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SLOWLY CHANGING FUNCTION ORIENTED GROWTH MEASUREMENT OF DIFFERENTIAL POLYNOMIAL AND DIFFERENTIAL MONOMIAL

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Abstract. In the paper we establish some new results depending on the comparative growth properties of composite entire and meromorphic functions using relative ${}_pL^*$-order, relative ${}_pL^*$-lower order and differential monomials, differential polynomials generated by one of the factors.

1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [9, 11, 19, 20]. We also use the standard notations and definitions of the theory of entire functions which are available in [16] and therefore we do not explain those in details. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r} |f(z)|$. In this connection the following definition is relevant:

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DEFINITION 1. [6] A non-constant entire function f is said to have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may see [6].

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$. Now we just recall the following properties of meromorphic functions which will be needed in the sequel.

Let $n_{0j}, n_{1j}, ... n_{kj} (k \geq 1)$ be non-negative integers such that for each j, $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. For a non-constant meromorphic function f , we call $M_j[f] = A_j(f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} ... \left(f^{(k)}\right)^{n_{kj}}$ where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{Mj} = \sum\limits_{i=0}^{k} n_{ij}$ and $\Gamma_{Mj} = \sum\limits_{i=0}^{k} (i+1) n_{ij}$ are called respectively the degree and weight of $M_j[f]$ ([7], [15]). The expression $P[f] = \sum\limits_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max\limits_{1 \leq j \leq s} \gamma_{Mj}$ and $\Gamma_P = \max\limits_{1 \leq j \leq s} \Gamma_{Mj}$ are called respectively the degree and weight of $P[f]$ ([7], [15]). Also we call the numbers $\underline{\gamma_P} = \min\limits_{1 \leq j \leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_p} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper, we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f, i.e., for which $n_{0j} = 0$ for $j = 1, 2, ...s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

However, the Nevanlinna's Characteristic function of a meromorphic function f is define as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)$ $(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may employ $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover for any non-constant entire function f, $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (|T_f(0)|,\infty) \to (0,\infty)$ is exists where $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection we immediately remind the following definitions which are relevant:

Definition 2. Let 'a' be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of 'a' with respect to a meromorphic function f are defined as

$$\delta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \underline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}$$

and

$$\Delta(a;f) = 1 - \underline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \overline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}.$$

Definition 3. The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows

$$\Theta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_f(r,a)}{T_f(r)}.$$

Definition 4. [18] For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n_{f|=1}(r,a)$, the number of simple zeros of $f-a$ in $|z| \leq r$. $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way. We put

$$\delta_1(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{f|=1}(r,a)}{T_f(r)},$$

the deficiency of 'a' corresponding to the simple a-points of f i.e., simple zeros of $f-a$.

Yang [17] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C}\cup\{\infty\}$ for which $\delta_1(a;f) > 0$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \leq 4$.

Definition 5. [10] For $a \in \mathbb{C}\cup\{\infty\}$, let $n_p(r,a;f)$ denotes the number of zeros of $f-a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define

$$\delta_p(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_p(r,a;f)}{T_f(r)}.$$

Definition 6. [1] $P[f]$ is said to be admissible if

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.

However in case of any two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called as the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions. Further the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders are normally defined in terms of their growth with respect to the exp function which are shown in the following definition:

Definition 7. The order $\rho(f)$ (the lower order $\lambda(f)$) of a meromorphic function f is defined as

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log T_{\exp z}(r)}=\overline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)}=\overline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log(r)+O(1)}$$

$$\left(\lambda(f)=\underline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log T_{\exp z}(r)}=\underline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)}=\underline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log(r)+O(1)}\right).$$

If f is entire, then

$$\rho(f)=\overline{\lim_{r\to\infty}}\frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)}=\overline{\lim_{r\to\infty}}\frac{\log^{[2]}M_f(r)}{\log r}$$

$$\left(\lambda(f)=\underline{\lim_{r\to\infty}}\frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)}=\underline{\lim_{r\to\infty}}\frac{\log^{[2]}M_f(r)}{\log r}\right).$$

Somasundaram and Thamizharasi [14] introduced the notions of L-order and L-lower order for entire functions where $L\equiv L(r)$ is a positive continuous function increasing slowly, i.e., $L(ar)\sim L(r)$ as $r\to\infty$ for every positive constant "a". The more generalized concept of L-order and L-lower order of meromorphic functions are L^*-order and L^*-lower order respectively which are as follows:

Definition 8. [14] The L^*-order $\rho^{L^*}(f)$ and the L^*-lower order $\lambda^{L^*}(f)$ of a meromorphic function f are defined by

$$\rho^{L^*}(f)=\overline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}\text{ and }\lambda^{L^*}(f)=\underline{\lim_{r\to\infty}}\frac{\log T(r,f)}{\log\left[re^{L(r)}\right]}.$$

If f is entire, then

$$\rho^{L^*}(f)=\overline{\lim_{r\to\infty}}\frac{\log^{[2]}M_f(r)}{\log\left[re^{L(r)}\right]}\text{ and }\lambda^{L^*}(f)=\underline{\lim_{r\to\infty}}\frac{\log^{[2]}M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Extending the notion of Somasundaram and Thamizharasi [14], one may introduce concept of ${}_pL^*$-order and ${}_pL^*$-lower order of a meromorphic function f which are as follows:

Definition 9. For any positive integer p, the ${}_pL^*$-order $\rho_p^{L^*}(f)$ and the ${}_pL^*$-lower order $\lambda_p^{L^*}(f)$ of a meromorphic function f are defined by

$$\rho_p^{L^*}(f)=\overline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log\left[r\exp^{[p]}L(r)\right]}\text{ and }\lambda_p^{L^*}(f)=\underline{\lim_{r\to\infty}}\frac{\log T_f(r)}{\log\left[r\exp^{[p]}L(r)\right]}.$$

If f is entire, then

$$\rho_p^{L^*}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log\left[r \exp^{[p]} L(r)\right]} \text{ and } \lambda_p^{L^*}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log\left[r \exp^{[p]} L(r)\right]}.$$

Lahiri and Banerjee [12] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 10. [12] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as

$$\begin{aligned}\rho(f,g) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\}\\ &= \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1}(T_f(r))}{\log r}.\end{aligned}$$

The definition coincides with the classical one [12] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1}(T_f(r))}{\log r}.$$

In order to make some progress in the study of relative order, one may introduce the definitions of relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of a meromorphic function f with respect to an entire g which are as follows:

Definition 11. [3] The relative ${}_pL^*$-order denoted as $\rho_p^{L^*}(f,g)$ and relative ${}_pL^*$-lower order denoted as $\lambda_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire g are defined as

$$\rho_p^{L^*}(f,g) = \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1}(T_f(r))}{\log\left[r \exp^{[p]} L(r)\right]} \text{ and } \lambda_p^{L^*}(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1}(T_f(r))}{\log\left[r \exp^{[p]} L(r)\right]},$$

where p is any positive integers.

In the paper we establish some new results depending on the comparative growth properties of composite entire and meromorphic functions using relative ${}_pL^*$-order (respectively, relative ${}_pL^*$-lower order) and differential monomials, differential polynomials generated by one of the factors. Indeed some works on relative ${}_pL^*$-order (respectively, relative ${}_pL^*$-lower order) related to the growth of composite entire and meromorphic functions have also been explored in [2–4].

