NORMALITY AND QUOTIENT IN CROSSED MODULES OVER GROUPOIDS AND 2-GROUPOIDS

Sedat Temel

Abstract. The aim of this paper is to consider the categorical equivalence between crossed modules within groupoids and 2-groupoids; and then relate normality and quotient in these two categories.

1. Introduction

A *group-groupoid*, which is also named $\mathcal{G}$-*groupoid* [4] or 2-group [1,2], can be defined in various equivalent ways, for example as a group object in the category Gpd of groupoids or as an internal category in the category Gp of groups. A 2-groupoid with a unique object is also a group-groupoid [2]. Group-groupoids can be thought as semi-abelian categories, see [14]. As algebraic models for homotopy 2-types, crossed modules over groups were defined by Whitehead in 1946 during his investigation of second relative homotopy groups for topological spaces [25, 26]. Crossed modules can be thought as 2-dimensional groups [5] and are used in homotopy theory [6], homological algebra [12] and algebraic K-theory [16], etc. The well known equivalence between crossed modules and group-groupoids was proved by Brown and Spencer [4]. This equivalence is presented in [2] by obtaining a group-groupoid as a 2-category with one object. In [23], this equivalence was extended by Porter in some algebraic categories which are include group structures. The groupoid

version of this equivalence is given in [13] by replacing crossed modules over groups with crossed modules over groupoids and group-groupoids with 2-groupoids.

The notions of *subcrossed modules* and *normal subcrossed modules* are introduced by Norrie in [22]. In [19], the concept of *normal subgroup-groupoids* and of *quotient group-groupoids* are obtained by using the equivalence between group-groupoids and crossed modules. Recently normal and quotient objects in the category of crossed modules over groupoids and of double groupoids were compared and the corresponding objects in the category of double groupoids were characterized in [20] using the categorical equivalence between double groupoids with thin structures and crossed modules over groupoids given in [6]. The equivalence between 2-groupoids and crossed modules over groupoids is proved by Icen in [13]. However there is a gap in this equivalence in terms of normality and quotient structures. In this paper, we consider the structures of normal subcrossed modules and quotient crossed modules for groupoids; and then we obtain the associated concepts in 2-groupoids. Since a 2-groupoid with one object is a group-groupoid, the results in this paper are the extention of the results in [19].

2. Preliminaries

A groupoid is a category in which every morphism has an inverse. In more details, a groupoid G consists of the set of objects G_0 and the set of morphisms G_1 with the source and the target maps $s, t \colon G_1 \to G_0$, $s(g) = x$, $t(g) = y$, respectively, where $x \xrightarrow{g} y$, the composition map $m(g, h) = h \circ g$ which is defined on the pullback $G_{1s} \times_t G_1$, the identity map $\varepsilon \colon G_0 \to G_1$, $\varepsilon(x) = 1_x$ such that $g \circ 1_x = 1_y \circ g = g$ for $g \in G(x, y)$ and the inverse map $\eta \colon G_1 \to G_1, \eta(g) = \overline{g}$ such that $g \circ \overline{g} = 1_y, \overline{g} \circ g = 1_x$. We write $G(x, y)$ for the set of morphisms from x to y, and $G(x)$ for the set of morphisms from x to x.

A subgroupoid $\mathcal{H}$ of $\mathcal{G}$ is a subcategory $\mathcal{H}$ of $\mathcal{G}$ such that $h \in H_1 \Rightarrow \overline{h} \in H_1$. We say $\mathcal{H}$ is wide if $H_0 = G_0$ [7].

Let $\mathcal{G}$ be a groupoid and $\mathcal{N}$ be a wide subgroupoid of $\mathcal{G}$. Then $\mathcal{N}$ is called *normal* if

$$g \circ N(x) \circ \overline{g} \subseteq N(y)$$

that is

$$g \circ N(x) = N(y) \circ g,$$

for any $x, y \in G_0$ and $g \in G(x, y)$ [7].

