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IMEX METHODS FOR PRICING FIXED STRIKE ASIAN OPTIONS WITH JUMP-DIFFUSION MODELS

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Abstract. In this paper we study implicit-explicit (IMEX) methods combined with a semi-Lagrangian scheme to evaluate the prices of fixed strike arithmetic Asian options under jump-diffusion models. An Asian option is described by a two-dimensional partial integro-differential equation (PIDE) that has no diffusion term in the arithmetic average direction. The IMEX methods with the semi-Lagrangian scheme to solve the PIDE are discretized along characteristic curves and performed without any fixed point iteration techniques at each time step. We implement numerical simulations for the prices of a European fixed strike arithmetic Asian put option under the Merton model to demonstrate the second-order convergence rate.

1. Introduction

An Asian option is a path-dependent option where the terminal payoff depends on the average price of an underlying asset over a prescribed period of time. The average price is calculated using either an arithmetic average or a geometric average. Since under the Black-Scholes model the geometric average price is log-normally distributed and the arithmetic average price is not log-normally distributed, it is difficult to price the arithmetic Asian option rather than the geometric Asian option.

Many researchers have developed a number of numerical methods for pricing Asian options. In the case of the Black-Scholes model, Zvan, Forsyth, and Vetzal [10] suggested an implicit method in conjunction with the van Leer flux limiter which leads to oscillation free results. Večeř [8] derived a one-dimensional partial differential equation for pricing arithmetic Asian options and it is extended by Cen, Le, and Xu [3]. In the case of a jump-diffusion model, Tangman, Peer, Rambeerich, and Bhuruth [7] proposed an exponential integration time

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(ETI) scheme and it is also applied to an infinite-activity Lévy process. A matrix exponential has to be evaluated in the ETI scheme. Zhang and Oosterlee [9] developed an Asian cosine method when the characteristic function for an exponential Lévy model is known. A semi-Lagrangian approach is suggested by d'Halluin, Forsyth, and Labahn [5] and a subquadratic convergence is observed with an iteration scheme at each time step.

The purpose of this paper is to construct three IMEX methods combined with a semi-Lagrangian scheme to solve the PIDE for pricing a fixed strike arithmetic Asian option under a jump-diffusion model. The three IMEX methods are focused on the inverse of the tridiagonal linear system at each time step so that it can be computed without any fixed point iteration techniques. A fast Fourier transform (FFT) is applied to evaluate an integral operator over a truncated domain in the PIDE and a quadratic interpolation is used to compute the option prices at non-grid points.

This paper is structured as follows. In section 2 we introduce the fixed strike Asian option with the arithmetic average price under the jump-diffusion model. In section 3 we propose the three IMEX methods with the semi-Lagrangian scheme to solve the PIDE numerically. Some numerical results in section 4 show that the proposed methods for the prices of a European fixed strike arithmetic Asian put option under the Merton model have the second-order convergence rate in the discrete ℓ^2-norm. Section 5 contains concluding remarks.

2. Fixed strike Asian options with arithmetic average price

Let $(S_t)_{t\in[0,T]}$ be the stock price of an underlying asset which follows a jump-diffusion model. Then the stochastic differential equation of S_t in a risk-neutral world is given by

$$\frac{dS_t}{S_{t-}} = (r - d - \lambda\zeta)dt + \sigma dW_t + \eta dN_t, \tag{1}$$

where r is the risk-free interest rate, d is the continuous dividend yield, σ is the volatility, W_t is the Wiener process, N_t is the Poisson process with intensity λ, and ζ is the expectation of η which is the random variable causing a jump from S_{t-} to $(\eta+1)S_{t-}$. The continuous arithmetic average price A_t we consider is defined as

$$A_t = \frac{1}{t}\int_0^t S_u du \quad \text{and} \quad dA_t = \frac{S_t - A_t}{t}dt. \tag{2}$$

By applying the Itô's formula, the price of an Asian option u satisfies the following PIDE

$$\begin{aligned} u_\tau(\tau,x,a) =& \frac{\sigma^2}{2} u_{xx}(\tau,x,a) + \frac{1}{A_0 e^a} \cdot \frac{S_0 e^x - A_0 e^a}{T-\tau} u_a(\tau,x,a) \\ &+ \left(r - d - \frac{\sigma^2}{2} - \lambda\zeta \right) u_x(\tau,x,a) \\ &- (r+\lambda) u(\tau,x,a) + \lambda \int_{\mathbb{R}} u(\tau,z,a) f(z-x) dz \end{aligned} \tag{3}$$

for all $(\tau,x,a) \in (0,T] \times \mathbb{R} \times \mathbb{R}$, where $\tau = T - t$ is the time to maturity, $x = \ln(S/S_0)$ is the log price of the underlying asset S, $a = \ln(A/A_0)$ is the log price of the arithmetic average A, and $f(x)$ is the probability density function of the random variable $\ln(\eta+1)$.

