{-\frac{2\tau}{\varepsilon^2}} + (\phi^*)^2(1 - e^{-\frac{2\tau}{\varepsilon^2}})}}.$$

Step 2: Update ρ^{n+1} by

$$\rho^{n+1} = \frac{\rho_M + \rho_m}{2} + \frac{\rho_M - \rho_m}{2} \phi^{n+1},$$

where $\rho_M = \max \{\rho\}$ and $\rho_m = \min \{\rho\}$.

Step 3: Find $\hat{\mathbf{u}}^{n+1}$ as the solution of (2.4).

Step 4: Find ψ^{n+1} as the solution of (2.5).

Step 5: Update \mathbf{u}^{n+1} and s^{n+1} by (2.6).

Step 6: Update p^{n+1} by (2.7).

REMARK 2.3. Recall that ϕ is a piecewise constant function valued -1 or 1 . In computation, interfaces of ϕ are connected by a rotationally symmetric function. So,

$$\int_{\Omega} \left[\frac{1}{2}(\bar{\mathbf{u}}^{n+1} \cdot \nabla \phi^* + \bar{\mathbf{u}}^{n+1} \cdot \nabla \phi^n) - \frac{1}{2}(\Delta \phi^* + \Delta \phi^n) \right] dx = 0.$$

The Lagrange multiplier ξ^n acts to correct the computation error and keep the volume of ϕ^* to constant.

3. The Stability Analysis

In this section, we prove Theorems 2.1 and 2.2. We start this section with introduce well-known lemmas:

LEMMA 3.1 (Orthogonality between divergence free and curl free functions). *Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and $q \in L^2(\Omega)$. If $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} \cdot \nu = 0$ on $\partial\Omega$, then*

$$(3.1) \quad \langle \mathbf{u}, \nabla q \rangle = 0.$$

In [17], we can find out the following crucial inequality.

LEMMA 3.2 (div-grad relation). *If $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, then*

$$\|\nabla \cdot \mathbf{v}\|_0 \leq \|\nabla \mathbf{v}\|_0.$$

In order to treat time derivative terms, we will use the well-known lemma in [3, 5, 6, 13].

LEMMA 3.3 (Inner product of time derivative terms). *For any sequence $\langle z^n \rangle$, we have*

$$(3.2) \quad \begin{aligned} 2 \langle 3z^{n+1} - 4z^n + z^{n-1}, z^{n+1} \rangle &= \delta \|z^{n+1}\|^2 + \delta \|2z^{n+1} - z^n\|^2 + \|\delta z^{n+1}\|^2, \\ 2 \langle z^{n+1} - z^n, z^{n+1} \rangle &= \|z^{n+1}\|^2 - \|z^n\|^2 + \|z^{n+1} - z^n\|^2, \end{aligned}$$

and

$$2 \langle z^{n+1} - z^n, z^n \rangle = \|z^{n+1}\|^2 - \|z^n\|^2 - \|z^{n+1} - z^n\|^2.$$

We now introduce the discrete Gronwall lemma.

LEMMA 3.4 (Discrete Gronwall inequality). *Let $\langle a^n \rangle$ and $\langle b^n \rangle$ be non-negative sequences, and let c be a non-negative real number. If we have*

$$a^{n+1} \leq c + \sum_{k=0}^n a^k b^k,$$

then

$$a^{n+1} \leq c \exp \left(\sum_{k=0}^n b^k \right).$$

At last, we will prove Theorem 2.1 and 2.2.

3.1. Proof of Theorem 2.1. We start to prove (2.8) by taking the inner product of (2.3) with $4\tau\rho^{n+1}$ to obtain

$$\langle 3\rho^{n+1} - 4\rho^n + \rho^{n-1}, \rho^{n+1} \rangle = 0.$$

We note here that the convection term disappears by the orthogonality (1.4). Applying (3.2) and then summing up over n from 0 to N lead to (2.8).

In order to prove (2.9), We take an inner product of (2.4) with $4\tau\hat{\mathbf{u}}^{n+1}$ to obtain

$$\begin{aligned}
 (3.3) \quad & \|\sigma^{n+1}\hat{\mathbf{u}}^{n+1}\|_0^2 - \|\sigma^{n+1}\mathbf{u}^n\|_0^2 + \|\sigma^{n+1}(2\hat{\mathbf{u}}^{n+1} - \mathbf{u}^n)\|_0^2 + 4\mu\tau\|\nabla\hat{\mathbf{u}}^{n+1}\|_0^2 \\
 & - \|\sigma^{n+1}(2\mathbf{u}^n - \mathbf{u}^{n-1})\|_0^2 + \|\sigma^{n+1}(\hat{\mathbf{u}}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\|_0^2 \\
 & + 4\tau\langle\rho^{n+1}(\bar{\mathbf{u}}^{n+1} \cdot \nabla)\hat{\mathbf{u}}^{n+1}, \hat{\mathbf{u}}^{n+1}\rangle \\
 & = 4\tau\langle\mathbf{f}^{n+1}, \hat{\mathbf{u}}^{n+1}\rangle - 4\tau\langle\nabla p^n, \hat{\mathbf{u}}^{n+1}\rangle.
 \end{aligned}$$

In order to attack (3.3), we do first inner product of (2.3) with $2\tau\hat{\mathbf{u}}^{n+1} \cdot \hat{\mathbf{u}}^{n+1}$ to get an equation

$$(3.4) \quad \langle 3\rho^{n+1} - 4\rho^n + \rho^{n-1}, \hat{\mathbf{u}}^{n+1} \cdot \hat{\mathbf{u}}^{n+1} \rangle + 2\tau\langle\nabla \cdot (\rho^{n+1}\bar{\mathbf{u}}^{n+1})\hat{\mathbf{u}}^{n+1}, \hat{\mathbf{u}}^{n+1}\rangle = 0.$$

In conjunction with (1.3), (3.4) leads us to

$$\begin{aligned}
 (3.5) \quad & 4\tau\langle\rho^{n+1}(\bar{\mathbf{u}}^{n+1} \cdot \nabla)\hat{\mathbf{u}}^{n+1}, \hat{\mathbf{u}}^{n+1}\rangle = -2\tau\langle\nabla \cdot (\rho^{n+1}\bar{\mathbf{u}}^{n+1})\hat{\mathbf{u}}^{n+1}, \hat{\mathbf{u}}^{n+1}\rangle \\
 & = \langle 3\rho^{n+1} - 4\rho^n + \rho^{n-1}, \hat{\mathbf{u}}^{n+1} \cdot \hat{\mathbf{u}}^{n+1} \rangle.
 \end{aligned}$$