Let $\mathcal{N}$ be a normal subgroupoid of $\mathcal{G}$. Then $\mathcal{N}$ defines an equivalence relation on the objects of $\mathcal{G}$ by $x \sim x'$ if and only if there is a morphism n of $\mathcal{N}$ such that $s(n) = x, t(n) = x'$. The equivalence classes are denoted by $[x]$, for $x \in G_0$ and the set of equivalence classes by G_0/N. $\mathcal{N}$ defines an equivalence relation on morphisms of $\mathcal{G}$ by $g \sim g'$ if and only if there are morphisms $m, n \in N$ such that $g = m \circ g' \circ n$, for $g, g' \in G$. Since $\mathcal{N}$ is a subgroupoid of $\mathcal{G}$, $\sim$ is an equivalence relation on $\mathcal{G}$. The equivalence classes are denoted by $[g]$, for $g \in G_1$ and the set of equivalence classes by G_1/N. Then $\mathcal{G}/\mathcal{N} = (G_0/N, G_1/N)$ is a groupoid with the structure maps $s([g]) = [s(g)], t([g]) = [t(g)], 1_{[x]} = [1_x], \overline{[g]} = [\overline{g}]$ and the product $[g_1] \circ [g] = [g_1 \circ n \circ g]$ where $s(g_1) \sim t(g)$ and $s(n) = t(g), t(n) = s(g_1)$. For further details see [17, p. 9], [11, p. 86] and [7, p. 420].

A crossed module consists of groups A, B with an action $\bullet \colon B \times A \to A$ of B on A and a homomorphism $\partial \colon A \to B$ such that $\partial(b \bullet a) = b\partial(a)b^{-1}$ and $\partial(a) \bullet a' = aa'a^{-1}$ [25, 26].

The structure of 2-categories was first introduced by Bénabou in 1967 [3]. For the basic references, see [2, 15]. A *2-category* $\mathcal{C} = (C_0, C_1, C_2)$ has a set C_0 of objects, a set of 1-morphisms C_1 and a set of 2-morphisms C_2 as follows

$$x \underset{g}{\overset{f}{\rightrightarrows}} \Downarrow\alpha \; y$$

with

- the source and the target maps
  $s \colon C_1 \to C_0, \; s(f) = x, \; s_h \colon C_2 \to C_0, \; s_h(\alpha) = x, \; s_v \colon C_2 \to C_1, \; s_v(\alpha) = f,$
  $t \colon C_1 \to C_0, \; t(f) = y, \; t_h \colon C_2 \to C_0, \; t_h(\alpha) = y, \; t_v \colon C_2 \to C_1, \; t_v(\alpha) = g,$
- the composition of 1-morphisms as in a classical category,

- the associative horizontal composition of 2-morphisms

$$x \underset{g}{\overset{f}{\Downarrow \alpha}} y \underset{g_1}{\overset{f_1}{\Downarrow \delta}} z \;=\; x \underset{g_1 \circ g}{\overset{f_1 \circ f}{\Downarrow \circ_h \alpha}} z \;,$$

- the associative vertical composition of 2-morphisms

$$x \overset{\overset{f}{\Downarrow \alpha}}{\underset{\underset{h}{\Downarrow \beta}}{\longrightarrow g}} y \;=\; x \underset{h}{\overset{f}{\Downarrow \circ_v \alpha}} y$$

- the identity maps $\varepsilon(x) = 1_x$, $\varepsilon_h(x) = 1_{1_x}$ as $x \underset{1_x}{\overset{1_x}{\Downarrow 1_{1_x}}} x$ and

$\varepsilon_v(f) = 1_f$ as $x \underset{f}{\overset{f}{\Downarrow 1_f}} y$

whenever the following diagram is commutative
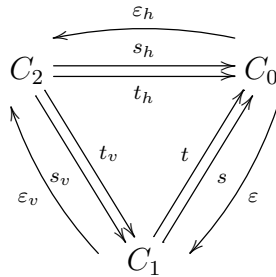
From the above diagram we can see that the construction of a 2-category $\mathcal{C} = (C_0, C_1, C_2)$ contains compatible category structures $\mathcal{C}_1 = (C_0, C_1, s, t, \varepsilon, \circ)$, $\mathcal{C}_2 = (C_0, C_2, s_h, t_h, \varepsilon_h, \circ_h)$ and $\mathcal{C}_3 = (C_1, C_2, s_v, t_v, \varepsilon_v, \circ_v)$ [13, 24]. In a 2-category, the horizontal composition and the vertical composition of 2-morphisms must satisfy the following interchange rule

$$(\theta \circ_v \delta) \circ_h (\beta \circ_v \alpha) = (\theta \circ_h \beta) \circ_v (\delta \circ_h \alpha)$$

whenever compositions are defined.