The payoff functions of fixed strike Asian call and put options are given by

$$h_c(a) = (A_0 e^a - K)^+ \quad \text{and} \quad h_p(a) = (K - A_0 e^a)^+, \tag{4}$$

respectively, where K is the strike price and $(x)^+$ denotes the maximum of 0 and x. To solve the PIDE numerically, we truncate an infinite domain $\mathbb{R} \times \mathbb{R}$ to bounded intervals $\Omega_x \times \Omega_a = (-X, X) \times (-Y, Y)$ with $X > 0$ and $Y > 0$. In order to set the boundary conditions in the stock price direction, we consider the put-call parity of the form

$$u_p = u_c - \frac{S_0 e^x}{T} \left(\frac{e^{-d\tau} - e^{-r\tau}}{r-d} \right) + e^{-r\tau} \left(K - \frac{T-\tau}{T} A_0 e^a \right), \tag{5}$$

where u_c and u_p are the prices of European fixed strike arithmetic Asian call and put options, respectively. We refer to [1, 6] for more details about the put-call parity. Then the boundary behaviors of the European fixed strike arithmetic Asian call option are

$$\lim_{x\to-\infty} \left\{ u_c(\tau,x,a) - e^{-r\tau} \left(\frac{T-\tau}{T} A_0 e^a - K \right)^+ \right\} = 0$$

and

$$\lim_{x\to\infty} \left\{ u_c(\tau,x,a) - \frac{S_0 e^x}{T} \left(\frac{e^{-d\tau} - e^{-r\tau}}{r-d} \right) + e^{-r\tau} \left(K - \frac{T-\tau}{T} A_0 e^a \right) \right\} = 0.$$

In the case of the European fixed strike arithmetic Asian put option, the boundary behaviors are given by

$$\lim_{x\to-\infty} \left\{ u_p(\tau,x,a) - e^{-r\tau} \left(K - \frac{T-\tau}{T} A_0 e^a \right)^+ \right\} = 0$$

and

$$\lim_{x\to\infty} u_p(\tau,x,a) = 0.$$

3. Discretization

In this section, we present IMEX methods to evaluate the prices of the European fixed strike arithmetic Asian option under the jump-diffusion model. The IMEX methods are combined with a semi-Lagrangian scheme to solve the PIDE (3) along characteristic curves in the a-direction. The semi-Lagrangian approach is suggested by d'Halluin, Forsyth, and Labahn [5]. Then the PIDE (3) can be reformulated as

$$\frac{Du}{D\tau}(\tau, x, a) - \mathcal{L}u(\tau, x, a) = 0, \tag{6}$$

where the time derivative with the semi-Lagrangian scheme is

$$\frac{Du}{D\tau} = \frac{\partial u}{\partial \tau} + \frac{\partial u}{\partial a}\frac{da}{d\tau} \quad \text{with} \quad \frac{da}{d\tau} = \frac{1}{A_0 e^a}\frac{A_0 e^a - S_0 e^x}{T - \tau} \tag{7}$$

and the integro-differential operator $\mathcal{L}u$ is given by

$$\mathcal{L}u(\tau, x, a) = \mathcal{D}u(\tau, x, a) + \mathcal{I}u(\tau, x, a) \tag{8}$$

with the differential operator $\mathcal{D}u$ being

$$\begin{aligned}\mathcal{D}u(\tau, x, a) =& \frac{\sigma^2}{2}u_{xx}(\tau, x, a) + \left(r - d - \frac{\sigma^2}{2} - \lambda\zeta\right)u_x(\tau, x, a) \\ &- (r + \lambda)u(\tau, x, a)\end{aligned} \tag{9}$$

and the integral operator $\mathcal{I}u$ being

$$\mathcal{I}u(\tau, x, a) = \lambda\int_{-\infty}^{\infty} u(\tau, z, a)f(z - x)dz. \tag{10}$$

In order to solve the PIDE numerically, we use an equally spaced points on the truncated domain $(0, T] \times \Omega_x \times \Omega_a$. For three positive integers $N > 0$, $M > 0$ and $L > 0$, let $\tau_n = n\Delta\tau$ for $n = 0, 1, \ldots, N$ be the time grid points with $\Delta\tau = T/N$, $x_m = -X + m\Delta x$ be the grid points in the x-direction with $\Delta x = 2X/M$, and $a_l = -Y + l\Delta a$ for $l = 0, 1, \ldots, L$ be the grid points in the a-direction with $\Delta a = 2Y/L$.