Since we have the orthogonality $\langle\mathbf{u}^{n+1}, \nabla\delta\psi^{n+1}\rangle = 0$, the equation $\hat{\mathbf{u}}^{n+1} = \mathbf{u}^{n+1} - \frac{1}{\rho^{n+1}}\nabla\delta\psi^{n+1}$ yields

$$\begin{aligned}
 (3.6) \quad & \|\sigma^{n+1}\hat{\mathbf{u}}^{n+1}\|_0^2 = \|\sigma^{n+1}\mathbf{u}^{n+1}\|_0^2 + \left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2, \\
 & \|\sigma^{n+1}(2\hat{\mathbf{u}}^{n+1} - \mathbf{u}^n)\|_0^2 = \|\sigma^{n+1}(2\mathbf{u}^{n+1} - \mathbf{u}^n)\|_0^2 + 4\left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2, \\
 & \|\sigma^{n+1}(\hat{\mathbf{u}}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\|_0^2 = \|\sigma^{n+1}(\mathbf{u}^{n+1} - 2\mathbf{u}^n + \mathbf{u}^{n-1})\|_0^2 \\
 & \quad + \left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2.
 \end{aligned}$$

In addition, $p^{n+1} = -\frac{3}{2\tau}\psi^{n+1} + \mu s^{n+1}$ in (2.7) yields

$$(3.7) \quad -4\tau\langle\nabla p^n, \hat{\mathbf{u}}^{n+1}\rangle = 6\langle\nabla\psi^n, \hat{\mathbf{u}}^{n+1}\rangle - 4\mu\tau\langle\nabla p^n, \hat{\mathbf{u}}^{n+1}\rangle.$$

In light of (3.5)~(3.7), (3.3) can be rewritten by

$$\begin{aligned}
 (3.8) \quad & \left\| \sigma^{n+1} \mathbf{u}^{n+1} \right\|_0^2 + \left\| \sigma^{n+1} (2\mathbf{u}^{n+1} - \mathbf{u}^n) \right\|_0^2 + \left\| \sigma^{n+1} \delta \delta \mathbf{u}^{n+1} \right\|_0^2 - \left\| \sigma^{n+1} \mathbf{u}^n \right\|_0^2 \\
 & - \left\| \sigma^{n+1} (2\mathbf{u}^n - \mathbf{u}^{n-1}) \right\|_0^2 + 4\tau\mu \left\| \nabla \hat{\mathbf{u}}^{n+1} \right\|_0^2 + 6 \left\| \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\|_0^2 \\
 & = 4\tau \left\langle \mathbf{f}^{n+1}, \hat{\mathbf{u}}^{n+1} \right\rangle + 6 \left\langle \nabla \psi^n, \hat{\mathbf{u}}^{n+1} \right\rangle - 4\mu\tau \left\langle \nabla s^n, \hat{\mathbf{u}}^{n+1} \right\rangle \\
 & + \left\langle 3\rho^{n+1} - 4\rho^n + \rho^{n-1}, \hat{\mathbf{u}}^{n+1} \cdot \hat{\mathbf{u}}^{n+1} \right\rangle = \sum_{i=1}^4 A_i
 \end{aligned}$$

where

$$\begin{aligned}
 A_1 &:= 4\tau \left\langle \mathbf{f}^{n+1}, \hat{\mathbf{u}}^{n+1} \right\rangle, \quad A_2 := 6 \left\langle \nabla \psi^n, \hat{\mathbf{u}}^{n+1} \right\rangle, \quad A_3 := -4\mu\tau \left\langle \nabla s^n, \hat{\mathbf{u}}^{n+1} \right\rangle, \\
 A_4 &:= \left\langle 3\rho^{n+1} - 4\rho^n + \rho^{n-1}, \hat{\mathbf{u}}^{n+1} \cdot \hat{\mathbf{u}}^{n+1} \right\rangle.
 \end{aligned}$$

We now estimate each term respectively. The first term can be bounded by

$$(3.9) \quad A_1 \leq \frac{C\tau}{\mu} \left\| \mathbf{f}^{n+1} \right\|_{-1}^2 + \mu\tau \left\| \nabla \hat{\mathbf{u}}^{n+1} \right\|_0^2.$$

The equation $\mathbf{u}^{n+1} = \hat{\mathbf{u}}^{n+1} + \frac{1}{\rho^{n+1}} \nabla(\psi^{n+1} - \psi^n)$ in (2.6) gives us

$$\begin{aligned}
 A_2 &= 6 \left\langle \nabla \psi^n, \mathbf{u}^{n+1} - \frac{1}{\rho^{n+1}} \nabla \delta \psi^{n+1} \right\rangle \\
 &= -6 \left\langle \frac{1}{\sigma^{n+1}} \nabla \psi^n, \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\rangle \\
 &= -3 \left(\left\| \frac{1}{\sigma^{n+1}} \nabla \psi^{n+1} \right\|_0^2 - \left\| \frac{1}{\sigma^{n+1}} \nabla \psi^n \right\|_0^2 - \left\| \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\|_0^2 \right).
 \end{aligned}$$

Since we have $\frac{1}{\rho^{n+1}} - \frac{1}{\rho^n} = \left(\frac{\rho^n}{\rho^{n+1}} - 1\right) \frac{1}{\rho^n} = \frac{\rho^n - \rho^{n+1}}{\rho^{n+1}} \frac{1}{\rho^n}$, by Assumption 2 we arrive at

$$\begin{aligned}
\left\| \frac{1}{\sigma^{n+1}} \nabla \psi^n \right\|_0^2 &= \int_{\Omega} \frac{1}{\rho^{n+1}} (\nabla \psi^n)^2 dx \\
&= \int_{\Omega} \left(\frac{1}{\rho^{n+1}} - \frac{1}{\rho^n} \right) (\nabla \psi^n)^2 dx + \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 \\
&\leq \left\| \frac{\rho^{n+1} - \rho^n}{\rho^{n+1}} \right\|_{L^\infty(\Omega)} \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 + \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 \\
&\leq M\tau \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 + \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2
\end{aligned}$$

and A_2 becomes

$$\begin{aligned}
(3.10) \quad A_2 &\leq -3 \left(\left\| \frac{1}{\sigma^{n+1}} \nabla \psi^{n+1} \right\|_0^2 - \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 \right) \\
&\quad + 3M\tau \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 + 3 \left\| \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\|_0^2.
\end{aligned}$$

Invoking Lemma 3.2, the second equation $s^{n+1} = s^n - \nabla \cdot \hat{\mathbf{u}}^{n+1}$ in (2.6) derives

$$\|s^{n+1} - s^n\|_0^2 = \|\nabla \cdot \hat{\mathbf{u}}^{n+1}\|_0^2 \leq \|\nabla \hat{\mathbf{u}}^{n+1}\|_0^2$$

and so we conclude

$$\begin{aligned}
(3.11) \quad A_3 &= 4\mu\tau \langle s^n, \nabla \cdot \hat{\mathbf{u}}^{n+1} \rangle \\
&= -4\mu\tau \langle s^n, s^{n+1} - s^n \rangle \\
&= -2\mu\tau \left(\|s^{n+1}\|_0^2 - \|s^n\|_0^2 - \|s^{n+1} - s^n\|_0^2 \right) \\
&\leq -2\mu\tau \left(\|s^{n+1}\|_0^2 - \|s^n\|_0^2 \right) + 2\mu\tau \|\nabla \hat{\mathbf{u}}^{n+1}\|_0^2.
\end{aligned}$$