Let $\mathcal{C}, \mathcal{C}'$ be 2-categories. A *2-functor* F is a mapping from $\mathcal{C}$ to $\mathcal{C}'$ sending each object of $\mathcal{C}$ to an object of $\mathcal{C}'$, each 1-morphism of $\mathcal{C}$ to

1-morphism of $\mathcal{C}'$ and each 2-morphism of $\mathcal{C}$ to 2-morphism of $\mathcal{C}'$ as follows

$$x \overset{f}{\underset{g}{\Downarrow\alpha}} y \;\mapsto\; F(x) \overset{F(f)}{\underset{F(g)}{\Downarrow F(\alpha)}} F(y)$$

such that $F(h \circ f) = F(h) \circ F(f)$, $F(\delta \circ_h \alpha) = F(\delta) \circ_h F(\alpha)$, $F(\beta \circ_v \alpha) = F(\beta) \circ_v F(\alpha)$, $F(1_{1_x}) = 1_{F(1_x)} = 1_{1_{F(x)}}$ and $F(1_f) = 1_{F(f)}$ whenever compositions are defined. Thus, 2-categories and 2-functors form a category which is denoted by 2CAT [21].

A 2-groupoid is a 2-category whose all 1-morphisms and 2-morphisms are invertible as follows

$$x \overset{f}{\underset{g}{\Downarrow\alpha}} y \overset{\bar{f}}{\underset{\bar{g}}{\Downarrow\bar\alpha^h}} x \;=\; x \overset{1_x}{\underset{1_x}{\Downarrow 1_{1_x}}} x \;,$$

$$x \overset{f}{\underset{f}{\overset{\Downarrow\alpha}{\underset{\Downarrow\bar\alpha^v}{\longrightarrow g \longrightarrow}}}} y \;=\; x \overset{f}{\underset{f}{\Downarrow 1_f}} y \;.$$

Let $\mathcal{G}, \mathcal{G}'$ be 2-groupoids. A *morphism of 2-groupoids* is a 2-functor $F \colon \mathcal{G} \to \mathcal{G}'$ such that F preserves 2-groupoid structures. Thus 2-groupoids form a category which is denoted by 2GPD [21].

EXAMPLE 2.1. Let X be a set and G be a group. Then $\mathcal{G} = (X, X \times G \times X, X \times G \times G \times X)$ is a 2-groupoid in the following way. We assume that triples (x, g, y) and (x, h, y) are 1-morphisms from x to y and the 4-tuple (x, g, h, y) is 2-morphism from (x, g, y) to (x, h, y) as follows:

$$x \overset{(x,g,y)}{\underset{(x,h,y)}{\Downarrow (x,g,h,y)}} y \;.$$

We have these structure maps $s(x, g, y) = x$, $s_h(x, g, h, y) = x$, $s_v(x, g, h, y) = (x, g, y)$, $t(x, g, y) = y$, $t_h(x, g, h, y) = y$, $t_v(x, g, h, y) = (x, h, y)$, $\varepsilon(x) = (x, e, x)$, $\varepsilon_h(x) = (x, e, e, x)$, $\varepsilon_v(x, g, x) = (x, g, g, x)$, $\eta(x, g, y) = (y, g^{-1}, x)$, $\eta_h(x, g, h, y) = (y, g^{-1}, h^{-1}, x)$, $\eta_v(x, g, h, y) = (x, h, g, y)$ and compositions defined by $(y, g_1, z) \circ (x, g, y) = (x, g_1 g, z)$, $(y, g_1, h_1, z) \circ_h (x, g, h, y) = (x, g_1 g, h_1 h, z)$, $(x, h, k, y) \circ_v (x, g, h, y) = (x, g, k, y)$ where all compositions are defined. It is routine to check that the vertical composition and the horizontal composition satisfy the usual interchange law.

Now we recall the definitions of an action of groupoids and of crossed modules over groupoids from [6,10,13]. Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids over the same object set X and let $\mathcal{H}$ be totally disconnected. An action of $\mathcal{G}$ on $\mathcal{H}$ is a partially defined map

$$\bullet \colon G \times H \to H, \ (g, h) \mapsto g \bullet h$$

such that the following conditions holds

[AG 1] $g \bullet h$ is defined iff $t(h) = s(g)$ and $t(g \bullet h) = t(g)$,
[AG 2] $(g_2 \circ g_1) \bullet h = g_2 \bullet (g_1 \bullet h)$,
[AG 3] $g \bullet (h_2 \circ h_1) = (g \bullet h_2) \circ (g \bullet h_1)$, for $h_1, h_2 \in H(x)$ and $g \in G(x, y)$,
[AG 4] $1_x \bullet h = h$, for $h \in H(x)$.

