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GLOBAL EXISTENCE OF STRONG SOLUTION FOR SOME CONTROLLED ODE-PDE SYSTEMS

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Abstract. This paper is concerned with the global existence of strong solution for the controlled ode-pde systems. Also, we consider the continuous dependence of solution on the control.

1. Introduction

The modelling of forest age structure dynamics is one of the most important problems of mathematical ecology. The following model is introduced as base mathematical model of mono-species forest with two age classes ([1], [2], [7]).

$$\begin{aligned}
&\frac{\partial y}{\partial t} = d\frac{\partial^2 y}{\partial x^2} - \gamma(\rho)y - fy + g\rho \qquad \text{in } I \times (0,T],\\
&\frac{\partial \rho}{\partial t} = fy - h\rho - u(t)\rho \qquad \text{in } I \times (0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = 0 \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \qquad \text{in } I.
\end{aligned} \tag{1.1}$$

Here, $I = (0, L)$ is a bounded interval in R. $y = y(x,t)$ denotes tree density of young age class in I at time t and $\rho = \rho(x,t)$ is tree density of old age class in I at time t. $d > 0$ is a diffusion rate. $g > 0$ is fertility of the species. $h > 0$ and $f > 0$ denote death and aging rates. $\gamma(\rho)$ denotes a mortality rate function of the young trees with $\gamma(\rho) = a(\rho - b)^2 + c$ $(a, b, c > 0)$. $u(t)$ denotes the control term.

In [3] and [5], the authors studied the global existence of strong solution and the optimal control problem for prey-predator reaction diffusion model. In [4], the existence of strong solution and the control problem for FitzHugh-Nagumo

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system were studied. In [6], the author considered the local existence of strong solution and the existence of optimal control for (1.1). In this paper, we show the global existence of strong solution and the continuous dependence of solution on the control.

The paper is organized as follows. Section 2 is a preliminary section. In Section 3, we obtain the global existence of strong solution. Also, we show the continuous dependence of solution on the control.

Notation. $L^p(I;\mathcal{H})$, $1 \le p \le \infty$, denotes the L^p space of measurable functions in I with values in a Hilbert space $\mathcal{H}$. $\mathcal{C}(I;\mathcal{H})$ denotes the space of continuous functions in I with values in $\mathcal{H}$. $W^{1,2}(I;\mathcal{H}) = \{y; D^j y \in L^2(I;\mathcal{H}), j = 0, 1\}$, where D is the derivative in the sense of distributions. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by a, b, c, d, f, g, h, m, l and I.

2. Preliminaries

In this section, we recall the existence and uniqueness of a local strong solution for (1.1) as in [6].

We rewrite (1.1) as an abstract problem (2.1) in the Hilbert spaces $\mathcal{H} = L^2(I) \times L^2(I)$. To this end, let us define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ as follows:

$$\mathcal{A}Y = \begin{pmatrix} d\frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \rho \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in D(\mathcal{A}).$$

Here, $D(\mathcal{A}) = \Big\{ Y = \binom{y}{\rho} \in H^2(I) \times L^\infty(I), \frac{\partial y}{\partial x}(0) = \frac{\partial y}{\partial x}(L) = 0 \Big\}$. Then $\mathcal{A}$ is a self adjoint dissipative operator in $\mathcal{H}$.

Thus, (1.1) is formulated to the following abstract form

$$\begin{aligned} &\frac{dY}{dt} + AY = F(t, Y(t)), \quad 0 < t \le T, \\ &Y(0) = Y_0 \end{aligned} \tag{2.1}$$

in the space $\mathcal{H}$. Here, $F(t, Y(t)) : [0, T] \times \mathcal{H} \to \mathcal{H}$ is the mapping

$$F(t, Y(t)) = \begin{pmatrix} f(t, y, \rho) \\ g(t, y, \rho) \end{pmatrix} = \begin{pmatrix} -\gamma(\rho)y - fy + g\rho \\ fy - h\rho - u(t)\rho \end{pmatrix}$$

and Y_0 is defined by $Y_0 = \binom{y_0}{\rho_0}$. $\mathcal{K} = \Big\{ \binom{y_0}{\rho_0} \in D(\mathcal{A}); y_0 \ge 0 \text{ and } \rho_0 \ge 0 \Big\}$ and $U_{ad} = \{u \in H^1(0, T); \|u\|_{H^1(0,T)} \le m,\ 0 \le u(t) \le l\}$.