Finally, by Assumption 2 we get

$$\begin{aligned}
 (3.12) \quad A_4 &= \langle 3\rho^{n+1} - 4\rho^n + \rho^{n-1}, \hat{\mathbf{u}}^{n+1} \cdot \hat{\mathbf{u}}^{n+1} \rangle \\
 &\leq \left\| \frac{3\rho^{n+1} - 4\rho^n + \rho^{n-1}}{\rho^{n+1}} \right\|_{L^\infty(\Omega)} \|\sigma^{n+1} \hat{\mathbf{u}}^{n+1}\|_0^2 \\
 &\leq M\tau \|\sigma^{n+1} \hat{\mathbf{u}}^{n+1}\|_0^2 = M\tau \left(\|\sigma^{n+1} \mathbf{u}^{n+1}\|_0^2 + \left\| \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\|_0^2 \right).
 \end{aligned}$$

Since we can obtain

$$\begin{aligned}
 -\|\sigma^{n+1} \mathbf{u}^n\|_0^2 &= -\int_{\Omega} \rho^{n+1} (\mathbf{u}^n)^2 d\mathbf{x} \\
 &= -\int_{\Omega} (\rho^{n+1} - \rho^n) (\mathbf{u}^n)^2 d\mathbf{x} - \|\sigma^n \mathbf{u}^n\|_0^2 \\
 &\geq -\left\| \frac{\rho^{n+1} - \rho^n}{\rho^n} \right\|_{L^\infty(\Omega)} \|\sigma^n \mathbf{u}^n\|_0^2 - \|\sigma^n \mathbf{u}^n\|_0^2 \\
 &\geq -M\tau \|\sigma^n \mathbf{u}^n\|_0^2 - \|\sigma^n \mathbf{u}^n\|_0^2
 \end{aligned}$$

as well as

$$-\|\sigma^{n+1} (2\mathbf{u}^n - \mathbf{u}^{n-1})\|_0^2 \geq -M\tau \|\sigma^n (2\mathbf{u}^n - \mathbf{u}^{n-1})\|_0^2 - \|\sigma^n (2\mathbf{u}^n - \mathbf{u}^{n-1})\|_0^2,$$

the estimations (3.9)-(3.12) make (3.8) become

$$\begin{aligned}
 &\delta \|\sigma^{n+1} \mathbf{u}^{n+1}\|_0^2 + \delta \|\sigma^{n+1} (2\mathbf{u}^{n+1} - \mathbf{u}^n)\|_0^2 + \|\sigma^{n+1} \delta \delta \mathbf{u}^{n+1}\|_0^2 \\
 &+ 3\delta \left\| \frac{1}{\sigma^{n+1}} \nabla \psi^{n+1} \right\|_0^2 + 3 \left\| \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\|_0^2 + 2\mu\tau \delta \|s^{n+1}\|_0^2 + \mu\tau \|\nabla \hat{\mathbf{u}}^{n+1}\|_0^2 \\
 &\leq M\tau \left(\|\sigma^n \mathbf{u}^n\|_0^2 + \|\sigma^n (2\mathbf{u}^n - \mathbf{u}^{n-1})\|_0^2 \right) \\
 &\quad + M\tau \left(\|\sigma^{n+1} \mathbf{u}^{n+1}\|_0^2 + \left\| \frac{1}{\sigma^{n+1}} \nabla \delta \psi^{n+1} \right\|_0^2 \right) + \frac{C\tau}{\mu} \|\mathbf{f}^{n+1}\|_{-1}^2.
 \end{aligned}$$

If τ is small enough, then summing over n from 1 to N yields

$$\begin{aligned}
& \|\sigma^{N+1}\mathbf{u}^{N+1}\|_0^2 + \|\sigma^{N+1}(2\mathbf{u}^{N+1} - \mathbf{u}^N)\|_0^2 + 3\left\|\frac{1}{\sigma^{N+1}}\nabla\psi^{N+1}\right\|_0^2 + 2\mu\tau\|s^{N+1}\|_0^2 \\
& + \sum_{n=1}^N \left(\|\sigma^{n+1}\delta\mathbf{u}^{n+1}\|_0^2 + 3\left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2 + \mu\tau\|\nabla\hat{\mathbf{u}}^{n+1}\|_0^2 \right) \\
& \leq \|\sigma^1\mathbf{u}^1\|_0^2 + \|\sigma^1(2\mathbf{u}^1 - \mathbf{u}^0)\|_0^2 + 3\left\|\frac{1}{\sigma^1}\nabla\psi^1\right\|_0^2 + 2\mu\tau\|s^1\|_0^2 \\
& + M\tau \sum_{n=1}^N \left(\|\sigma^n\mathbf{u}^n\|_0^2 + \|\sigma^n(2\mathbf{u}^n - \mathbf{u}^{n-1})\|_0^2 \right) \\
& + M\tau \sum_{n=1}^N \left(\|\sigma^{n+1}\mathbf{u}^{n+1}\|_0^2 + \left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2 \right) + \frac{C\tau}{\mu} \sum_{n=1}^N \|\mathbf{f}^{n+1}\|_{-1}^2.
\end{aligned}$$

Invoking Lemma 3.4, we arrive at (2.9) and finish this proof. \square

3.2. Proof of Theorem 2.2. Because the proof of (2.11) is exactly same with that of (2.8), we will prove here (2.12) only. We now take the inner product of (2.10) with $4\tau\hat{\mathbf{u}}^{n+1}$, and then apply (1.3) and (3.2) to obtain

$$\begin{aligned}
(3.13) \quad & \|\sigma^{n+1}\hat{\mathbf{u}}^{n+1}\|_0^2 + \|2\sigma^{n+1}\hat{\mathbf{u}}^{n+1} - \sigma^n\mathbf{u}^n\|_0^2 - \|2\sigma^n\mathbf{u}^n - \sigma^{n-1}\mathbf{u}^{n-1}\|_0^2 \\
& - \|\sigma^n\mathbf{u}^n\|_0^2 + \|\sigma^{n+1}\hat{\mathbf{u}}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1}\|_0^2 + 4\tau\mu\|\nabla\hat{\mathbf{u}}^{n+1}\|_0^2 \\
& = 4\tau\langle \mathbf{f}^{n+1}, \hat{\mathbf{u}}^{n+1} \rangle - 4\tau\langle \nabla p^n, \hat{\mathbf{u}}^{n+1} \rangle.
\end{aligned}$$