From this conditions, it can be easily obtain that $g \bullet 1_x = 1_y$, for $g \in G(x, y)$.

Let $\mathcal{G} = (X, G)$ and $\mathcal{H} = (X, H)$ be groupoids and let $\mathcal{H}$ be totally disconnected. A crossed module $K = (\mathcal{H}, \mathcal{G}, \partial, \bullet)$ of groupoids consists of a morphism $\partial = (1, \partial) \colon \mathcal{H} \to \mathcal{G}$ of groupoids which is the identity on objects together with an action $\bullet \colon G \times H \to H$ of groupoids which satisfy

[CMG 1] $\partial(g \bullet h) = g \circ \partial(h) \circ \overline{g}$,
[CMG 2] $\partial(h) \bullet h_1 = h \circ h_1 \circ \overline{h}$, for $h, h_1 \in H(x)$ and $g \in G(x, y)$.

Let $K = (\mathcal{H}, \mathcal{G}, \partial, \bullet)$ and $K' = (\mathcal{H}', \mathcal{G}', \partial', \bullet')$ be crossed modules of groupoids. A *morphism of crossed modules of groupoids* is a triple $\lambda = (\lambda_0, \lambda_1, \lambda_2) \colon K \to K'$ such that $(\lambda_0, \lambda_1) \colon \mathcal{H} \to \mathcal{H}'$ and $(\lambda_0, \lambda_2) \colon \mathcal{G} \to \mathcal{G}'$ are morphisms of groupoids which satisfy $\lambda_2 \partial = \partial' \lambda_1$ and $\lambda_1(g \bullet h) = \lambda_2(g) \bullet' \lambda_1(h)$ [10, 13]. Hence crossed modules of groupoids and their morphisms form a category which we denoted by CMG.

The following theorem was proved by Icen in [13]. We give a sketch proof in terms of our notation.

THEOREM 2.2. *The category of 2-groupoids and the category of crossed module over groupoids are equivalent.*

Proof. Given any 2-groupoid $\mathcal{G} = (G_0, G_1, G_2)$, we know that $\mathcal{B} = (G_0, G_1)$ is a category. Let $A(x) = \{\alpha \in G_2 | s_v(\alpha) = \varepsilon(x)\}$, for $x \in G_0$ and $A = \{A(x)\}_{x \in G_0}$. Then $\mathcal{A} = (G_0, A)$ is a category. Now we can define a functor $\gamma \colon 2\text{GPD} \to \text{CMG}$ as equivalence of categories such

that $\gamma(\mathcal{G}) = (\mathcal{A}, \mathcal{B}, \partial)$ is a crossed module of groupoids with $\partial: \mathcal{A} \to \mathcal{B}$, $\partial(\alpha) = t_v(\alpha)$ and $\bullet: G_1 \times A \to A$, $(f \bullet \alpha) = 1_f \circ_h \alpha \circ_h 1_{\overline{f}}$.

$$y \underset{\partial(f\bullet\alpha)}{\overset{1_y}{\rightrightarrows}} {}_{\Downarrow f\bullet\alpha}\, y \;=\; y \underset{\overline{f}}{\overset{\overline{f}}{\rightrightarrows}} {}_{\Downarrow 1_{\overline{f}}}\, x \underset{\partial(\alpha)}{\overset{1_x}{\rightrightarrows}} {}_{\Downarrow \alpha}\, x \underset{f}{\overset{f}{\rightrightarrows}} {}_{\Downarrow 1_f}\, y$$

Clearly $\bullet$ is an action of $\mathcal{B}$ on $\mathcal{A}$ and

$$\partial(f \bullet \alpha) = f \circ \partial(\alpha) \circ \overline{f}, \ \partial(\alpha) \bullet \alpha_1 = \alpha \circ_h \alpha_1 \circ_h \overline{\alpha}^h,$$

for $f \in G_1(x, y)$ and $\alpha, \alpha_1 \in A(x)$.

Let $F = (F_0, F_1, F_2)$ be a morphism of 2-groupoids. Then $\gamma(F) = (F_2\big|_A, F_1, F_0)$ is a morphism of crossed modules over groupoids.