Now, we have the followig result for the local strong solution to (1.1)(for the proof, see [6]).

Theorem 2.1. *For $Y_0 \in \mathcal{K}$ and $u \in U_{ad}$, (1.1) has a unique strong solution $Y = \binom{y}{\rho} \in W^{1,2}(0,S;\mathcal{H})$ such that*

$$0 \leq y \in L^\infty((0,S)) \times I) \cap L^\infty(0,S;H^1(I)) \cap L^2(0,S;H^2(I)),$$
$$0 \leq \rho \in L^\infty((0,S) \times I) \cap L^\infty(0,S;L^2(I)).$$

Here, the time $S \in (0,T]$ is determined by $\|y_0\|_{L^\infty(I)}$ and $\|\rho_0\|_{L^\infty(I)}$.

3. Global existence of strong solution

Theorem 3.1. *For any $0 \leq y_0 \in H^2(I)$ and $0 \leq \rho_0 \in H^1(I)$ and $u \in U_{ad}$, (1.1) has a unique global strong solution $Y = \binom{y}{\rho} \in W^{1,2}(0,T;\mathcal{H})$ such that*

$$0 \leq y \in L^\infty((0,T) \times I) \cap L^\infty(0,T;H^1(I)) \cap L^2(0,T;H^2(I)),$$
$$0 \leq \rho \in L^\infty((0,T) \times I) \cap L^\infty(0,T;L^2(I)).$$

Moreover, the estimates

$$\left\|\frac{\partial y}{\partial t}\right\|_{L^2(0,T;L^2(I))} + \|y\|_{L^2(0,T;H^2(I))} + \|y\|_{H^1(I)} + \|y\|_{L^\infty((0,T)\times I)} \leq C \quad (3.1)$$

and

$$\left\|\frac{\partial \rho}{\partial t}\right\|_{L^2(0,T;L^2(I))} + \|\rho\|_{L^\infty((0,T)\times I)} + \|\rho\|_{L^2(I)} \leq C \quad (3.2)$$

hold, where C is also determined by $\|y_0\|_{L^\infty(I)}$ and $\|\rho_0\|_{L^\infty(I)}$.

Proof. Let y, ρ be any local strong solution of (1.1) on an interval $[0,S]$ as in Theorem 2.1. We shall obtain the desired result by three steps.

Step 1. Multifly the first equation of (1.1) by y and integrate the product in I. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L y^2 dx + f\int_0^L y^2 dx + d\int_0^L \left|\frac{\partial y}{\partial x}\right|^2 dx$$
$$= g\int_0^L \rho y dx - \int_0^L \gamma(\rho) y^2 dx. \quad (3.3)$$

Multifly the second equation of (1.1) by ρ and integrate the product in I. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \rho^2 dx + h\int_0^L \rho^2 dx = f\int_0^L y\rho dx - \int_0^L u(t)\rho^2 dx \quad (3.4)$$
$$\leq f\int_0^L y\rho dx.$$

Thus, we obtain from (3.3) and (3.4) that

$$\frac{1}{2}\frac{d}{dt}\int_0^L (y^2+\rho^2)dx + (f+h)\int_0^L (y^2+\rho^2)dx + d\int_0^L \left|\frac{\partial y}{\partial x}\right|^2 dx \\ \leq (f+g)\int_0^L \rho y dx - \int_0^L \gamma(\rho)y^2 dx. \quad (3.5)$$