2. Preliminaries

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [5] *If f is meromorphic and g is entire then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leq \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 2. [13] *Let f and g be any two entire functions. Then for all $r > 0$,*

$$T_{f\circ g}(r) \geq \frac{1}{3} \log M_f \left\{ \frac{1}{8} M_g\left(\frac{r}{4}\right) + o(1) \right\}.$$

Lemma 3. [8] *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

Lemma 4. [3] *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and g be an entire function with regular growth having nonzero finite order and $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $P_0[f]$ with respect to $P_0[g]$ are same as those of f with respect to g for homogeneous $P_0[f]$ and $P_0[g]$, i.e.,*

$$\rho_p^{L^*}(P_0[f], P_0[g]) = \rho_p^{L^*}(f, g) \text{ and } \lambda_p^{L^*}(P_0[f], P_0[g]) = \lambda_p^{L^*}(f, g).$$

Lemma 5. [3] *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and g be a transcendental entire function with regular growth and nonzero finite order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $M[f]$ with respect to $M[g]$ are same as those of f with respect to g, i.e.,*

$$\rho_p^{L^*}(M[f], M[g]) = \rho_p^{L^*}(f, g) \text{ and } \lambda_p^{L^*}(M[f], M[g]) = \lambda_p^{L^*}(f, g).$$

3. Main results

In this section we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ for meromorphic f will be needed as per the requirements of the theorems.

THEOREM 1. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let g, k be any two entire functions, $\rho_p^{L^*}(g) < \rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g, k) < \infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) \cdot K(r, g; L)} = 0 \ , \ \textit{where}$$

$$K(r, g; L) = \begin{cases} 1 \textit{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \textit{ as } r\to\infty \\ \qquad\qquad\qquad\qquad \textit{and for some } \beta < \rho_p^{L^*}(f, h) \\ \exp^{[p-1]} L(M_g(r)) \textit{ otherwise.} \end{cases}$$

Proof. Let us consider that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 3. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 1, Lemma 3 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [9] } for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}(T_{f\circ g}(r)) &\leqslant T_h^{-1}(\{1+o(1)\} T_f(M_g(r))) \\ \textit{i.e., } T_h^{-1}(T_{f\circ g}(r)) &\leqslant \alpha\left(T_h^{-1} T_f(M_g(r))\right) \\ \textit{i.e., } \log T_h^{-1}(T_{f\circ g}(r)) &\leqslant \log T_h^{-1}(T_f(M_g(r))) + O(1) \end{aligned}$$

(1)
$$\log T_h^{-1}(T_{f\circ g}(r)) \leqslant \left(\rho_p^{L^*}(f, h) + \varepsilon\right)\left[\log M_g(r) + \exp^{[p-1]} L(M_g(r))\right] + O(1).$$

Now from the definition of $\rho_p^{L^*}(g)$, we obtain for all sufficiently large positive numbers of r that

(2) $$\log^{[2]} M_g(r) \leq \left(\rho_p^{L^*}(g) + \varepsilon\right)\left[\log r + \exp^{[p-1]} L(r)\right].$$

Also from the definition of $\rho_p^{L^*}(g,k)$, we get for all sufficiently large positive numbers of r that

(3) $$\log T_k^{-1}(T_g(r)) \leq \left(\rho_p^{L^*}(g,k)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right].$$

Therefore from (1) and in view of (2), we get for all sufficiently large positive numbers of r that

(4) $$\log T_h^{-1}(T_{f\circ g}(r))$$
$$\leq O(1)+\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\left[\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}+\exp^{[p-1]}L(M_g(r))\right].$$

Now from (3) and (4), it follows for all sufficiently large positive numbers of r that

(5) $$\log T_h^{-1}(T_{f\circ g}(r))+\log T_k^{-1}(T_g(r)) \leq$$
$$\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\left[\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}+\exp^{[p-1]}L(M_g(r))\right]$$
$$O(1)+\left(\rho_p^{L^*}(g,k)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right].$$

Also from the definition of $\rho_p^{L^*}(P_0[f],P_0[h])$ and in view of Lemma 4, we obtain for a sequence of positive numbers of r tending to infinity that

$$\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) \geq \left(\rho_p^{L^*}(P_0[f],P_0[h])-\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]$$
$$i.e.,\ \log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) \geq \left(\rho_p^{L^*}(f,h)-\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]$$

(6) $$i.e.,\ T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) \geq \left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(f,h)-\varepsilon\right)}.$$

Now from (5) and (6), we get for a sequence of positive numbers of r tending to infinity that

(7) $$\frac{\log T_h^{-1}(T_{f\circ g}(r))+\log T_k^{-1}(T_g(r))}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)} \leq$$
$$\frac{O(1)+\left(\rho_p^{L^*}(g,k)+\varepsilon\right)\log\left[r\exp^{[p]}L(r)\right]}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)}+$$
$$\frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\left[\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}+\exp^{[p-1]}L(M_g(r))\right]}{\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(f,h)-\varepsilon\right)}}.$$

Since $\rho_p^{L^*}(g) < \rho_p^{L^*}(f,h)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\rho_p^{L^*}(g) + \varepsilon < \rho_p^{L^*}(f,h) - \varepsilon. \tag{8}$$

Case I. Let $\exp^{[p-1]} L(M_g(r)) = o\left\{\left[r\exp^{[p]} L(r)\right]^{\beta}\right\}$ as $r \to \infty$ and for some $\beta < \rho_p^{L^*}(f,h)$.
As $\beta < \rho_p^{L^*}(f,h)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\beta < \rho_p^{L^*}(f,h) - \varepsilon. \tag{9}$$

Since $\exp^{[p-1]} L(M_g(r)) = o\left\{\left[r\exp^{[p]} L(r)\right]^{\beta}\right\}$ as $r \to \infty$ we get on using (9) that

$$\frac{\exp^{[p-1]} L(M_g(r))}{\left[r\exp^{[p]} L(r)\right]^{\beta}} \to 0 \text{ as } r \to \infty$$

$$i.e., \quad \frac{\exp^{[p-1]} L(M_g(r))}{\left[r\exp^{[p]} L(r)\right]^{\left(\rho_p^{L^*}(f,h)-\varepsilon\right)}} \to 0 \text{ as } r \to \infty. \tag{10}$$

Now in view of (7), (8) and (10) we get that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)} = 0\ . \tag{11}$$

Case II. If $\exp^{[p-1]} L(M_g(r)) \neq o\left\{\left[r\exp^{[p]} L(r)\right]^{\beta}\right\}$ as $r \to \infty$ and for some $\beta < \rho_p^{L^*}(f,h)$ then we get from (7) for a sequence of positive numbers of r tending to infinity that

$$\frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)\cdot \exp^{[p-1]} L(M_g(r))} \leq \tag{12}$$

$$\frac{O(1) + \left(\rho_p^{L^*}(g,k)+\varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]}{\left[r\exp^{[p]} L(r)\right]^{\left(\rho_p^{L^*}(f,h)-\varepsilon\right)}\cdot \exp^{[p-1]} L(M_g(r))} +$$

$$\frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\left[\left[r\exp^{[p]} L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)} + \exp^{[p-1]} L(M_g(r))\right]}{\left[r\exp^{[p]} L(r)\right]^{\left(\rho_p^{L^*}(f,h)-\varepsilon\right)}\cdot \exp^{[p-1]} L(M_g(r))}.$$

Now using (8), it follows from (12) that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)\cdot \exp^{[p-1]} L(M_g(r))} = 0\ . \tag{13}$$

Combining (11) and (13) we obtain that

$$\lim_{r\to\infty}\frac{\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)+\log T_k^{-1}\left(T_g\left(r\right)\right)}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)\cdot K\left(r,g;L\right)}=0,$$

$$\text{where } K\left(r,g;L\right)=\begin{cases}1 \text{ if } \exp^{[p-1]}L\left(M_g\left(r\right)\right)=o\left\{\left[r\exp^{[p]}L\left(r\right)\right]^{\beta}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad\qquad \text{and for some } \beta<\ \rho_p^{L^*}\left(f,h\right) \\ \exp^{[p-1]}L\left(M_g\left(r\right)\right) \text{ otherwise.}\end{cases}$$