Conversely we define a functor $\theta: \mathrm{CMG} \to 2\mathrm{GPD}$ which is an equivalence of categories. Given a crossed module $K = (\mathcal{A}, \mathcal{B}, \partial)$ over groupoids $\mathcal{A} = (X, A)$ and $\mathcal{B} = (X, B)$, a 2-groupoid $\theta(K) = (X, B, B \ltimes A)$ can be constructed as in the following way where the set of 2-morphisms is the semi-direct product $B \ltimes A = \{(b, a) | b \in B, a \in A, s(a) = t(a) = t(b)\}$ of A and B. If $x \xrightarrow{b} y \xrightarrow{a} y$, then (b, a) is a 2-morphism as follows

$$x \underset{\partial(a)\circ b}{\overset{b}{\rightrightarrows}} {}_{\Downarrow (b,a)}\, y$$

with the horizontal composition of 2-morphisms

$$(b_1, a_1) \circ_h (b, a) = (b_1 \circ b, a_1 \circ (b_1 \bullet a))$$

when $y \xrightarrow{b_1} z \xrightarrow{a_1} z$ and the vertical composition of 2-morphisms is defined by

$$\left(\partial(a) \circ b, a'\right) \circ_v (b, a) = (b, a' \circ a)$$

when $y \xrightarrow{a'} y$. The structure maps are defined by

$$s_v(b, a) = b, \ t_v(b, a) = \partial(a) \circ b, \ \varepsilon_v(b) = (b, 1_y), \ \overline{(b, a)}^v = (\partial(a) \circ b, \overline{a}),$$

$$s_h(b, a) = s(b), \ t_h(b, a) = t(b), \ \varepsilon_h(x) = (1_x, 1_x), \ \overline{(b, a)}^h = (\overline{b}, \overline{b} \bullet \overline{a}).$$

Let $\lambda = (\lambda_2, \lambda_1, \lambda_0)$ be a morphism of crossed modules of groupoids. Then $\theta(\lambda) = (\lambda_0, \lambda_2, \lambda_2 \times \lambda_1)$ is morphism of 2-groupoids. $\qquad\square$

3. Normality and Quotients in Crossed Modules over Groupoids and 2-Groupoids

In this section using definitions of subcrossed modules and of normal subcrossed modules over groupoids and the equivalence of the categories as given in Theorem 2.2, we will define the normal and quotient objects in the category 2GPD of 2-groupoids. We recall the following two definitions are adopted in [20] from those in [22].

DEFINITION 3.1. Let $\mathcal{A} = (X, A)$, $\mathcal{B} = (X, B)$ be groupoids over the same object set, $\mathcal{A}$ be totally disconnected and $(\mathcal{A}, \mathcal{B}, \partial)$ be a crossed module of groupoids. A crossed module of groupoids $(\mathcal{M}, \mathcal{N}, \sigma)$ is called a subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$ if

[SCMG 1] $\mathcal{M} = (Y, M)$ is a subgroupoid of $\mathcal{A} = (X, A)$,

[SCMG 2] $\mathcal{N} = (Y, N)$ is a subgroupoid of $\mathcal{B} = (X, B)$,

[SCMG 3] σ is the restriction of ∂ to M,

[SCMG 4] the action of $\mathcal{N}$ on $\mathcal{M}$ is the restriction of the action of $\mathcal{B}$ on $\mathcal{A}$.

If $X = Y$ then $(\mathcal{M}, \mathcal{N}, \sigma)$ is called a wide subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$.

DEFINITION 3.2. A normal subcrossed module is a subcrossed module $(\mathcal{M}, \mathcal{N}, \sigma)$ of $(\mathcal{A}, \mathcal{B}, \partial)$ over groupoids which satisfies

[NCMG 1] $\mathcal{N}$ is normal subgroupoid of $\mathcal{B}$,

[NCMG 2] $b \bullet m \in M(y)$, for all $b \in B(x, y), m \in M(x)$,

[NCMG 3] $(n \bullet a) \circ \bar{a} \in M(x)$, for all $n \in N(x), a \in A(x)$.

From [NCMG2] we have that $\partial(a) \bullet m = a \circ m \circ \bar{a} \in M$ and so $\mathcal{M}$ is normal subgroupoid of $\mathcal{A}$.