Here, we notice that

$$\begin{aligned}(f+g)\rho y - \gamma(\rho)y^2 = & -\Big[a(\rho-b)^2y^2 - (f+g)(\rho-b)y + \frac{(f+g)^2}{4a}\Big] \\ & - \Big(cy^2 - (f+g)by + \frac{(f+g)^2b^2}{4c}\Big) + \frac{(f+g)^2}{4}\Big(\frac{1}{a}+\frac{b^2}{c}\Big) \\ \leq & \frac{(f+g)^2}{4}\Big(\frac{1}{a}+\frac{b^2}{c}\Big).\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\int_0^L (y^2+\rho^2)dx + 2(f+h)\int_0^L (y^2+\rho^2)dx \leq C. \quad (3.6)$$

If we solve (3.6), we have

$$\|y(t)\|^2_{L^2(I)} + \|\rho(t)\|^2_{L^2(I)} \\ \leq C\Big[e^{-2(f+h)t}\big(\|y_0\|^2_{L^2(I)} + \|\rho_0\|^2_{L^2(I)}\big) + 1\Big], \quad 0 \leq t \leq S. \quad (3.7)$$

If we use (3.5), we obtain

$$\int_0^t \|y(s)\|^2_{H^1(I)}ds \leq \big(\|y_0\|^2_{L^2(I)} + \|\rho_0\|^2_{L^2(I)}\big) + Ct, \quad 0 \leq t \leq S. \quad (3.8)$$

Step 2. We will estimate the norm $\|\rho\|_{L^\infty(I)}$. Since ρ_0, $y(x,t)$ and $u(t)$ are non-negative, we have

$$\begin{aligned}\rho(t) = & e^{-\int_0^t (h+u(\tau))d\tau}\rho_0 + f\int_0^t e^{-\int_s^t (h+u(\tau))d\tau} y(x,s)ds \quad (3.9) \\ \leq & e^{-ht}\rho_0 + f\int_0^t e^{-h(t-s)}y(x,s)ds.\end{aligned}$$

Now, we obtain several estimates in order to obtain $\|\rho\|_{L\infty(I)}$. Firstly, we have

$$\begin{aligned}\|e^{-ht}\rho_0\|_{H^1(I)} = & \|e^{-ht}\rho_0\|_{L^2(I)} + \Big\|e^{-ht}\frac{\partial \rho_0}{\partial x}\Big\|_{L^2(I)} \quad (3.10) \\ \leq & e^{-2ht}\|\rho_0\|_{H^1(I)}.\end{aligned}$$

Since

$$\Big|\int_0^t e^{-h(t-s)}y(x,s)ds\Big| \le \Big(\int_0^t e^{-2h(t-s)}ds\Big)^{\frac{1}{2}}\Big(\int_0^t y^2(x,s)ds\Big)^{\frac{1}{2}} \le \frac{1}{\sqrt{2h}}\Big(\int_0^t y^2(x,s)ds\Big)^{\frac{1}{2}},$$

we obtain

$$\Big\|\int_0^t e^{-h(t-s)}y(x,s)ds\Big\|^2_{L^2(I)} = \int_0^L \Big|\int_0^t e^{-h(t-s)}y(x,s)ds\Big|^2 dx \le \int_0^L \frac{1}{2h}\int_0^t y^2(x,s)dsdx = \frac{1}{2h}\|y\|^2_{L^2((0,S;L^2(I))}. \quad (3.11)$$

Similarily, we have

$$\Big\|\frac{\partial}{\partial x}\int_0^t e^{-h(t-s)}y(x,s)ds\Big\|^2_{L^2(I)} \le \frac{1}{2h}\Big\|\frac{\partial y}{\partial x}\Big\|^2_{L^2((0,S;L^2(I))}. \quad (3.12)$$

Therefore, we obtain from (3.11) and (3.12) that

$$\Big\|\int_0^t e^{-h(t-s)}y(x,s)ds\Big\|_{H^1(I)} \le \frac{1}{\sqrt{2h}}\|y\|_{L^2((0,S;H^1(I))}. \quad (3.13)$$

From (3,9), (3.10) and (3.13) we have

$$\|\rho\|_{H^1(I)} \le e^{-2ht}\|\rho_0\|_{H^1(I)} + \frac{f}{\sqrt{2h}}\|y\|_{L^2((0,S;H^1(I))}.$$