By the same manner as (3.6), we obtain

$$\begin{aligned}
& \|\sigma^{n+1}\hat{\mathbf{u}}^{n+1}\|_0^2 = \|\sigma^{n+1}\mathbf{u}^{n+1}\|_0^2 + \left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2, \\
& \|2\sigma^{n+1}\hat{\mathbf{u}}^{n+1} - \sigma^n\mathbf{u}^n\|_0^2 = \|2\sigma^{n+1}\mathbf{u}^{n+1} - \sigma^n\mathbf{u}^n\|_0^2 + 4\left\|\frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}\right\|_0^2 \\
& + 4\left\langle \frac{1}{\sigma^{n+1}}\nabla\delta\psi^{n+1}, \sigma^n\mathbf{u}^n \right\rangle.
\end{aligned}$$

In conjunction with (2.7), the two equations above lead (3.13) to become (3.14)

$$\begin{aligned} & \delta \left\| 2\sigma^{n+1}\mathbf{u}^{n+1} - \sigma^n\mathbf{u}^n \right\|_0^2 + \left\| \sigma^{n+1}\widehat{\mathbf{u}}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1} \right\|_0^2 \\ & + \delta \left\| \sigma^{n+1}\mathbf{u}^{n+1} \right\|_0^2 + 5 \left\| \frac{1}{\sigma^{n+1}} \nabla \delta\psi^{n+1} \right\|_0^2 + 4\tau\mu \left\| \nabla \widehat{\mathbf{u}}^{n+1} \right\|_0^2 = \sum_{i=1}^4 A_i, \end{aligned}$$

where

$$\begin{aligned} A_1 &:= 4\tau \left\langle \mathbf{f}^{n+1}, \widehat{\mathbf{u}}^{n+1} \right\rangle, \quad A_2 := 6 \left\langle \nabla \psi^n, \widehat{\mathbf{u}}^{n+1} \right\rangle, \\ A_3 &:= -4\mu\tau \left\langle \nabla s^n, \widehat{\mathbf{u}}^{n+1} \right\rangle, \quad A_4 := -4 \left\langle \frac{1}{\sigma^{n+1}} \nabla \delta\psi^{n+1}, \sigma^n\mathbf{u}^n \right\rangle. \end{aligned}$$

Because A_1 , A_2 and A_3 are the same as in (3.9)~(3.11), respectively, it is enough to estimate A_4 . Since $\langle \nabla \delta\psi^{n+1}, \mathbf{u}^n \rangle = 0$, we obtain

$$\begin{aligned} A_4 &= 4 \left\langle \frac{\sigma^{n+1} - \sigma^n}{\sigma^{n+1}} \nabla \delta\psi^{n+1}, \mathbf{u}^n \right\rangle \\ &\leq 4 \left\| \frac{\sigma^{n+1} - \sigma^n}{\sigma^n} \right\|_{L^\infty(\Omega)} \left\| \frac{1}{\sigma^{n+1}} \nabla \delta\psi^{n+1} \right\|_0 \left\| \sigma^n \mathbf{u}^n \right\|_0 \\ &\leq \left\| \frac{1}{\sigma^{n+1}} \nabla \delta\psi^{n+1} \right\|_0^2 + 4 \left\| \frac{\sigma^{n+1} - \sigma^n}{\sigma^n} \right\|_{L^\infty(\Omega)}^2 \left\| \sigma^n \mathbf{u}^n \right\|_0^2 \\ &\leq \left\| \frac{1}{\sigma^{n+1}} \nabla \delta\psi^{n+1} \right\|_0^2 + 4M\tau \left\| \sigma^n \mathbf{u}^n \right\|_0^2. \end{aligned}$$

In conjunction with the Assumption 2, inserting estimates of A_i 's, $i = 1, 2, 3$, and 4 into (3.14) gives us

$$\begin{aligned} & \delta \left\| \sigma^{n+1}\mathbf{u}^{n+1} \right\|_0^2 + \delta \left\| 2\sigma^{n+1}\mathbf{u}^{n+1} - \sigma^n\mathbf{u}^n \right\|_0^2 + 3 \left(\left\| \frac{1}{\sigma^{n+1}} \nabla \psi^{n+1} \right\|_0^2 - \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 \right) \\ & + \left\| \sigma^{n+1}\widehat{\mathbf{u}}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1} \right\|_0^2 + 2\mu\tau \left(\left\| s^{n+1} \right\|_0^2 - \left\| s^n \right\|_0^2 \right) + \mu\tau \left\| \nabla \widehat{\mathbf{u}}^{n+1} \right\|_0^2 \\ & + \left\| \frac{1}{\sigma^{n+1}} \nabla \delta\psi^{n+1} \right\|_0^2 \leq CM\tau \left\| \sigma^n \mathbf{u}^n \right\|_0^2 + CM\tau \left\| \frac{1}{\sigma^n} \nabla \psi^n \right\|_0^2 + \frac{C\tau}{\mu} \left\| \mathbf{f}^{n+1} \right\|_{-1}^2. \end{aligned}$$

Summing over n from 1 to N leads to

$$\begin{aligned}
& \|\sigma^{N+1}\mathbf{u}^{N+1}\|_0^2 + \|2\sigma^{N+1}\mathbf{u}^{N+1} - \sigma^N\mathbf{u}^N\|_0^2 + 2\mu\tau\|s^{n+1}\|_0^2 + \mu\tau\sum_{n=1}^N \|\nabla\hat{\mathbf{u}}^{n+1}\|_0^2 \\
& + \sum_{n=1}^N \|\sigma^{n+1}\hat{\mathbf{u}}^{n+1} - 2\sigma^n\mathbf{u}^n + \sigma^{n-1}\mathbf{u}^{n-1}\|_0^2 + \sum_{n=1}^N \left\| \frac{1}{\sigma^{n+1}} \nabla\delta\psi^{n+1} \right\|_0^2 \\
& + 3\left\| \frac{1}{\sigma^{N+1}} \nabla\psi^{N+1} \right\|_0^2 \leq \|\sigma^1\mathbf{u}^1\|_0^2 + 2\mu\tau\|s^1\|_0^2 + \|2\sigma^1\mathbf{u}^1 - \sigma^0\mathbf{u}^0\|_0^2 \\
& + 3\left\| \frac{1}{\sigma^1} \nabla\psi^1 \right\|_0^2 + \frac{C\tau}{\mu}\|\mathbf{f}^{n+1}\|_{-1}^2 + CM\tau\sum_{n=1}^N \left(\|\sigma^n\mathbf{u}^n\|_0^2 + \left\| \frac{1}{\sigma^n} \nabla\psi^n \right\|_0^2 \right).
\end{aligned}$$