Since the characteristic curve $a = a(\tau; \tau_{n+1}, x_m, a_l)$ for the fixed x_m passing through the grid point a_l at $\tau = \tau_{n+1}$ satisfies the condition (7), the grid points of this trajectory $a^n_{l(m,n+1)}$ at $\tau = \tau_n$ and $a^{n-1}_{l(m,n+1)}$ at $\tau = \tau_{n-1}$ are obtained by

$$A_0 e^{a^n_{l(m,n+1)}} = A_0 e^{a_l} + \frac{S_0 e^{x_m} - A_0 e^{a_l}}{N - n} \quad \text{for } 0 \le n \le N - 1$$

and

$$A_0 e^{a^{n-1}_{l(m,n+1)}} = A_0 e^{a_l} + \frac{2(S_0 e^{x_m} - A_0 e^{a_l})}{N - n + 1} \quad \text{for } 1 \le n \le N - 1,$$

respectively.

Three IMEX methods are introduced to approximate the PIDE (6) along the characteristic curve in the a-direction. In a leapfrog (LF) method, the discrete equation is

$$\frac{u_{m,l}^{n+1} - u_{m,l(m,n+1)}^{n-1}}{2\Delta\tau} = \mathcal{D}_\Delta \left(\frac{u_{m,l}^{n+1} + u_{m,l(m,n+1)}^{n-1}}{2} \right) + \mathcal{I}_\Delta u_{m,l(m,n+1)}^{n}, \tag{11}$$

where $\mathcal{D}_\Delta$ and $\mathcal{I}_\Delta$ are the discrete differential and integral operators of the forms

$$\begin{aligned} \mathcal{D}_\Delta u_{m,l}^n =& \frac{\sigma^2}{2} \frac{u_{m+1,l}^n - 2u_{m,l}^n + u_{m-1,l}^n}{\Delta x^2} \\ &+ \left(r - d - \frac{\sigma^2}{2} - \lambda\zeta \right) \frac{u_{m+1,l}^n - u_{m-1,l}^n}{2\Delta x} - (r+\lambda) u_{m,l}^n \end{aligned}$$

and

$$\begin{aligned} \mathcal{I}_\Delta u_{m,l}^n =& \frac{\lambda \Delta x}{2} \left(u_{0,l}^n f_{m,0} + 2 \sum_{j=1}^{M-1} u_{j,l}^n f_{m,j} + u_{M,l}^n f_{m,M} \right) \\ &+ \lambda R(\tau_n, x_m, a_l, X), \end{aligned}$$

where $u_{m,l}^n = u(\tau_n, x_m, a_l)$, $f_{m,j} = f(x_j - x_m)$, and $R(\tau_n, x_m, a_l, X)$ denotes the integral over the outside of Ω_x in (10). We next deal with a Crank-Nicolson (CN) method given by

$$\begin{aligned} \frac{u_{m,l}^{n+1} - u_{m,l(m,n+1)}^{n}}{\Delta\tau} =& \mathcal{D}_\Delta \left(\frac{u_{m,l}^{n+1} + u_{m,l(m,n+1)}^{n}}{2} \right) \\ &+ \mathcal{I}_\Delta \left(\frac{3u_{m,l}^{n} - u_{m,l(m,n+1)}^{n-1}}{2} \right) \end{aligned} \tag{12}$$

and a 2-step backward differentiation formula (BDF2 method) given by

$$\begin{aligned} &\frac{3u_{m,l}^{n+1} - 4u_{m,l(m,n+1)}^{n} + u_{m,l(m,n+1)}^{n-1}}{2\Delta\tau} \\ &\qquad = \mathcal{D}_\Delta u_{m,l}^{n+1} + \mathcal{I}_\Delta \left(2u_{m,l(m,n+1)}^{n} - u_{m,l(m,n+1)}^{n-1} \right). \end{aligned} \tag{13}$$

We note that the global discretization error, which is discussed in detail in [4], is

$$\mathcal{O}\left(\Delta\tau^2 + \Delta x^2 + \frac{\Delta a^p}{\Delta\tau} \right), \tag{14}$$

where p is derived from an interpolation. Thus one can expect that the three IMEX methods have the second-order convergence rate with respect to the time and spatial variables when a quadratic interpolation is used. Furthermore, since any fixed point iteration techniques at each time step are not needed in the three

TABLE 1. The prices of the European fixed strike arithmetic Asian put option at $S = 100$ under the Merton model. These prices are obtained by using the LF method, the CN method, and the BDF2 method with the parameters in (15). N is the number of time steps, M is the number of spatial steps, and L is the number of arithmetic average steps.

			LF method		CN method		BDF2 method	
N	M	L	Price	Error	Price	Error	Price	Error
25	32	32	2.8175	-	2.8218	-	2.8217	-
50	64	64	3.1920	0.3744	3.1939	0.3721	3.1954	0.3737
100	128	128	3.2582	0.0662	3.2590	0.0651	3.2597	0.0643
200	256	256	3.2702	0.0120	3.2704	0.0114	3.2706	0.0109
400	512	512	3.2730	0.0028	3.2730	0.0026	3.2731	0.0025
800	1024	1024	3.2736	0.0007	3.2736	0.0006	3.2737	0.0006

IMEX methods, it is efficient to price the fixed strike arithmetic Asian option under the jump-diffusion model.