Thus the theorem is established. □

Theorem 2. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta\left(\infty;f\right)=\sum_{a\neq\infty}\delta_p\left(a;f\right)=1$ or $\delta\left(\infty;f\right)=\sum_{a\neq\infty}\delta\left(a;f\right)=1$ and h be an entire function having regular growth and nonzero finite order with $\Theta\left(\infty;h\right)=\sum_{a\neq\infty}\delta_p\left(a;h\right)=1$ or $\delta\left(\infty;h\right)=\sum_{a\neq\infty}\delta\left(a;h\right)=1$. Also let g, k be any two entire functions, $\rho_p^{L^*}\left(g\right)<\lambda_p^{L^*}\left(f,h\right)<\infty$ and $\rho_p^{L^*}\left(g,k\right)<\infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)+\log T_k^{-1}\left(T_g\left(r\right)\right)}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)\cdot K\left(r,g;L\right)}=0,$$

$$\text{where } K\left(r,g;L\right)=\begin{cases}1 \text{ if } \exp^{[p-1]}L\left(M_g\left(r\right)\right)=o\left\{\left[r\exp^{[p]}L\left(r\right)\right]^{\beta}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad\qquad \text{and for some } \beta<\ \lambda_p^{L^*}\left(f,h\right) \\ \exp^{[p-1]}L\left(M_g\left(r\right)\right) \text{ otherwise.}\end{cases}$$

Theorem 3. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta\left(\infty;f\right)=\sum_{a\neq\infty}\delta_p\left(a;f\right)=1$ or $\delta\left(\infty;f\right)=\sum_{a\neq\infty}\delta\left(a;f\right)=1$ and h be an entire function having regular growth and nonzero finite order with $\Theta\left(\infty;h\right)=\sum_{a\neq\infty}\delta_p\left(a;h\right)=1$ or $\delta\left(\infty;h\right)=\sum_{a\neq\infty}\delta\left(a;h\right)=1$. Also let g, k be any two entire functions, $\lambda_p^{L^*}\left(g\right)<\lambda_p^{L^*}\left(f,h\right)\leq\rho_p^{L^*}\left(f,h\right)<\infty$ and $\rho_p^{L^*}\left(g,k\right)<\infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)+\log T_k^{-1}\left(T_g\left(r\right)\right)}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)\cdot K\left(r,g;L\right)}=0,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \text{ as } r \to \infty \\ \qquad \text{and for some } \beta < \lambda_p^{L^*}(f,h) \\ \exp^{[p-1]} L(M_g(r)) \text{ otherwise.} \end{cases}$$

Theorem 4. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ or $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$. Also let g, k be any two entire functions, $\rho_p^{L^*}(g) < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(g,k) < \infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) \cdot K(r,g;L)} = 0,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \text{ as } r \to \infty \\ \qquad \text{and for some } \beta < \lambda_p^{L^*}(f,h) \\ \exp^{[p-1]} L(M_g(r)) \text{ otherwise.} \end{cases}$$

The proofs of Theorem 2, Theorem 3 and Theorem 4 are omitted because those can be carried out in the line of Theorem 1.

In the line of Theorem 1, Theorem 2, Theorem 3 and Theorem 4 respectively and with the help of Lemma 5, one can easily prove the following four theorems and therefore their proofs are omitted:

Theorem 5. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. Also let g, k be any two entire functions, $\rho_p^{L^*}(g) < \rho_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(g,k) < \infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{M[h]}^{-1}\left(T_{M[f]}(r)\right) \cdot K(r,g;L)} = 0,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad\qquad\qquad \text{and for some } \beta < \rho_p^{L^*}(f,h) \\ \exp^{[p-1]} L(M_g(r)) \text{ otherwise.} \end{cases}$$

Theorem 6. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. Also let g, k be any two entire functions, $\rho_p^{L^*}(g) < \lambda_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(g,k) < \infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)\cdot K(r,g;L)} = 0 \ ,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad\qquad\qquad \text{and for some } \beta < \lambda_p^{L^*}(f,h) \\ \exp^{[p-1]} L(M_g(r)) \text{ otherwise.} \end{cases}$$

Theorem 7. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. Also let g, k be any two entire functions, $\lambda_p^{L^*}(g) < \lambda_p^{L^*}(f,h) \le \rho_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(g,k) < \infty$ where p is any positive integer. If h satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)\cdot K(r,g;L)} = 0,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad\qquad\qquad \text{and for some } \beta < \lambda_p^{L^*}(f,h) \\ \exp^{[p-1]} L(M_g(r)) \text{ otherwise.} \end{cases}$$

Theorem 8. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. Also let g, k be any two entire functions,*

$\rho_p^{L^*}(g) < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$ *and* $\rho_p^{L^*}(g,k) < \infty$ *where* p *is any positive integer. If* h *satisfies the Property (A), then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r)) + \log T_k^{-1}(T_g(r))}{T_{M[h]}^{-1}\left(T_{M[f]}(r)\right) \cdot K(r,g;L)} = 0,$$

where
$$K(r,g;L) = \begin{cases} 1 \text{ if } \exp^{[p-1]} L(M_g(r)) = o\left\{\left[r \exp^{[p]} L(r)\right]^{\beta}\right\} \text{ as } r\to\infty \\ \qquad\qquad \text{and for some } \beta < \lambda_p^{L^*}(f,h) \\ \exp^{[p-1]} L(M_g(r)) \text{ otherwise.}\end{cases}$$

Theorem 9. *Let* f *be an entire function either of finite order or of nonzero lower order such that* $\Theta(\infty;f) = \sum\limits_{a\neq\infty} \delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum\limits_{a\neq\infty} \delta(a;f) = 1$ *and* h *be an entire function having regular growth and nonzero finite order with* $\Theta(\infty;h) = \sum\limits_{a\neq\infty} \delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum\limits_{a\neq\infty} \delta(a;h) = 1$*. Also let* g *be any entire function,* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$*,* $0 < \lambda_p^{L^*}(g) < \infty$ *where* p *is any positive integer. If* h *satisfies the Property (A), then for every constant* A *and for any real number* x*,*

$$\lim_{r\to\infty} \frac{\log T_h^{-1}(T_{f\circ g}(r))}{\left\{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r^A)\right)\right\}^{1+x}} = \infty .$$

Proof. If x is such that $1+x \leq 0$, then the theorem is obvious. So we suppose that $1+x>0$. Let us consider that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 3. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 2, Lemma 3 and the inequality $T_g(r) \leq \log M_g(r)$ {cf. [9] } for all sufficiently large positive numbers of r that

$$T_h^{-1}(T_{f\circ g}(r)) \geq T_h^{-1}\left(\frac{1}{3}T_f\left(\frac{1}{8}M_g\left(\frac{r}{4}\right) + o(1)\right)\right)$$

$$i.e.,\ T_h^{-1}(T_{f\circ g}(r)) \geq \left(\frac{1}{\alpha}T_h^{-1}\left(T_f\left(\frac{1}{8}M_g\left(\frac{r}{4}\right) + o(1)\right)\right)\right)$$

$$i.e.,\ \log T_h^{-1}(T_{f\circ g}(r)) \geq \log\left(\frac{1}{\alpha}T_h^{-1}T_f\left(\frac{1}{8}M_g\left(\frac{r}{4}\right) + o(1)\right)\right)$$

(14)

$$i.e.,\ \log T_h^{-1}(T_{f\circ g}(r)) \geq O(1) + \log T_h^{-1}\left(T_f\left(\frac{1}{8}M_g\left(\frac{r}{4}\right) + o(1)\right)\right).$$

$$i.e.,\ \log T_h^{-1}\left(T_{f\circ g}(r)\right)\geq O(1)+\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[\log\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)+o(1)\right)\right.$$
$$\left.+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]$$

$$i.e.,\ \log T_h^{-1}\left(T_{f\circ g}(r)\right)\geq O(1)+\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[\log M_g\left(\frac{r}{4}\right)\right.$$
$$\left.+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]$$