Invoking Lemma 3.4, we arrive at (2.12) and finish this proof. \square

4. Finite Element Discretization

In order to introduce finite element discretization, let $\mathfrak{T} = \{K\}$ be a shape regular quasi-uniform partition of Ω with a mesh size h . We define the spaces

$$\begin{cases} \mathbb{V}_h^b = \{\mathbf{v}_h \in \mathbf{C}(\Omega) : \mathbf{v}_h|_K \in \mathcal{R}(K), \quad \forall K \in \mathfrak{T}; \mathbf{v}_h|_\Gamma = b\}, \\ \mathbb{Q}_h = \{q_h \in L_0^2(\Omega) \cap C(\Omega) : q_h|_K \in \mathcal{L}(K), \quad \forall K \in \mathfrak{T}\}, \\ \mathbb{W}_h = \{\phi_h \in C(\Omega) : \phi_h|_K \in \mathcal{P}(K), \quad \forall K \in \mathfrak{T}\}, \end{cases}$$

where, for all $K \in \mathfrak{T}$, $\mathcal{P}(K)$, $\mathcal{L}(K)$ and $\mathcal{R}(K)$ are spaces of polynomials with degree \mathcal{P} , \mathcal{L} and \mathcal{R} , respectively. Because all algorithms in this paper can be applied by the same finite element technique, we consider fully discretization only for convective form SGUM in Algorithm 1. One of the difficult problems in numerical study is hyperbolic partial differential equations and one of them is density equation (2.3). It is well known that the standard FEM solution of (2.3) does not satisfy mass conservation. Moreover, the system is not symmetric. In order to overcome the weakness, various techniques to solve such the first order problems are developed, for example, streamline diffusion [8], discontinuous Galerkin [8], artificial diffusion [8], sub-grid or least-squares [2], etc. We selected a least-squares method to solve (2.3). To simple explanation, we consider

$$(4.1) \quad \rho + \alpha \mathbf{U} \cdot \nabla \rho = f,$$

where α is a given constant and \mathbf{U} is a given velocity with $\nabla \cdot \mathbf{U} = 0$ and $\mathbf{U} \cdot \nu|_{\Gamma} = 0$. By taking the inner product of (4.1) with $\phi + \alpha \mathbf{U} \cdot \nabla \phi$, we derive

$$(4.2) \quad \langle \rho + \alpha \mathbf{U} \cdot \nabla \rho, \phi + \alpha \mathbf{U} \cdot \nabla \phi \rangle = \langle f, \phi + \alpha \mathbf{U} \cdot \nabla \phi \rangle.$$

Due to $\nabla \cdot \mathbf{U} = 0$ and $\mathbf{U} \cdot \nu = 0$, we get $\langle \mathbf{U} \cdot \nabla \rho, \phi \rangle = -\langle \mathbf{U} \cdot \nabla \phi, \rho \rangle$. So (4.2) can be rewritten by

$$\langle \rho, \phi \rangle + \alpha^2 \langle \mathbf{U} \cdot \nabla \rho, \mathbf{U} \cdot \nabla \phi \rangle = \langle f, \phi + \alpha \mathbf{U} \cdot \nabla \phi \rangle.$$

Then, we define the least-squares method as: find $\rho_h \in \mathbb{W}_h$ such that

$$\langle \rho_h, \phi_h \rangle + \alpha^2 \langle \mathbf{U} \cdot \nabla \rho_h, \mathbf{U} \cdot \nabla \phi_h \rangle = \langle f, \phi_h + \alpha \mathbf{U} \cdot \nabla \phi_h \rangle, \quad \forall \phi_h \in \mathbb{W}_h.$$

In contrast to the standard Galerkin formulation, the above linear system is symmetric and we have the following error bound (cf. [2]):

$$\|\rho - \rho_h\|_0 + \|\mathbf{U} \cdot \nabla(\rho - \rho_h)\|_0 \leq Ch^\gamma \|\rho\|_{\gamma+1}.$$

Note that this estimate is only sub-optimal in the L^2 -norm as is in the standard Galerkin method since it is optimal in the norm induced by the stream-wise derivative.

FEM stabilized Gauge-Uzawa Method for variable density problems. Let $\rho_{0h}, \mathbf{u}_{0h}$ be a suitable approximation of ρ_0 and \mathbf{u}_0 , respectively. Set $\rho_h^0 = \rho_{0h}, \mathbf{u}_h^0 = \mathbf{u}_{0h}$ and $s_h^0 = 0$; compute \mathbf{u}_h^1, ρ_h^1 and p_h^1 via any first order projection method. Set $\psi_h^1 = -\frac{2\tau}{3}p_h^1$ and $s_h^1 = 0$ and then repeat for $2 \leq n \leq N \leq T/\tau - 1$:

Step 1: Set $\bar{\mathbf{u}}_h^{n+1} = 2\mathbf{u}_h^n - \mathbf{u}_h^{n-1}$ and find $\rho_h^{n+1} \in \mathbb{W}_h$ such that,

$$\left\langle \frac{3\rho_h^{n+1} - 4\rho_h^n + \rho_h^{n-1}}{2\tau}, \phi_h \right\rangle + \langle \bar{\mathbf{u}}_h^{n+1} \cdot \nabla \rho_h^{n+1}, \bar{\mathbf{u}}_h^{n+1} \cdot \nabla \phi_h \rangle = 0,$$

$$\rho_h^{n+1}|_{\Gamma_{\mathbf{u}_h^n}} = r_h^{n+1}.$$

Step 2: Find $\hat{\mathbf{u}}_h^{n+1} \in \mathbb{V}_h^0$ such that

$$\left\langle \rho_h^{n+1} \frac{3\hat{\mathbf{u}}_h^{n+1} - 4\mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{2\tau}, \mathbf{w}_h \right\rangle + \langle \rho_h^{n+1} (\bar{\mathbf{u}}_h^{n+1} \cdot \nabla) \hat{\mathbf{u}}_h^{n+1}, \mathbf{w}_h \rangle$$

$$- \langle p_h^n, \nabla \cdot \mathbf{w}_h \rangle + \mu \langle \nabla \hat{\mathbf{u}}_h^{n+1}, \nabla \mathbf{w}_h \rangle = \langle \mathbf{f}_h^{n+1}, \mathbf{w}_h \rangle, \quad \forall \mathbf{w}_h \in \mathbb{V}_h^0.$$

Step 3: Find $\psi_h^{n+1} \in \mathbb{Q}_h$ such that

$$(4.3) \quad \left\langle \frac{1}{\rho_h^{n+1}} \nabla(\psi_h^{n+1} - \psi_h^n), \nabla q_h \right\rangle = - \langle \hat{\mathbf{u}}_h^{n+1}, \nabla q_h \rangle, \quad \forall q_h \in \mathbb{Q}_h.$$

Step 4: Update \mathbf{u}_h^{n+1} and $s_h^{n+1} \in \mathbb{Q}_h$ by

$$(4.4) \quad \begin{aligned} \mathbf{u}_h^{n+1} &= \hat{\mathbf{u}}_h^{n+1} + \frac{1}{\rho_h^{n+1}} \nabla(\psi_h^{n+1} - \psi_h^n), \\ \langle s_h^{n+1}, q_h \rangle &= \langle s_h^n, q_h \rangle - \langle \nabla \cdot \hat{\mathbf{u}}_h^{n+1}, q_h \rangle, \quad \forall q_h \in \mathbb{Q}_h. \end{aligned}$$

Step 5: Update p_h^{n+1} by

$$p_h^{n+1} = -\frac{3}{2\tau} \psi_h^{n+1} + \mu s_h^{n+1}.$$

REMARK 4.1. We note that

$$\langle \mathbf{u}_h^{n+1}, \nabla q_h \rangle = 0 \quad \forall q_h \in \mathbb{Q}_h,$$

by inserting the first equation of (4.4) into (4.3).