Since $\rho_0 \in H^1(I)$ and $H^1(I) \subset L^\infty(I)$, we obtain from (3.8) that

$$\|\rho\|_{L^\infty(I)} \le C. \quad (3.14)$$

Step 3. We will estimate the norm $\|y\|_{L^\infty(I)}$. Multifly the first equation of (1.1) by $-\frac{\partial^2 y}{\partial x^2}$ and integrate the product in I. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Big|\frac{\partial y}{\partial x}\Big|^2 dx + d\int_0^L \Big|\frac{\partial^2 y}{\partial x^2}\Big|^2 dx = f\int_0^L y\frac{\partial^2 y}{\partial x^2}dx - g\int_0^L \rho\frac{\partial^2 y}{\partial x^2}dx + \int_0^L \gamma(\rho)y\frac{\partial^2 y}{\partial x^2}dx.$$

Therefore, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_0^L \Big|\frac{\partial y}{\partial x}\Big|^2 dx + \frac{d}{2}\int_0^L \Big|\frac{\partial^2 y}{\partial x^2}\Big|^2 dx + f\int_0^L \Big|\frac{\partial y}{\partial x}\Big|^2 dx \le C\int_0^L \big(\rho^2 + \gamma(\rho)^2 y^2\big)dx.$$

Since

$$\int_0^L \big(\rho^2 + \gamma(\rho)^2 y^2\big)dx \le C(\|\rho\|^4_{L^\infty(I)} + \|\rho\|^2_{L^\infty(I)} + 1)(\|y\|^2_{L^2(I)} + \|\rho\|^2_{L^2(I)}),$$

it follows from (3.7) and (3.14) that

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left|\frac{\partial y}{\partial x}\right|^2 dx + \frac{d}{2}\int_0^L \left|\frac{\partial^2 y}{\partial x^2}\right|^2 dx + f\int_0^L \left|\frac{\partial y}{\partial x}\right|^2 dx \le C.$$

If we solve the differential inequality

$$\frac{d}{dt}\int_0^L \left|\frac{\partial y}{\partial x}\right|^2 dx + 2f\int_0^L \left|\frac{\partial y}{\partial x}\right|^2 dx \le C, \quad 0 \le t \le S.$$

we have

$$\left\|\frac{\partial y}{\partial x}\right\|^2_{L^2(I)} \le C(e^{-2ft}\|y_0\|^2_{H^1(I)} + 1), \quad 0 \le t \le S. \tag{3.15}$$

From (3.7) and (3.15). we obtain

$$\|y\|_{L^\infty(I)} \le C\|y\|_{H^1(I)} \le C.$$

Step 2 and Step 3 show that y and ρ are uniformly bounded on $(0,S)\times I$ with respect to S. Hence, y and ρ can be extended as a strong solution beyond the S. By the standard argument on the extension of the strong solutions, we can then prove the global exisrence of the strong solution. The estimates (3.1) and (3.2) can obtain as in [6]. □

Moreover, we obtain the continuous dependence of solution on the control.

Theorem 3.2. *For any $u_1, u_2 \in U_{ad}$, we have*

$$\begin{aligned}&\|y_1(t) - y_2(t)\|^2_{L^2(I)} + \|\rho_1(t) - \rho_2(t)\|^2_{L^2(I)} \\ &\qquad + \int_0^t \|y_1(s) - y_2(s)\|^2_{H^1(I)} ds \le C\|u_1(t) - u_2(t)\|^2_{H^1(0,T)}, \quad 0 \le t \le T,\end{aligned}$$

where (y_1, ρ_1) and (y_2, ρ_2) are the solutions of (1.1) *with respect to u_1 and u_2, respectively.*