(15) $i.e.,\ \log T_h^{-1}\left(T_{f\circ g}(r)\right)$
$$\geq O(1)+\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[\left[\left(\frac{r}{4}\right)\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(g)-\varepsilon}\right.$$
$$\left.+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]$$

where we choose $0<\varepsilon<\min\left\{\lambda_p^{L^*}(f,h),\lambda_p^{L^*}(g)\right\}$.
Also for all sufficiently large positive numbers of r, we get in view of Lemma 4, that

$$\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r^A\right)\right)\leq\left(\rho_p^{L^*}\left(P_0[f],P_0[h]\right)+\varepsilon\right)\log\left[r^A\exp^{[p]}L\left(r^A\right)\right]$$
$$i.e.,\ \log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r^A\right)\right)\leq\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\log\left[r^A\exp^{[p]}L\left(r^A\right)\right]$$

(16) $i.e.,\ \left\{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r^A\right)\right)\right\}^{1+x}\leq$
$$\left(\rho_p^{L^*}(f,h)+\varepsilon\right)^{1+x}\left(\log\left[r^A\exp^{[p]}L\left(r^A\right)\right]\right)^{1+x}.$$

Therefore from (15) and (16) it follows for all sufficiently large positive numbers of r that

$$\frac{\log T_h^{-1}\left(T_{f\circ g}(r)\right)}{\left\{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r^A\right)\right)\right\}^{1+x}}$$
$$\geq\frac{O(1)+\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[\left[\left(\frac{r}{4}\right)\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(g)-\varepsilon}+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]}{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)^{1+x}\left(\log\left[r^A\exp^{[p]}L\left(r^A\right)\right]\right)^{1+x}}$$

Thus from above the theorem follows. □

THEOREM 10. *Let k be an entire function either of finite order or of nonzero lower order such that* $\Theta(\infty;k) = \sum_{a\neq\infty} \delta_p(a;k) = 1$ *or* $\delta(\infty;k) = \sum_{a\neq\infty} \delta(a;k) = 1$ *and g be an entire function having regular growth and nonzero finite order with* $\Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g) = 1$. *Also let f, h be any two entire functions,* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, $0 < \lambda_p^{L^*}(g) < \infty$, $0 < \rho_p^{L^*}(g,k) < \infty$ *where p is any positive integer. If h satisfies the Property (A), then for every constant A and for any real number x,*

$$\lim_{r\to\infty} \frac{\log T_h^{-1}\left(T_{f\circ g}(r)\right)}{\left\{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}\left(r^A\right)\right)\right\}^{1+x}} = \infty.$$

The proof of Theorem 10 is omitted as it can be carried out in the line of Theorem 9.

In the line of Theorem 9 and Theorem 10 and with the help of Lemma 5, one can easily prove the following two theorems and therefore their proofs are omitted:

THEOREM 11. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ *and h be a transcendental entire function with regular growth and nonzero finite order. Also let* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. *Also let g be any entire function,* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, $0 < \lambda_p^{L^*}(g) < \infty$ *where p is any positive integer. If h satisfies the Property (A), then for every constant A and for any real number x,*

$$\lim_{r\to\infty} \frac{\log T_h^{-1}\left(T_{f\circ g}(r)\right)}{\left\{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r^A\right)\right)\right\}^{1+x}} = \infty.$$

THEOREM 12. *Let k be a transcendental entire function of finite order or of nonzero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$ *and g be a transcendental entire function having regular growth and nonzero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. *Also let f, h be any two entire functions,* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, $0 < \lambda_p^{L^*}(g) < \infty$, $0 < \rho_p^{L^*}(g,k) < \infty$ *where* *p is any positive integer. If h satisfies the Property (A), then for every constant A and for any real number x,*

$$\lim_{r\to\infty}\frac{\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\left\{\log T_{M[k]}^{-1}\left(T_{M[g]}\left(r^A\right)\right)\right\}^{1+x}}=\infty.$$

THEOREM 13. *Let f be a meromorphic function either of finite order or of nonzero lower order such that* $\Theta\left(\infty;f\right)=\sum_{a\neq\infty}\delta_p\left(a;f\right)=1$ *or* $\delta\left(\infty;f\right)=\sum_{a\neq\infty}\delta\left(a;f\right)=1$ *and h be an entire function having regular growth and nonzero finite order with* $\Theta\left(\infty;h\right)=\sum_{a\neq\infty}\delta_p\left(a;h\right)=1$ *or* $\delta\left(\infty;h\right)=\sum_{a\neq\infty}\delta\left(a;h\right)=1$. *Also let g be an entire function,* $0<\rho_p^{L^*}\left(f,h\right)<\infty$ *and* $\rho_p^{L^*}\left(g\right)$ *is nonzero finite where p is any positive integer. If h satisfies the Property (A), then for each* $\alpha\in\left(-\infty,\infty\right)$,

$$\varliminf_{r\to\infty}\frac{\left\{\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\right\}^{1+\alpha}}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)}=0 \text{ where } A>\left(1+\alpha\right)\cdot\rho_p^{L^*}\left(g\right).$$

Proof. If $1+\alpha<0$, then the theorem is trivial. So we take $1+\alpha>0$. Now from (4) we obtain for all sufficiently large positive numbers of r that

$$\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\leq\left[r\exp^{[p]}L\left(r\right)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}\cdot\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)+O(1)+\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)\cdot\exp^{[p-1]}L\left(M_g\left(r\right)\right)$$

(17) *i.e.,* $\left\{\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\right\}^{1+\alpha}$

$$\leq\left[\left[r\exp^{[p]}L\left(r\right)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}\cdot\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)+O(1)+\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)\cdot\exp^{[p-1]}L\left(M_g\left(r\right)\right)\right]^{1+\alpha}.$$

Again in view of Lemma 4, we have for a sequence of positive numbers of r tending to infinity and for $\varepsilon(>0)$,

$$\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)\geq\left(\rho_p^{L^*}\left(P_0\left[f\right],P_0\left[h\right]\right)-\varepsilon\right)\log\left[\exp\left(r^A\right)\exp^{[p]}L\left(\exp\left(r^A\right)\right)\right]$$

i.e., $\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)\geq$

$$\left(\rho_p^{L^*}(f,h)-\varepsilon\right)\log\left[\exp\left(r^A\right)\exp^{[p]}L\left(\exp\left(r^A\right)\right)\right]$$

(18)
$i.e.,\ \log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)$

$$\geq\left(\rho_p^{L^*}(f,h)-\varepsilon\right)\left[r^A+\exp^{[p-1]}L\left(\exp\left(r^A\right)\right)\right].$$

Now let

$$\left(\rho_p^{L^*}(f,h)+\varepsilon\right)=k_1,\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\cdot\exp^{[p-1]}L\left(M_g(r)\right)+O(1)=k_2,$$

$$\left(\rho_p^{L^*}(f,h)-\varepsilon\right)=k_3\text{ and }\left(\rho_p^{L^*}(f,h)-\varepsilon\right)\exp^{[p-1]}L\left(\exp\left(r^A\right)\right)=k_4.$$

Then from (17), (18) and above we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)}\leq\frac{\left[\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}k_1+k_2\right]^{1+\alpha}}{k_3r^A+k_4}$$

$$i.e.,\ \frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)}$$

$$\leq\frac{\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)(1+\alpha)}\left[k_1+\frac{k_2}{\left[r\exp^{[p]}L(r)\right]^{\left(\rho_p^{L^*}(g)+\varepsilon\right)}}\right]^{1+\alpha}}{k_3r^A+k_4}$$

where k_1,k_2,k_3 and k_4 are all finite.