5. Numerical experiments

In this section, we present error decays of Algorithms 1~3 in §5.1 and numerical simulation of Algorithm 4 for Rayleigh-Taylor instability problem in §5.2 to assert unconditionally stable. For computing the numerical experiments, we use a partial differential equation solver FreeFem++ [7].

5.1. Accuracy check using an exact solution. In order to check the convergence rate of our numerical algorithms, we employ the known solution in [3]. We consider Taylor-Hood finite element for (\mathbf{u}, p) and linear element for ρ , i.e., $(\mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) . We choose an exact solution of (1.1) to be:

$$\begin{aligned} \mathbf{u}(x, y, t) &= (-y \cos t, x \cos t), \\ p(x, y, t) &= \sin x \sin y \sin t, \\ \rho(x, y, t) &= 2 + r \cos(\theta - \sin t), \end{aligned}$$

for the unit circle $|r| \leq 1$ where $r = \sqrt{x^2 + y^2}$ and $\tan \theta = \frac{y}{x}$. We set $\mu = 1$ and then, the force function \mathbf{f} can be calculated:

$$\mathbf{f}(x, y, t) = \begin{pmatrix} (y \sin t - x \cos^2 t) \rho(x, y, t) + \cos x \sin y \sin t \\ -(x \sin t + y \cos^2 t) \rho(x, y, t) + \sin x \cos y \sin t \end{pmatrix}.$$

We choose the mesh size for time and space $\tau = 0.1 \times h$ and we denote error functions

$$\varepsilon^{n+1} = \rho(t^{n+1}) - \rho^{n+1}, \mathbf{E}^{n+1} = \mathbf{u}(t^{n+1}) - \mathbf{u}^{n+1}, e^{n+1} = p(t^{n+1}) - p^{n+1}.$$

Table 1 and 2 show the errors and convergence rates of Algorithm 1 and 2, respectively. we also compute the errors of Algorithm 3 to compare numerical performance. Since the density formulation is not defined in Algorithm 3, we use the same density equation in Algorithm 1. We can conclude that the errors of SGUMs are smaller than that of the fractional time-stepping method.

$h = 10\tau$	1/16	1/32	1/64	1/128	1/256	1/512
$\ \varepsilon\ _0$	2.49397E-03	6.15006E-04	1.47959E-04	3.63742E-05	9.01843E-06	2.22973E-06
Order		2.020	2.055	2.024	2.012	2.016
$\ \varepsilon\ _{L^\infty}$	3.34546E-03	7.02556E-04	2.13751E-04	9.29533E-05	2.89400E-05	8.85893E-06
Order		2.252	1.717	1.201	1.683	1.708
$\ \mathbf{E}\ _0$	7.05987E-04	1.58097E-04	3.94170E-05	1.01579E-05	2.61464E-06	6.63486E-07
Order		2.159	2.004	1.956	1.958	1.978
$\ \mathbf{E}\ _{L^\infty}$	1.48873E-03	4.04066E-04	1.10927E-04	2.95630E-05	7.59910E-06	1.93690E-06
Order		1.881	1.865	1.908	1.960	1.972
$\ \mathbf{E}\ _{H^1}$	7.11544E-03	1.96983E-03	4.79020E-04	1.30469E-04	3.41915E-05	9.11180E-06
Order		1.853	2.040	1.876	1.932	1.908
$\ e\ _0$	8.60452E-03	2.07820E-03	5.17036E-04	1.31254E-04	3.31296E-05	8.03693E-06
Order		2.050	2.007	1.978	1.986	2.043
$\ e\ _{L^\infty}$	6.82909E-03	2.64522E-03	7.43633E-04	1.66197E-04	4.16335E-05	1.24547E-05
Order		1.368	1.831	2.162	1.997	1.741

TABLE 1. Error and convergence rate of Algorithm 1 with finite element $(\mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) , $\mu = 1$ and $\tau = 0.1 \times h$

5.2. Rayleigh-Taylor instability. The Rayleigh-Taylor instability documented by Tryggvason in [19] is very unstable problem. We perform the simulation for Algorithm 1 with same conditions within [3] and find out similar results that the computation can not reach the end of step. Since Algorithms 1 and 2 display very similar behavior, we present only the

$h = 10\tau$	1/16	1/32	1/64	1/128	1/256	1/512
$\ \varepsilon\ _0$	1.81516E-03	4.53901E-04	1.14066E-04	2.92608E-05	9.00982E-06	2.22745E-06
Order		2.000	1.993	1.963	1.699	2.016
$\ \varepsilon\ _{L^\infty}$	2.48022E-03	6.23530E-04	2.24056E-04	9.85715E-05	2.89404E-05	8.85898E-06
Order		1.992	1.477	1.185	1.768	1.708
$\ \mathbf{E}\ _0$	6.93054E-04	1.23266E-04	2.79597E-05	6.97177E-06	2.63486E-06	6.68252E-07
Order		2.491	2.140	2.004	1.404	1.979
$\ \mathbf{E}\ _{L^\infty}$	1.26361E-03	2.88777E-04	7.23136E-05	1.90954E-05	7.59912E-06	1.93691E-06
Order		2.130	1.998	1.921	1.329	1.972
$\ \mathbf{E}\ _{H^1}$	7.69852E-03	2.00431E-03	4.58577E-04	1.20388E-04	3.42031E-05	9.11386E-06
Order		1.941	2.128	1.929	1.815	1.908
$\ e\ _0$	1.23320E-02	2.89864E-03	7.10530E-04	1.78920E-04	3.31869E-05	8.05001E-06
Order		2.089	2.028	1.990	2.431	2.044
$\ e\ _{L^\infty}$	1.30654E-02	3.18190E-03	9.72251E-04	2.85822E-04	4.17483E-05	1.24092E-05
Order		2.038	1.710	1.766	2.775	1.750

TABLE 2. Error and convergence rate of Algorithm 2 with finite element $(\mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) , $\mu = 1$ and $\tau = 0.1 \times h$