EXAMPLE 3.3. Let $\lambda = (\lambda_0, \lambda_1, \lambda_2) \colon (\mathcal{A}, \mathcal{B}, \partial) \to (\mathcal{A}', \mathcal{B}', \partial')$ be morphism of crossed modules over groupoids. Then $(\operatorname{Ker} \lambda_1, \operatorname{Ker} \lambda_2, \partial^*)$ is a normal subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$ where ∂^* is the restriction of ∂. Since

$$\lambda_1(b \bullet m) = \lambda_2(b) \bullet \lambda_1(m) = \lambda_2(b) \bullet 1_x = 1_y,$$

$b \bullet m \in \operatorname{Ker} \lambda_1$, for $b \in B(x, y)$ and $m \in \operatorname{Ker} \lambda_1$. Now, for $n \in \operatorname{Ker} \lambda_2$ and $a \in A(x, x)$,

$$\lambda_1\Big((n \bullet a) \circ \bar{a}\Big) = \Big(\lambda_2(n) \bullet \lambda_1(a)\Big) \circ \lambda_1(\bar{a}) = (1_x \bullet \lambda_1(a)) \circ \lambda_1(\bar{a}) = \lambda_1(a) \circ \lambda_1(\bar{a}) = 1_x$$

and so $(n \bullet a) \circ \bar{a} \in \operatorname{Ker} \lambda_2$.

The following theorem is proved in [20].

THEOREM 3.4. *Let $(\mathcal{M}, \mathcal{N}, \sigma)$ be normal subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$ over groupoids and N be totally disconnected. Then $(\mathcal{A}/\mathcal{M}, \mathcal{B}/\mathcal{N}, \rho)$ is a crossed module of groupoids.*

The crossed module $(\mathcal{A}/\mathcal{M}, \mathcal{B}/\mathcal{N}, \rho)$ over groupoids is called *the quotient crossed module of groupoids*.

We now give the definition of sub2-groupoid and normal sub2-groupoid as follows.

DEFINITION 3.5. Let $\mathcal{G} = (G_0, G_1, G_2)$ be a 2-groupoid. We say $\mathcal{H} = (H_0, H_1, H_2)$ is a *sub2-groupoid* of $\mathcal{G}$ if

[S2G 1] $H_0 \subseteq G_0$,
[S2G 2] $H_1(x, y) \subseteq G_1(x, y)$, $H_2(x, y) \subseteq G_2(x, y)$, $H_2(f, g) \subseteq G_2(f, g)$,
[S2G 3] $1_x \in H_1$, $1_{1_x} \in H_2$, for each $x \in H_0$ and $1_f \in H_2$, for each $f \in H_1$,
[S2G 4] all compositions in $\mathcal{H}$ are the same as those for $\mathcal{G}$.

The sub2-groupoid $\mathcal{H}$ of $\mathcal{G}$ is full if $H_1(x, y) = G_1(x, y)$ and $H_2(x, y) = G_2(x, y)$, for all $x, y \in H_0$; and $\mathcal{H}$ is a wide sub2-groupoid of $\mathcal{G}$ if $H_0 = G_0$. Hence, if $\mathcal{H} = (H_0, H_1, H_2)$ is a full(wide) sub2-groupoid of $\mathcal{G}$, then (H_0, H_1) is full(wide) subgroupoid of (G_0, G_1) and (H_0, H_2) is full(wide) subgroupoid of (G_0, G_2).

DEFINITION 3.6. Let $\mathcal{G} = (G_0, G_1, G_2)$ be a 2-groupoid and $\mathcal{N} = (G_0, N_1, N_2)$ be a wide sub2-groupoid of $\mathcal{G}$. $\mathcal{N}$ is called *normal sub2-groupoid* of $\mathcal{G}$ if

[N2G 1] (G_0, N_1) is normal subgroupoid of (G_0, G_1), that is $f \circ N_1(x) \circ \overline{f} \subseteq N_1(y)$,
[N2G 2] (G_0, N_2) is normal subgroupoid of (G_0, G_2), that is $\alpha \circ_h N_2(x) \circ_h \overline{\alpha}^h \subseteq N_2(y)$

for any objects x, y of $\mathcal{G}$ and $f \in N_1(x, y), \alpha \in N_2(x, y)$.

The following corollary agree with the one in [19, 20].

COROLLARY 3.7. *Let $\mathcal{G}$ be a 2-groupoid with a single object and $\mathcal{N}$ be a normal sub2-groupoid of $\mathcal{G}$. Since $\mathcal{G}$ is a group-groupoid, $\mathcal{N}$ is a normal subgroup-groupoid of $\mathcal{G}$.*

EXAMPLE 3.8. Consider any normal subgroup N of a group G. Then we can consruct a 2-groupoid with a set X as in Example 2.1. Since

$$(x, g, y) \circ (x, N, x) = (x, gN, y) = (x, Ng, y) = (y, N, y) \circ (x, g, y)$$

and

$$(x, g, h, y) \circ_h (x, N, N, x) = (x, gN, hN, y)$$
$$= (x, Ng, Nh, y) = (y, N, N, y) \circ_h (x, g, h, y),$$

$\mathcal{N}$ is a normal sub2-groupoid of $\mathcal{G}$.