4. Numerical results

In this section, we employ the three IMEX methods coupled with the semi-Lagrangian scheme to price a European fixed strike Asian put option with an arithmetic average price under the Merton model. By using the boundary behaviors of the Asian put option, the integral over the domain $\mathbb{R} \setminus \Omega_x$ is obtained by

$$R(\tau, x, a, X) = e^{-r\tau} \left(K - \frac{T-\tau}{T} A_0 e^a \right)^+ \cdot \Phi\left(-\frac{X + x + \mu_J}{\sigma_J} \right).$$

The quadratic interpolation is applied to compute the prices of the Asian put option at the non-grid points $(\tau_n, x_m, a_{l(m,n+1)})$ and $(\tau_{n-1}, x_m, a_{l(m,n+1)})$ with the global second-order accuracy. The FFT is carried out for the computation of the discrete integral operator over the bounded domain Ω_x. To illustrate the performance of these methods, the parameters used in the numerical test are

$$r = 0.15, \quad d = 0, \quad \sigma = 0.1, \quad \lambda = 1, \tag{15}$$

$$\mu_J = -0.1, \quad \sigma_J = 0.3, \quad T = 1, \quad K = 100,$$

where the probability density function of $\ln(\eta + 1)$ is a normal distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_J^2}} e^{-\frac{(x-\mu_J)^2}{2\sigma_J^2}}.$$

The bounded domain is given by $\Omega_x \times \Omega_a = (-2, 2) \times (-2, 2)$ and we set $S_0 = K$ and $A_0 = K$.

The numerical results are obtained with MATLAB on a computer with Intel(R) Core(TM) i5-6200U CPU 2.30GHz. It is shown in Table 1 that the

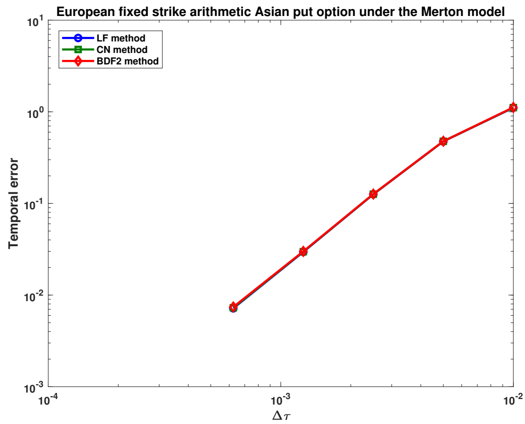

FIGURE 1. The discrete ℓ^2-errors for the European fixed strike arithmetic Asian put option under the Merton model by using the LF method, the CN method, and the BDF2 method.

pointwise errors at $S = 100$ converge with the second-order accuracy. The reference price at $S = 100$ is 3.272 which is evaluated in [2] and then the prices in Table 1 are the same as the reference price with 2 digits after the decimal point. In Figure 1, the temporal error is computed by

$$\text{Temporal error} = \|u(\Delta\tau, \Delta x, \Delta a) - u(\Delta\tau/2, \Delta x/2, \Delta a/2)\|_{\ell^2},$$

where $u(\Delta\tau, \Delta x, \Delta a)$ is the Asian option price on the final time level $\tau = T$ and $\|u^n\|_{\ell^2}$ is given by

$$\|u^n\|_{\ell^2} = \sqrt{\Delta x \Delta a \sum_{l=0}^{L} \sum_{m=1}^{M-1} \left(u^n_{m,l}\right)^2}$$

with

$$u^n = (u^n_{1,0}, u^n_{2,0}, \ldots, u^n_{M-1,0}, \ldots, u^n_{1,L}, u^n_{2,L}, \ldots, u^n_{M-1,L})^{\mathrm{T}}.$$

We observe that the slopes of the three lines are about 2 and the LF, CN, and BDF2 methods have the second-order convergence in the discrete ℓ^2-norm.

5. Conclusion

We consider the three IMEX methods combined with the semi-Lagrangian scheme to solve the PIDE for the prices of the European fixed strike arithmetic Asian option under the jump-diffusion model. These proposed methods are performed without any fixed point iteration techniques at each time step. The quadratic interpolation is used to obtain the prices of the Asian option along the characteristic curve and the FFT is also used to compute the discrete integral operator over the domain Ω_x. Numerical simulations show that the three IMEX methods for the European fixed strike arithmetic Asian put option under the Merton model have the second-order convergence rate with respect to the time, spatial, and arithmetic average variables.

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