$h = 10\tau$	1/16	1/32	1/64	1/128	1/256	1/512
$\ \varepsilon\ _0$	3.90655E-02	5.79541E-03	1.39108E-03	2.28473E-04	5.58765E-05	1.50257E-05
Order		2.753	2.059	2.606	2.032	1.895
$\ \varepsilon\ _{L^\infty}$	6.44929E-02	1.74411E-02	5.19524E-03	1.17272E-03	3.09818E-04	1.08871E-04
Order		1.887	1.747	2.147	1.920	1.509
$\ \mathbf{E}\ _0$	1.24355E-03	3.53870E-04	9.55603E-05	2.51958E-05	6.50843E-06	1.66653E-06
Order		1.813	1.889	1.923	1.953	1.965
$\ \mathbf{E}\ _{L^\infty}$	1.35165E-03	3.86329E-04	9.45502E-05	2.33722E-05	5.92847E-06	1.47243E-06
Order		1.807	2.031	2.016	1.979	2.009
$\ \mathbf{E}\ _{H^1}$	6.18888E-03	1.76044E-03	4.79671E-04	1.31618E-04	3.58001E-05	9.79751E-06
Order		1.814	1.876	1.866	1.878	1.869
$\ e\ _0$	9.35810E-03	2.26649E-03	5.82218E-04	1.50575E-04	3.85031E-05	9.50119E-06
Order		2.046	1.961	1.951	1.967	2.019
$\ e\ _{L^\infty}$	8.12956E-03	2.63253E-03	7.33496E-04	1.60061E-04	4.20599E-05	1.24434E-05
Order		1.627	1.844	2.196	1.928	1.757

TABLE 3. Error and convergence rate of Alrorphism 3 with finite element $(\mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) , $\mu = 1$ and $\tau = 0.1 \times h$

results of Algorithm 1. We try to compute the Rayleigh-Taylor instability by Algorithm 4 and finish the computation without lose of stability. So we conclude that Algorithm 4 is unconditionally stable numerically.

In this experiment, we define the Atwood ratio $A = (\rho_M - \rho_m)/(\rho_M + \rho_m)$. The equations are nondimensional forms using the following references: ρ_m for density, d for length, and $\sqrt{d/(Ag)}$ for time, where g is the

gravity. Then, reference velocity is \sqrt{Adg} and the Reynolds number is defined by $Re = \rho_m \sqrt{d^3 g} / \mu$. The domain is $[-\frac{d}{2}, \frac{d}{2}] \times [-2d, 2d]$. While in our computation, the inflow boundary is empty. The initial value of density is the following:

$$\rho(x, y, 0) = \frac{\rho_M + \rho_m}{2} + \frac{\rho_M - \rho_m}{2} \tanh\left(\frac{y - \eta(x)}{0.01}\right),$$

where $\eta(x)$ is the initial condition of the perturbed interface. We perform simulations for Algorithm 1 and 4 with $\tau = 5/10,000$ and $d = 1$. Since we focus on the motion of fluids in the simulations, we choose the finite element space $(\mathcal{P}^1, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) .

5.2.1. Convective Form(Algorithm 1). We assume that the flow remains to be symmetric so the computational domain can be restricted by half, i.e., $\Omega = [0, \frac{1}{2}] \times [-2, 2]$. Then, we compute following simulations under same conditions indicated in [3].

- A low Atwood ratio problem: For the mesh size for space $h = 1/128$, set $\rho_M = 3$, $\rho_m = 1$ and $\eta(x) = -0.1 \cos(2\pi x)$. Snapshots of the density field are displayed in Figure 1 and Figure 2 at $Re = 1000$ and $Re = 5000$, respectively.
- A high Atwood ratio problem: For the mesh size for space $h = 1/256$ set $\rho_M = 7$, $\rho_m = 1$ and $\eta(x) = -0.01 \cos(2\pi x)$. Pictures of the density field are plotted in Figure 3 at $Re = 1000$.

5.2.2. Allen-Cahn Form(Algorithm 4). In this test, we set same conditions of a low Atwood ratio problem. Then we simulate the problem for $\varepsilon = 0.01$ and $\eta(x) = -0.1 \cos(2\pi x)$. Pictures of the density field are showed in Figures 4 and 5 at $Re = 1000$ and $Re = 5000$, respectively.

6. Conclusions

We attempted several approaches about variable density problem. We introduced new Algorithm 1~2 of order 2 by developing previous methods and proved the stability conditions of these algorithms. We had showed that our accuracy tests got better results than others. Rayleigh-Taylor instability is most popular simulations of variable density. Figure 1~3 had been displayed the similar results to those in other studies. ([3], [4]). In addition, we built the new algorithm using Allen-Cahn ideas in order to overcome the unstability of algorithms. Figure 4~5 showed

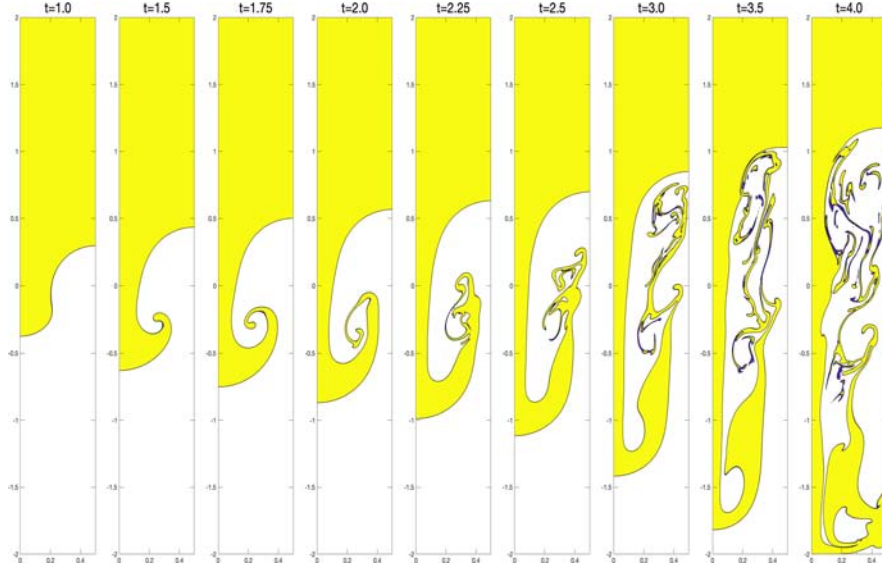


FIGURE 1. A low Atwood ratio problem of Algorithm 1 with finite element $(\mathcal{P}^1, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) at $Re = 1000$ (density contours $1.4 \leq \rho \leq 1.6$)