Since $\left(\rho_p^{L^*}(g)+\varepsilon\right)(1+\alpha)<A$, we obtain from above

$$\varliminf_{r\to\infty}\frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)}=0$$

where we choose $\varepsilon(>0)$ in such a way that

$$0<\varepsilon<\min\left\{\rho_p^{L^*}(f,h),\frac{A}{1+\alpha}-\rho_p^{L^*}(g)\right\}.$$

This proves the theorem. □

In the line of Theorem 13, the following theorem may be proved and therefore its proof is omitted:

THEOREM 14. *Let f be a meromorphic function either of finite order or of nonzero lower order such that* $\Theta(\infty; f) = \sum_{a \neq \infty} \delta_p(a; f) = 1$ *or* $\delta(\infty; f) = \sum_{a \neq \infty} \delta(a; f) = 1$ *and h be an entire function having regular growth and nonzero finite order with* $\Theta(\infty; h) = \sum_{a \neq \infty} \delta_p(a; h) = 1$ *or* $\delta(\infty; h) = \sum_{a \neq \infty} \delta(a; h) = 1$. *Also let g be an entire function,* $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$ *and $\rho_p^{L^*}(g)$ is nonzero finite where p is any positive integer. If h satisfies the Property (A), then for each* $\alpha \in (-\infty, \infty)$,

$$\lim_{r \to \infty} \frac{\left\{\log T_h^{-1}\left(T_{f \circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(\exp r^A\right)\right)} = 0 \text{ where } A > (1+\alpha) \cdot \rho_p^{L^*}(g).$$

In the line of Theorem 13 and Theorem 14 and with the help of Lemma 5, one can easily proof the following two theorems and therefore their proofs are omitted:

THEOREM 15. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ *and h be a transcendental entire function of regular growth having nonzero finite type with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. *Also let g be an entire function,* $0 < \rho_p^{L^*}(f, h) < \infty$ *and $\rho_p^{L^*}(g)$ is nonzero finite where p is any positive integer. If h satisfies the Property (A), then for each* $\alpha \in (-\infty, \infty)$,

$$\underline{\lim}_{r \to \infty} \frac{\left\{\log T_h^{-1}\left(T_{f \circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(\exp r^A\right)\right)} = 0 \text{ where } A > (1+\alpha) \cdot \rho_p^{L^*}(g).$$

THEOREM 16. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; f) = 4$ *and h be a transcendental entire function of regular growth having nonzero finite type with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$. *Also let g be an entire function,* $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$ *and $\rho_p^{L^*}(g)$ is nonzero finite where p is any positive integer. If h satisfies the Property (A), then for each*

$\alpha \in (-\infty, \infty)$,

$$\lim_{r\to\infty} \frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(\exp r^A\right)\right)} = 0 \text{ where } A > (1+\alpha)\cdot \rho_p^{L^*}(g).$$

Theorem 17. *Let k be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty; k) = \sum_{a\neq\infty} \delta_p(a; k) = 1$ or $\delta(\infty; k) = \sum_{a\neq\infty} \delta(a; k) = 1$ and g be an entire function having regular growth and nonzero finite order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Also let f be a meromorphic function, h be any entire function, $\rho_p^{L^*}(f, h) < \infty$, $\rho_p^{L^*}(g)$ is nonzero finite and $\lambda_p^{L^*}(g, k) > 0$ where p is any positive integer. If h satisfies the Property (A), then for each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}\left(\exp r^A\right)\right)} = 0 \text{ where } A > (1+\alpha)\cdot \rho_p^{L^*}(g).$$

Theorem 18. *Let k be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty; k) = \sum_{a\neq\infty} \delta_p(a; k) = 1$ or $\delta(\infty; k) = \sum_{a\neq\infty} \delta(a; k) = 1$ and g be an entire function having regular growth and nonzero finite order with $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$. Also let f be a meromorphic function, h be any entire function, $\rho_p^{L^*}(f, h) < \infty$, $0 < \rho_p^{L^*}(g) < \infty$ and $\rho_p^{L^*}(g, k) > 0$ where p is any positive integer. If h satisfies the Property (A), then for each $\alpha \in (-\infty, \infty)$,*

$$\underline{\lim}_{r\to\infty} \frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}\left(\exp r^A\right)\right)} = 0 \text{ where } A > (1+\alpha)\cdot \rho_p^{L^*}(g).$$

The proof of Theorem 17 and Theorem 18 are omitted because those can be carried out in the line of Theorem 14 and Theorem 13 respectively.

In the line of Theorem 17 and Theorem 18 and with the help of Lemma 5, one can easily prove the following two theorems and therefore their proofs are omitted:

THEOREM 19. *Let k be a transcendental entire function of finite order or of nonzero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;k)=4$ and g be an entire function having regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let f be a meromorphic function, h be any entire function, $\rho_p^{L^*}(f,h)<\infty$, $\rho_p^{L^*}(g)$ is nonzero finite and $\lambda_p^{L^*}(g,k)>0$ where p is any positive integer. If h satisfies the Property (A), then for each $\alpha\in(-\infty,\infty)$,*

$$\lim_{r\to\infty}\frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{M[k]}^{-1}\left(T_{M[g]}\left(\exp r^A\right)\right)}=0 \text{ where } A>(1+\alpha)\cdot\rho_p^{L^*}(g)\,.$$

THEOREM 20. *Let k be a transcendental entire function of finite order or of nonzero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;k)=4$ and g be an entire function having regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let f be a meromorphic function, h be any entire function, $\rho_p^{L^*}(f,h)<\infty$, $0<\rho_p^{L^*}(g)<\infty$ and $\rho_p^{L^*}(g,k)>0$ where p is any positive integer. If h satisfies the Property (A), then for each $\alpha\in(-\infty,\infty)$,*

$$\underline{\lim}_{r\to\infty}\frac{\left\{\log T_h^{-1}\left(T_{f\circ g}(r)\right)\right\}^{1+\alpha}}{\log T_{M[k]}^{-1}\left(T_{M[g]}\left(\exp r^A\right)\right)}=0 where A>(1+\alpha)\cdot\rho_p^{L^*}(g)\,.$$

THEOREM 21. *Let k be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty;k)=\sum_{a\neq\infty}\delta_p(a;k)=1$ or $\delta(\infty;k)=\sum_{a\neq\infty}\delta(a;k)=1$ and g be an entire function having regular growth and nonzero finite order with $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ or $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. Also let f be a meromorphic function, h be any entire function satisfying the Property (A), $\rho_p^{L^*}(f,h)<\infty$, $0<\rho_p^{L^*}(g)<\infty$ and $0<\lambda_p^{L^*}(g,k)<\infty$ where p is any positive integer. Then*
(a) if $\exp^{[p-1]}L\left(M_g(r)\right)=o\left\{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)\right\}$ then

$$\overline{\lim}_{r\to\infty}\frac{\log^{[2]}T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)+\exp^{[p-1]}L\left(M_g(r)\right)}\leq\frac{\rho_k^{L^*}(g)}{\lambda_p^{L^*}(g,k)}$$

and (*b*) *if* $\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) = o\left\{\exp^{[p-1]} L\left(M_g(r)\right)\right\}$ *then*

$$\lim_{r\to\infty} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L\left(M_g(r)\right)} = 0.$$