EXAMPLE 3.9. Let $F \colon \mathcal{G} \to \mathcal{H}$ be a morphism of 2-groupoids. Then $\mathcal{N} = \operatorname{Ker} F$ is a normal sub2-groupoid of $\mathcal{G}$ where $\operatorname{Ker} F$ is the wide sub2-groupoid of $\mathcal{G}$ whose 1-morphisms are all $f \in G_1$ such that $F(f) = 1$ and whose 2-morphisms are all $\alpha \in G_2$ such that $F(\alpha) = 1_1$.

THEOREM 3.10. *Let $\mathcal{N}$ be a normal sub2-groupoid of $\mathcal{G}$. Then $\mathcal{G}/\mathcal{N} = (G_0/N, G_1/N, G_2/N)$ is a 2-groupoid.*

Proof. Let $\mathcal{N}$ be a normal sub2-groupoid of $\mathcal{G}$. Then $\mathcal{N}$ defines an equivalence relation on the objects of $\mathcal{G}$ by $x \sim x'$ if and only if there is a 2-morphism n_2 of $\mathcal{N}$ such that $s_h(n_2) = x, t_h(n_2) = x'$. The equivalence classes are denoted by $[x]$, for $x \in G_0$ and the set of equivalence classes by G_0/N. On 2-morphisms, $\mathcal{N}$ defines an equivalence relation of $\mathcal{G}$ by $\alpha \sim \alpha'$ if and only if there are 2-morphisms $m_2, n_2 \in N_2$ such that $\alpha = m_2 \circ_h \alpha' \circ_h n_2$, for $\alpha, \alpha' \in G_2$. Since $\mathcal{N}$ is a sub2-groupoid of $\mathcal{G}$, $\sim$ is an equivalence relation on $\mathcal{G}$. The equivalence classes are denoted by $[\alpha]$, for $\alpha \in G_2$ and the set of equivalence classes by G_2/N. Then $\mathcal{G}/\mathcal{N} = (G_0/N, G_1/N, G_2/N)$ is a 2-groupoid with the structure maps $s_h([\alpha]) = [s_h(\alpha)], t_h([\alpha]) = [t_h(\alpha)], 1_{1_{[x]}} = [1_{1_x}], \overline{[\alpha]}^h = [\overline{\alpha}^h]$ and the product $[\alpha_1] \circ_h [\alpha] = [\alpha_1 \circ_h n_2 \circ_h \alpha]$ where $s_h(\alpha_1) \sim t_h(\alpha)$ and $s_h(n_2) = t_h(\alpha), t_h(n_2) = s(\alpha_1)$. Since $y \sim y'$ and

$$x \underset{g}{\overset{f}{\rightrightarrows}} {\Downarrow \alpha}\ y \underset{m_1}{\overset{n_1}{\rightrightarrows}} {\Downarrow n_2}\ y' \underset{g_1}{\overset{f_1}{\rightrightarrows}} {\Downarrow \alpha_1}\ z,$$

we can draw

$$[x] \underset{[g]}{\overset{[f]}{\rightrightarrows}} {\Downarrow [\alpha]}\ [y] \underset{[g_1]}{\overset{[f_1]}{\rightrightarrows}} {\Downarrow [\alpha_1]}\ [z] \;=\; [x] \underset{[g_1]\circ[g]}{\overset{[f_1]\circ[f]}{\rightrightarrows}} {[\alpha\Downarrow]\circ_h [\alpha]}\ [z].$$

The vertical composition of 2-morphisms is defined by $[\beta] \circ_v [\alpha] = [\beta \circ_v \alpha]$ where the structure maps are defined as $s_v([\alpha]) = [s_v(\alpha)], t_v([\alpha]) = [t_v(\alpha)], 1_{[f]} = [1_f], \overline{[\alpha]}^v = [\overline{\alpha}^v]$. It is easy to check that the vertical composition and the horizontal composition satisfy the usual interchange rule. $\qquad\square$

The 2-groupoid $\mathcal{G}/\mathcal{N}$ is called *quotient 2-groupoid*.

THEOREM 3.11. *Let* $(\mathcal{M}, \mathcal{N}, \sigma)$ *be a normal subcrossed module of* $(\mathcal{A}, \mathcal{B}, \partial)$ *over the same object set* X. *Then* $\mathcal{H} = (X, N, N \ltimes M)$ *is a normal sub2-groupoid of* $\mathcal{G} = (X, B, B \ltimes A)$.