Proof. Let (y_1, ρ_1) and (y_2, ρ_2) be the solutions of (1.1) with respect to u_1 and u_2, respectively. Then $\tilde{y} = y_1 - y_2$, $\tilde{\rho} = \rho_1 - \rho_2$ and $\tilde{u} = u_1 - u_2$ satisfiy the equations

$$\begin{aligned}&\frac{\partial \tilde{y}}{\partial t} = d\frac{\partial^2 \tilde{y}}{\partial x^2} - \gamma(\rho_1)\tilde{y} + [2ab - a(\rho_1 + \rho_2)]y_2\tilde{\rho} - f\tilde{y} + g\tilde{\rho} \qquad \text{in } I \times (0,T], \\ &\frac{\partial \tilde{\rho}}{\partial t} = f\tilde{y} - h\tilde{\rho} - u_1(t)\tilde{\rho} - \tilde{u}(t)\rho_2 \qquad \text{in } I \times (0,T], \\ &\frac{\partial \tilde{y}}{\partial x}(0,t) = \frac{\partial \tilde{y}}{\partial x}(L,t) = 0 \qquad \text{on } (0,T], \\ &\tilde{y}(x,0) = 0, \quad \tilde{\rho}(x,0) = 0 \qquad \text{in } I.\end{aligned} \tag{3.16}$$

Multifly the first equation of (3.16) by $\tilde{y}$ and integrate the product in I. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \tilde{y}^2 dx + f\int_0^L \tilde{y}^2 dx + d\int_0^L \Big|\frac{\partial \tilde{y}}{\partial x}\Big|^2 dx$$
$$= g\int_0^L \tilde{\rho}\tilde{y}dx - \int_0^L \gamma(\rho_1)\tilde{y}^2 dx + \int_0^L [2ab - a(\rho_1+\rho_2)]y_2\tilde{\rho}\tilde{y}dx.$$

Since $\rho_1, \rho_2, y_2 \in L^\infty((0,T)\times I)$, it follows that

$$\frac{d}{dt}\int_0^L \tilde{y}^2 dx + 2\delta\|\tilde{y}\|^2_{H^1(I)} \le C\int_0^L (\tilde{y}^2 + \tilde{\rho}^2)dx, \tag{3.17}$$

where $\delta = \min\{f, d\}$.

Multifly the second equation of (3.16) by $\tilde{\rho}$ and integrate the product in I. Then, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \tilde{\rho}^2 dx + h\int_0^L \tilde{\rho}^2 dx = f\int_0^L \tilde{y}\tilde{\rho}dx - \int_0^L u_1(t)\tilde{\rho}^2 dx - \int_0^L \tilde{u}(t)\rho_2\tilde{\rho}dx$$
$$\le \Big(\frac{h}{2}+\frac{h}{2}\Big)\int_0^L \tilde{\rho}^2 dx + C\Big(\int_0^L (\tilde{y}^2+\tilde{\rho}^2)dx + \tilde{u}^2(t)\|\rho_2\|^2_{L^2(I)}\Big).$$

Therefore, it follows that

$$\frac{d}{dt}\int_0^L \tilde{\rho}^2 dx \le C\Big(\int_0^L (\tilde{y}^2+\tilde{\rho}^2)dx + \tilde{u}^2(t)\|\rho_2\|^2_{L^2(I)}\Big). \tag{3.18}$$

Then, we obtain from (3.17) and (3.18) that

$$\frac{d}{dt}\int_0^L (\tilde{y}^2+\tilde{\rho}^2)dx \le C\Big(\int_0^L (\tilde{y}^2+\tilde{\rho}^2)dx + \tilde{u}^2(t)\|\rho_2\|^2_{L^2(I)}\Big).$$

By using Gronwall's Lemma and (3.2),

$$\int_0^L (\tilde{y}^2+\tilde{\rho}^2)dx \le C\|\rho_2\|^2_{L^\infty(0,T;L^2(I))}\|\tilde{u}(t)\|^2_{L^2(0,T)} \tag{3.19}$$
$$\le C\|\tilde{u}(t)\|^2_{H^1(0,T)}.$$

If we use (3.17) and (3.19), we obtain

$$\int_0^t \|\tilde{y}(s)\|^2_{H^1(I)}ds \le C\|\tilde{u}(t)\|^2_{H^1(0,T)}, \quad 0 \le t \le T.$$

□

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