Proof. Using $\log\left(1 + \frac{\exp^{[p-1]} L(M_g(r))+O(1)}{\log M_g(r)}\right) < \left(1 + \frac{\exp^{[p-1]} L(M_g(r))+O(1)}{\log M_g(r)}\right)$, we obtain from (1) for all sufficiently large positive numbers of r that

$$\log T_h^{-1}\left(T_{f\circ g}(r)\right) \leqslant \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\log M_g(r)\left[1 + \frac{\exp^{[p-1]} L\left(M_g(r)\right)+O(1)}{\log M_g(r)}\right]$$

$$i.e.,\ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \le \log\left(\rho_p^{L^*}(f,h)+\varepsilon\right) + \log^{[2]} M_g(r) + \log\left[1 + \frac{\exp^{[p-1]} L\left(M_g(r)\right)+O(1)}{\log M_g(r)}\right]$$

$$i.e.,\ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \le \log\left(\rho_p^{L^*}(f,h)+\varepsilon\right) + \left(\rho_k^{L^*}(g)+\varepsilon\right)\log\left[r\exp^{[p]} L(r)\right] + \log\left[1 + \frac{\exp^{[p-1]} L\left(M_g(r)\right)+O(1)}{\log M_g(r)}\right]$$

(19)
$$i.e.,\ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \le O(1) + \left(\rho_k^{L^*}(g)+\varepsilon\right)\left[\log r + \exp^{[p-1]} L(r)\right] + \frac{\log M_g(r) + \exp^{[p-1]} L\left(M_g(r)\right)+O(1)}{\log M_g(r)}.$$

Again from the definition of relative ${}_pL^*$-lower order and in view of Lemma 4, we get for all sufficiently large positive numbers of r that

$$\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) \ge \left(\lambda_p^{L^*}\left(P_0[g], P_0[k]\right) - \varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]$$

$$i.e.,\ \log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) \ge \left(\lambda_p^{L^*}(g,k) - \varepsilon\right)\log\left[r\exp^{[p]} L(r)\right]$$

(20)
$$i.e.,\ \left[\log r + \exp^{[p-1]} L(r)\right] \le \frac{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)}{\left(\lambda_p^{L^*}(g,k)-\varepsilon\right)}.$$

Hence from (19) and (20), it follows for all sufficiently large positive numbers of r that

$$\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \le O(1) + \left(\frac{\rho_k^{L^*}(g)+\varepsilon}{\lambda_p^{L^*}(g,k)-\varepsilon}\right)\cdot\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) +$$

$$\frac{\log M_g(r) + \exp^{[p-1]} L(M_g(r)) + O(1)}{\log M_g(r)}$$

$$i.e, \ \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))}$$
$$\leq \frac{O(1)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} +$$
$$\left(\frac{\rho_k^{L^*}(g)+\varepsilon}{\lambda_p^{L^*}(g,k)-\varepsilon}\right) \cdot \frac{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} +$$
$$\frac{\log M_g(r) + \exp^{[p-1]} L(M_g(r)) + O(1)}{\left[\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))\right] \log M_g(r)}$$

(21) $$i.e, \ \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\frac{O(1)}{\exp^{[p-1]} L(M_g(r))}}{\frac{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)}{\exp^{[p-1]} L(M_g(r))} + 1}$$
$$+ \frac{\left(\frac{\rho_k^{L^*}(g)+\varepsilon}{\lambda_p^{L^*}(g,k)-\varepsilon}\right)}{1 + \frac{\exp^{[p-1]} L(M_g(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)}} + \frac{1 + \frac{\log M_g(r)}{\exp^{[p-1]} L(M_g(r))}}{\left[1 + \frac{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)}{\exp^{[p-1]} L(M_g(r))}\right] \log M_g(r)}.$$

Since $\exp^{[p-1]} L(M_g(r)) = o\left\{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)\right\}$ as $r \to \infty$ and $\varepsilon\,(>0)$, is arbitrary we obtain from (21) that

(22) $$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_k^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

Again if $\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then from (21) we get that

(23) $$\lim_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} = 0.$$

Thus from (22) and (23) the theorem is established. □

In the line of Theorem 21 the following theorem may be proved and therefore its proof is omitted:

THEOREM 22. *Let k be an entire function either of finite order or of nonzero lower order such that* $\Theta(\infty;k)=\sum_{a\neq\infty}\delta_p(a;k)=1$ *or* $\delta(\infty;k)=\sum_{a\neq\infty}\delta(a;k)=1$ *and g be an entire function having regular growth and nonzero finite order with* $\Theta(\infty;g)=\sum_{a\neq\infty}\delta_p(a;g)=1$ *or* $\delta(\infty;g)=\sum_{a\neq\infty}\delta(a;g)=1$. *Also let f be a meromorphic function, h be any entire function satisfying the Property (A),* $\rho_p^{L^*}(f,h)<\infty$, $\rho_p^{L^*}(g)<\infty$ *and* $\rho_p^{L^*}(g,k)>0$ *where p is any positive integer. Then*
(*a*) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_k^{L^*}(g)}{\rho_p^{L^*}(g,k)}$$

and (*b*) *if* $\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right)+\exp^{[p-1]}L(M_g(r))}=0.$$

In the line of Theorem 21 and Theorem 22 and with the help of Lemma 5, one can easily proof the following two theorems and therefore their proofs are omitted:

THEOREM 23. *Let k be a transcendental entire function of finite order or of nonzero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;k)=4$ *and g be an entire function having regular growth and nonzero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. *Also let f be a meromorphic function, h be any entire function satisfying the Property (A),* $\rho_p^{L^*}(f,h)<\infty$, $0<\rho_p^{L^*}(g)<\infty$ *and* $0<\lambda_p^{L^*}(g,k)<\infty$ *where p is any positive integer. Then*
(*a*) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right)\right\}$ *then*

$$\varlimsup_{r\to\infty}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_k^{L^*}(g)}{\lambda_p^{L^*}(g,k)}$$

and (*b*) *if* $\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\lim_{r\to\infty}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right)+\exp^{[p-1]}L(M_g(r))}=0\ .$$

THEOREM 24. *Let k be a transcendental entire function of finite order or of nonzero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$ and g be an entire function having regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Also let f be a meromorphic function, h be any entire function satisfying the Property (A), $\rho_p^{L^*}(f,h) < \infty$, $\rho_p^{L^*}(g) < \infty$ and $\rho_p^{L^*}(g,k) > 0$ where p is any positive integer. Then*
(a) if $\exp^{[p-1]} L(M_g(r)) = o\left\{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right)\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_k^{L^*}(g)}{\rho_p^{L^*}(g,k)}$$

and (b) if $\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right) + \exp^{[p-1]} L(M_g(r))} = 0.$$

Now we state the following three theorems without their proofs as those can be carried out in the line of Theorem 21 and Theorem 22:

THEOREM 25. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ or $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$. Also let g be an entire function, h satisfying the Property (A), $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(g) < \infty$ where p is any positive integer. Then*
(a) if $\exp^{[p-1]} L(M_g(r)) = o\left\{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)\right\}$ then

$$\overline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}$$

and (b) if $\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\lim_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + \exp^{[p-1]} L(M_g(r))} = 0.$$

THEOREM 26. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let g be an entire function, h satisfying the Property (A), $0 < \rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g) < \infty$ where p is any positive integer. Then*
(a) if $\exp^{[p-1]} L(M_g(r)) = o\left\{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(g)}{\rho_p^{L^*}(f, h)}$$

and (b) if $\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + \exp^{[p-1]} L(M_g(r))} = 0.$$

THEOREM 27. *Let f be a meromorphic function either of finite order or of nonzero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let g be an entire function, h satisfying the Property (A), $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(g) < \infty$ where p is any positive integer. Then*
(a) if $\exp^{[p-1]} L(M_g(r)) = o\left\{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right)\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\lambda_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}$$

and (b) if $\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + \exp^{[p-1]} L(M_g(r))} = 0.$$

In the line of Theorem 25, Theorem 26 and Theorem 27 and with the help of Lemma 5, one can easily proof the following three theorems and therefore their proofs are omitted:

THEOREM 28. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and h be a transcendental entire function of regular growth having nonzero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. *Also let g be an entire function, h satisfying the Property (A),* $0<\lambda_p^{L^*}(f,h)\leq\rho_p^{L^*}(f,h)<\infty$ *and* $\rho_p^{L^*}(g)<\infty$ *where p is any positive integer. Then*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)\right\}$ *then*

$$\overline{\lim_{r\to\infty}}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}$$

and (b) *if* $\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\lim_{r\to\infty}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)+\exp^{[p-1]}L(M_g(r))}=0.$$

THEOREM 29. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and h be a transcendental entire function of regular growth having nonzero finite type with* $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. *Also let g be an entire function, h satisfying the Property (A),* $0<\rho_p^{L^*}(f,h)<\infty$ *and* $\rho_p^{L^*}(g)<\infty$ *where p is any positive integer. Then*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)\right\}$ *then*

$$\underline{\lim_{r\to\infty}}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(g)}{\rho_p^{L^*}(f,h)}$$

and (b) *if* $\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\underline{\lim_{r\to\infty}}\frac{\log^{[2]}T_h^{-1}(T_{f\circ g}(r))}{\log T_{M[h]}^{-1}\left(T_{M[f]}(r)\right)+\exp^{[p-1]}L(M_g(r))}=0.$$

Theorem 30. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ *and h be a transcendental entire function of regular growth having nonzero finite type with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. *Also let g be an entire function, h satisfying the Property (A),* $0<\lambda_p^{L^*}(f,h)\leq\rho_p^{L^*}(f,h)<\infty$ *and* $\lambda_p^{L^*}(g)<\infty$ *where p is any positive integer. Then*
(a) if $\exp^{[p-1]}L\left(M_g\left(r\right)\right)=o\left\{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right)\right\}$ *then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right)+\exp^{[p-1]}L\left(M_g\left(r\right)\right)}\leq\frac{\lambda_p^{L^*}\left(g\right)}{\lambda_p^{L^*}\left(f,h\right)}$$

and (b) if $\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right)=o\left\{\exp^{[p-1]}L\left(M_g\left(r\right)\right)\right\}$ *then*

$$\underline{\lim}_{r\to\infty}\frac{\log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right)+\exp^{[p-1]}L\left(M_g\left(r\right)\right)}=0.$$

Theorem 31. *Let f be an entire function either of finite order or of nonzero lower order such that* $\Theta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;f\right)=1$ *or* $\delta\left(\infty;f\right)=\sum\limits_{a\neq\infty}\delta\left(a;f\right)=1$ *and h be an entire function having regular growth and nonzero finite order with* $\Theta\left(\infty;h\right)=\sum\limits_{a\neq\infty}\delta_p\left(a;h\right)=1$ *or* $\delta\left(\infty;h\right)=\sum\limits_{a\neq\infty}\delta\left(a;h\right)=1$. *Also let g be an entire function, h satisfying the Property (A),* $0<\lambda_p^{L^*}(f,h)\leq\rho_p^{L^*}(f,h)<\infty$ *and* $\rho_p^{L^*}(g)>0$ *where p is any positive integer. Then*

$$\overline{\lim}_{r\to\infty}\frac{\log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)}\geq\frac{\rho_p^{L^*}\left(g\right)}{\rho_p^{L^*}\left(f,h\right)}.$$

Proof. From (14), we have for all sufficiently large positive numbers of r that

$$\log T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\geq O\left(1\right)+\left(\lambda_p^{L^*}\left(f,h\right)-\varepsilon\right)$$
$$\left(\log\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\left(1+\frac{o\left(1\right)}{\frac{1}{8}M_g\left(\frac{r}{4}\right)}\right)\right)+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right)$$

$$i.e., \ \log T_h^{-1}\left(T_{f\circ g}(r)\right) \geq \left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\log M_g\left(\frac{r}{4}\right)\cdot \left(\frac{\log M_g\left(\frac{r}{4}\right)+\log\left(1+\frac{o(1)}{\frac{1}{8}M_g\left(\frac{r}{4}\right)}\right)+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)}{\log M_g\left(\frac{r}{4}\right)}\right)$$

$$i.e., \ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+ \log\left(\frac{\log M_g\left(\frac{r}{4}\right)+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)}{\log M_g\left(\frac{r}{4}\right)}\right)$$

$$i.e., \ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+\left(\frac{\rho_p^{L^*}(g)-\varepsilon}{\rho_p^{L^*}(f,h)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right) - \log\left(\exp\left(\left(\frac{\rho_p^{L^*}(g)-\varepsilon}{\rho_p^{L^*}(f,h)+\varepsilon}\right)\cdot L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right)\right) + \log\left(\frac{\log M_g\left(\frac{r}{4}\right)+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)}{\log M_g\left(\frac{r}{4}\right)}\right)$$

$$i.e., \ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+\left(\frac{\rho_p^{L^*}(g)-\varepsilon}{\rho_p^{L^*}(f,h)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right) + \log\left(\frac{\log M_g\left(\frac{r}{4}\right)+\exp^{[p-1]}L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)+o(1)}{\exp\left(\left(\frac{\rho_p^{L^*}(g)-\varepsilon}{\rho_p^{L^*}(f,h)+\varepsilon}\right)\cdot L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right)\cdot\log M_g\left(\frac{r}{4}\right)}\right)$$

$$i.e., \ \log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \geq \log^{[2]} M_g\left(\frac{r}{4}\right)+\left(\frac{\rho_p^{L^*}(g)-\varepsilon}{\rho_p^{L^*}(f,h)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right).$$

Now from above it follows for a sequence of positive numbers of r tending to infinity that

$$\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right) \geq \left(\rho_p^{L^*}(g)-\varepsilon\right)\log\left[\frac{r}{4}\exp^{[p]}L\left(\frac{r}{4}\right)\right] + \left(\frac{\rho_p^{L^*}(g)-\varepsilon}{\rho_p^{L^*}(f,h)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right). \tag{24}$$

Further in view of Lemma 4, we get for all sufficiently large positive numbers of r that

$$\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)\leq\left(\rho_p^{L^*}\left(P_0\left[f\right],P_0\left[h\right]\right)+\varepsilon\right)\log\left[r\exp^{[p]}L\left(r\right)\right]$$

$$i.e.,\ \log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)\leq\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)\log\left[r\exp^{[p]}L\left(r\right)\right]$$

(25) $$i.e.,\ \log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)\leq\left(\rho_p^{L^*}\left(f,h\right)+\varepsilon\right)\log\left[\frac{r}{4}\exp^{[p]}L\left(\frac{r}{4}\right)\right]+\log 4.$$

Hence from (24) and (25) it follows for a sequence of positive numbers of r tending to infinity that

$$i.e.,\ \log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\geq\left(\frac{\rho_p^{L^*}\left(g\right)-\varepsilon}{\rho_p^{L^*}\left(f,h\right)+\varepsilon}\right)\left(\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(r\right)-\log 4\right)+\left(\frac{\rho_p^{L^*}\left(g\right)-\varepsilon}{\rho_p^{L^*}\left(f,h\right)+\varepsilon}\right)L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)$$

$$i.e.,\ \log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)\geq\left(\frac{\rho_p^{L^*}\left(g\right)-\varepsilon}{\rho_p^{L^*}\left(f,h\right)+\varepsilon}\right)\left[\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)\right]-\left(\frac{\rho_p^{L^*}\left(g\right)-\varepsilon}{\rho_p^{L^*}\left(f,h\right)+\varepsilon}\right)\log 4$$

$$i.e.,\ \frac{\log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)}\geq\left(\frac{\rho_p^{L^*}\left(g\right)-\varepsilon}{\rho_p^{L^*}\left(f,h\right)+\varepsilon}\right)-\frac{\left(\frac{\rho_p^{L^*}\left(g\right)-\varepsilon}{\rho_p^{L^*}\left(f,h\right)+\varepsilon}\right)\log 4}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to\infty}}\frac{\log^{[2]}T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}\left(r\right)\right)+L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)}\geq\frac{\rho_p^{L^*}\left(g\right)}{\rho_p^{L^*}\left(f,h\right)}.$$