Proof. We only need to show that the condition [N2G2] is satisfied. For $b \in B(x, y)$, $a \in A(y, y)$ and $(n_x, m_x) \in (N \ltimes M)(x) = N(x) \ltimes M(x)$,

$$(b, a) \circ_h (n_x, m_x) \circ_h \overline{(b, a)}^h = \left(b \circ n_x, a \circ (b \bullet m_x) \right) \circ_h (\bar{b}, \bar{b} \bullet \bar{a})$$

$$= \left(b \circ n_x \circ \bar{b}, a \circ (b \bullet m_x) \circ ((b \circ n_x) \bullet (\bar{b} \bullet \bar{a})) \right)$$

$$= \left(b \circ n_x \circ \bar{b}, a \circ (b \bullet m_x) \circ ((b \circ n_x \circ \bar{b}) \bullet \bar{a}) \right)$$

Let $b \bullet m_x = m_y$ and $b \circ n_x \circ \bar{b} = n_y$. Then, from [NCMG1] $n_y \in N(y)$, from [NCMG2] $m_y \in M(y)$ and from [NCMG3] $(n_y \bullet \bar{a}) \circ a = m'_y \in M(y)$. Now

$$(b, a) \circ_h (n_x, m_x) \circ_h \overline{(b, a)}^h = \left(n_y, a \circ m_y \circ (n_y \bullet \bar{a}) \circ a \circ \bar{a} \right)$$

$$= \left(n_y, a \circ m_y \circ m'_y \circ \bar{a} \right) \in (N \ltimes M)(y).$$

$\square$

THEOREM 3.12. *Let* $\mathcal{H} = (G_0, H_1, H_2)$ *be a normal sub2-groupoid of* $\mathcal{G} = (G_0, G_1, G_2)$. *Then the crossed module corresponding to* $\mathcal{H}$ *is a normal subcrossed module of the one corresponding to* $\mathcal{G}$.

Proof. Let $M = \{H_2(x)\}_{x \in G_0}$ and $A = \{G_2(x)\}_{x \in G_0}$. Then $(\mathcal{M}, \mathcal{N}, \sigma)$ is a crossed module over groupoids $\mathcal{M} = (G_0, M)$ and $\mathcal{N} = (G_0, H_1)$; $(\mathcal{A}, \mathcal{B}, \partial)$ is a crossed module over groupoids $\mathcal{A} = (G_0, A)$ and $\mathcal{B} = (G_0, G_1)$.

[NCMG 1] Clearly $\mathcal{N} = (G_0, H_1)$ is normal subgroupoid of $\mathcal{B} = (G_0, G_1)$.

[NCMG 2] Let $f \in G_1(x, y), \alpha \in H_2(x)$. Then

$$f \bullet \alpha = 1_f \circ_h \alpha \circ_h 1_{\bar{f}} = 1_f \circ_h \alpha \circ_h \overline{1_f} \in H_2(y).$$

[NCMG 3] Let $h \in H_1(x)$ and $\alpha \in G_2(x)$. Then

$$(h \bullet \alpha) \circ_h \bar{\alpha}^h = 1_h \circ_h \alpha \circ_h 1_{\bar{h}} \circ_h \bar{\alpha}^h$$

Since $\mathcal{N}$ is normal subgroupoid of $\mathcal{B}$, then $\alpha \circ_h 1_{\bar{f}} \circ_h \bar{\alpha}^h \in H_2(x)$. Since $1_h \in H_2(x)$ it implies that $(h \bullet \alpha) \circ_h \bar{\alpha}^h \in H_2(x)$. $\square$

COROLLARY 3.13. *Let $\mathcal{G}$ be a 2-groupoid and K be the crossed module of groupoids corresponding to $\mathcal{G}$. Then the category N2G/$\mathcal{G}$ of normal sub2-groupoids of $\mathcal{G}$ is equivalent to the category NCMG/K of normal subcrossed modules of K.*

PROPOSITION 3.14. *Let $(\mathcal{M}, \mathcal{N}, \sigma)$ be a normal subcrossed module of $(\mathcal{A}, \mathcal{B}, \partial)$. Then we have the isomorphism*

$$A/M \ltimes B/N \cong (A \ltimes B)/(M \ltimes N).$$

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Sedat Temel
Department of Mathematics
Recep Tayyip Erdoğan University
Rize, Turkey
E-mail: sedat.temel@erdogan.edu.tr