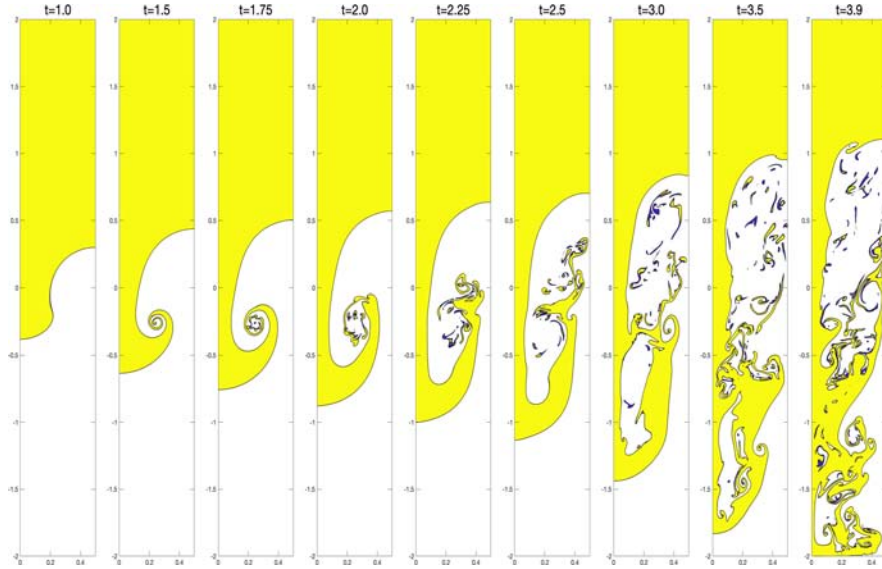


FIGURE 2. A low Atwood ratio problem of Algorithm 1 with finite element $(\mathcal{P}^1, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) at $Re = 5000$ (density contours $1.4 \leq \rho \leq 1.6$)

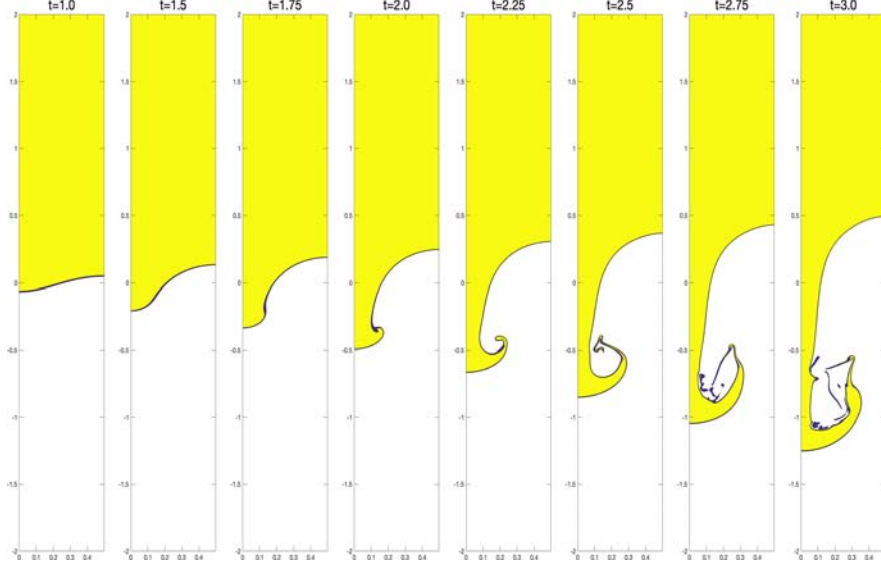


FIGURE 3. A high Atwood ratio problem of Algorithm 1 with finite element $(\mathcal{P}^1, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) at $Re = 1000$ (density contours $2 \leq \rho \leq 4$)

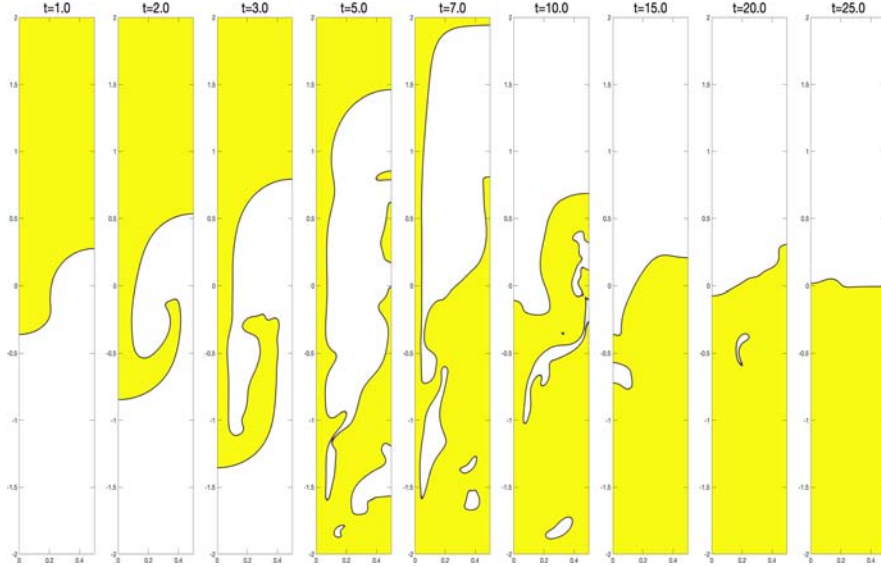


FIGURE 4. A low Atwood ratio problem of Algorithm 4 with finite element $(\mathcal{P}^1, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) at $Re = 1000$ (density contours $1.4 \leq \rho \leq 1.6$)

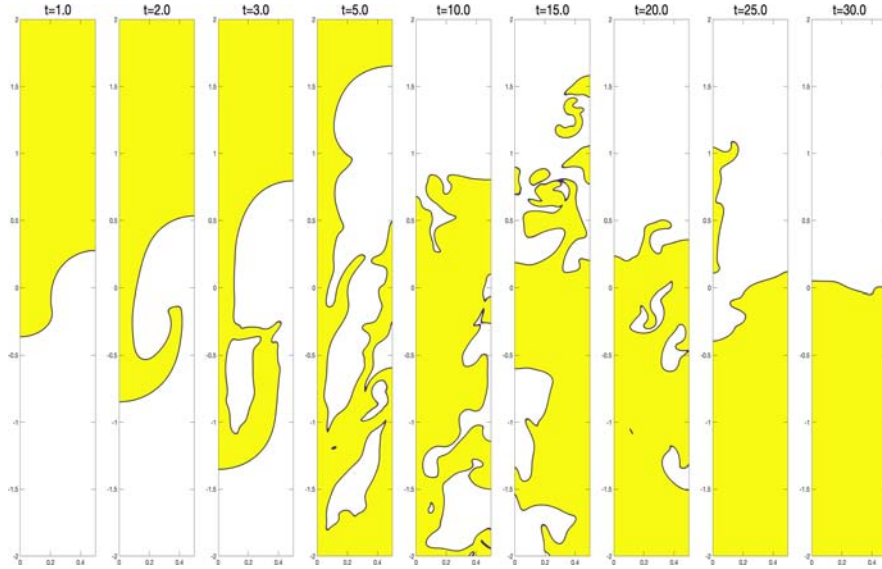


FIGURE 5. A low Atwood ratio problem of Algorithm 4 with finite element $(\mathcal{P}^1, \mathcal{P}^1, \mathcal{P}^1)$ for (\mathbf{u}, p, ρ) at $Re = 5000$ (density contours $1.4 \leq \rho \leq 1.6$)

the acceptable performance that Algorithm 4 arrives at the end of step unlike the others.

Our studies and results concluded that the new algorithms are suitable for simulation of flow problems with a variable density.

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