This proves the theorem. □

In the line of Theorem 31, the following two theorems may be proved and therefore their proofs are omitted:

Theorem 32. *Let f be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty; h) = \sum\limits_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum\limits_{a\neq\infty} \delta(a; h) = 1$. Also let g be an entire function, h satisfying the Property (A), $0 < \lambda_p^{L^*}(f, h) < \infty$ and $\lambda_p^{L^*}(g) > 0$ where p is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}(g)}{\lambda_p^{L^*}(f, h)}.$$

Theorem 33. *Let f be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty; f) = \sum\limits_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum\limits_{a\neq\infty} \delta(a; f) = 1$ and h be an entire function having regular growth and nonzero finite order with $\Theta(\infty; h) = \sum\limits_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum\limits_{a\neq\infty} \delta(a; h) = 1$. Also let g be an entire function, h satisfying the Property (A), $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) <$ and $\lambda_p^{L^*}(g) > 0$ where p is any positive integer. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}(T_{f\circ g}(r))}{\log T_{P_0[h]}^{-1}\left(T_{P_0[f]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}(g)}{\rho_p^{L^*}(f, h)}.$$

In the line of Theorem 31, Theorem 39 and Theorem 33 and with the help of Lemma 5, one can easily proof the following three theorems and therefore their proofs are omitted:

Theorem 34. *Let f be a transcendental meromorphic function of finite order or of nonzero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$ and h be a transcendental entire function of regular growth having nonzero finite type with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; h) = 4$. Also let g be an entire function, h satisfying the Property (A), $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$ and $\rho_p^{L^*}(g)$*

> 0 *where* p *is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\rho_p^{L^*}\left(g\right)}{\rho_p^{L^*}\left(f,h\right)}.$$

Theorem 35. *Let* f *be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ *and* h *be a transcendental entire function of regular growth having nonzero finite type with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$*. Also let* g *be an entire function,* h *satisfying the Property (A),* $0 < \lambda_p^{L^*}\left(f,h\right) < \infty$ *and* $\lambda_p^{L^*}\left(g\right) > 0$ *where* p *is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}\left(g\right)}{\lambda_p^{L^*}\left(f,h\right)}.$$

Theorem 36. *Let* f *be a transcendental meromorphic function of finite order or of nonzero lower order and* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ *and* h *be a transcendental entire function of regular growth having nonzero finite type with* $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$*. Also let* g *be an entire function,* h *satisfying the Property (A),* $0 < \lambda_p^{L^*}\left(f,h\right) \leq \rho_p^{L^*}\left(f,h\right) <$ *and* $\lambda_p^{L^*}\left(g\right) > 0$ *where* p *is any positive integer. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}\left(r\right)\right)}{\log T_{M[h]}^{-1}\left(T_{M[f]}\left(r\right)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}\left(g\right)}{\rho_p^{L^*}\left(f,h\right)}.$$

Now we state the following two theorems without their proofs as those can be carried out in the line of Theorem 31 and Theorem 33:

Theorem 37. *Let* k *be an entire function either of finite order or of nonzero lower order such that* $\Theta\left(\infty;k\right) = \sum\limits_{a\neq\infty} \delta_p\left(a;k\right) = 1$ *or* $\delta\left(\infty;k\right) = \sum\limits_{a\neq\infty} \delta\left(a;k\right) = 1$ *and* g *be an entire function having regular growth and nonzero finite order with* $\Theta\left(\infty;g\right) = \sum\limits_{a\neq\infty} \delta_p\left(a;g\right) = 1$ *or* $\delta\left(\infty;g\right) = \sum\limits_{a\neq\infty} \delta\left(a;g\right) = 1$*. Also let* f*,* h *be any two entire functions,* $\lambda_p^{L^*}\left(f,h\right) > 0$*,* $0 < \rho_p^{L^*}\left(g\right) < \infty$ *and* $0 < \rho_p^{L^*}\left(g,k\right) < \infty$ *where* p *is any positive integer.*

If h satisfies the Property (A). Then

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\rho_p^{L^*}(g)}{\rho_p^{L^*}(g,k)}.$$

Theorem 38. *Let k be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty;k) = \sum_{a\neq\infty} \delta_p(a;k) = 1$ or $\delta(\infty;k) = \sum_{a\neq\infty} \delta(a;k) = 1$ and g be an entire function having regular growth and nonzero finite order with $\Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g) = 1$. Also let f, h be any two entire functions, $\lambda_p^{L^*}(f,h) > 0$, $0 < \lambda_p^{L^*}(g) < \infty$ and $0 < \lambda_p^{L^*}(g,k) < \infty$ where p is any positive integer. If h satisfies the Property (A). Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

Theorem 39. *Let k be an entire function either of finite order or of nonzero lower order such that $\Theta(\infty;k) = \sum_{a\neq\infty} \delta_p(a;k) = 1$ or $\delta(\infty;k) = \sum_{a\neq\infty} \delta(a;k) = 1$ and g be an entire function having regular growth and nonzero finite order with $\Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g) = 1$. Also let f, h be any two entire functions, $\lambda_p^{L^*}(f,h) > 0$, $0 < \lambda_p^{L^*}(g) < \infty$ and $0 < \rho_p^{L^*}(g,k) < \infty$ where p is any positive integer. If h satisfies the Property (A). Then*

$$\underline{\lim}_{r\to\infty} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{P_0[k]}^{-1}\left(T_{P_0[g]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}(g)}{\rho_p^{L^*}(g,k)}.$$

In the line of Theorem 37, Theorem 38 and Theorem 39 and with the help of Lemma 5, one can easily proof the following three theorems and therefore their proofs are omitted:

Theorem 40. *Let k be a transcendental entire function of finite order or of nonzero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$ and g be a transcendental entire function having regular growth and nonzero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. Also let f, h be any two entire functions,*

$\lambda_p^{L^*}(f,h) > 0$, $0 < \rho_p^{L^*}(g) < \infty$ *and* $0 < \rho_p^{L^*}(g,k) < \infty$ *where* p *is any positive integer. If* h *satisfies the Property (A). Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\rho_p^{L^*}(g)}{\rho_p^{L^*}(g,k)}.$$

Theorem 41. *Let* k *be a transcendental entire function of finite order or of nonzero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$ *and* g *be a transcendental entire function having regular growth and nonzero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. *Also let* f, h *be any two entire functions,* $\lambda_p^{L^*}(f,h) > 0$, $0 < \lambda_p^{L^*}(g) < \infty$ *and* $0 < \lambda_p^{L^*}(g,k) < \infty$ *where* p *is any positive integer. If* h *satisfies the Property (A). Then*

$$\overline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

Theorem 42. *Let* k *be a transcendental entire function of finite order or of nonzero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$ *and* g *be a transcendental entire function having regular growth and nonzero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$. *Also let* f, h *be any two entire functions,* $\lambda_p^{L^*}(f,h) > 0$, $0 < \lambda_p^{L^*}(g) < \infty$ *and* $0 < \rho_p^{L^*}(g,k) < \infty$ *where* p *is any positive integer. If* h *satisfies the Property (A). Then*

$$\underline{\lim_{r\to\infty}} \frac{\log^{[2]} T_h^{-1}\left(T_{f\circ g}(r)\right)}{\log T_{M[k]}^{-1}\left(T_{M[g]}(r)\right) + L\left(\frac{1}{8}M_g\left(\frac{r}{4}\right)\right)} \geq \frac{\lambda_p^{L^*}(g)}{\rho_p^{L^*}(g,k)}.$$

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