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ON THE EQUATIONS DEFINING SOME CURVES OF MAXIMAL REGULARITY IN $\mathbb{P}^4$

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Abstract. For a nondegenerate irreducible projective variety, it is a classical problem to describe its defining equations. In this paper we precisely determine the defining equations of some rational curves of maximal regularity in $\mathbb{P}^4$ according to their rational parameterizations.

1. Introduction

Let $\mathbb{P}^r$ be the projective r-space over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic and $R = \mathbb{K}[X_0, X_1, \ldots, X_r]$ be the homogeneous coordinate ring of $\mathbb{P}^r$. Let $X \subset \mathbb{P}^r$ be a nondegenerate irreducible variety and let I_X be the homogeneous ideal of X in R. In projective algebraic geometry, it is a basic problem to describe the defining equations of X and its ideal I_X for a given embedding. This problem is well understood for Veronese varieties, rational normal scrolls and Segre varieties. For example, see [3]. Also this problem for non-normal del Pezzo varieties were completely solved in [4], [6] and [7].

Along this line, we continue the study of the problem to describe the equations defining rational curves begun in [8]. Let $T := \mathbb{K}[s, t]$ be the homogeneous coordinate ring of $\mathbb{P}^1$. For each $k \geq 1$, we denote by T_k the k-th graded component of T. Then the rational normal curve $\widetilde{C} \subset \mathbb{P}^d$ of degree d is defined to be the image of the embedding $\nu_d : \mathbb{P}^1 \to \mathbb{P}^d$ parameterized by

$$\widetilde{C} = \{[s^d(P) : s^{d-1}t(P) : \cdots : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}. \tag{1}$$

As is well-known, $\widetilde{C}$ is defined by the common zero locus of the polynomials $F_{i,j} = X_iX_j - X_{i-1}X_{j+1}$ for $1 \leq i \leq j \leq d-1$. Indeed the defining ideal $I_{\widetilde{C}}$ is minimally generated by the set $\{F_{i,j} \mid 1 \leq i \leq j \leq d-1\}$ in the sense of Definition 2.1. Let $C \subset \mathbb{P}^r$ be a nondegenerate rational curve of degree $d \geq r$. Since the normalization of C is the rational normal curve $\widetilde{C}$, it follows that C

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is given by a linear projection of $\widetilde{C} \subset \mathbb{P}^d$ from a linear subspace $\Lambda \cong \mathbb{P}^{d-r-1}$ of $\mathbb{P}^d$. In other words, there exists a subset $\{f_0, f_1, \ldots, f_r\} \subset T_d$ of $\mathbb{K}$-linearly independent forms of degree d such that C is a curve parameterized as

$$C = \{[f_0(P) : f_1(P) : \cdots : f_r(P)] \mid P \in \mathbb{P}^1\}.$$

The main purpose of this article is to determine a minimal generating set of the defining ideal I_C of C. In [8], the authors provide a complete description of defining equations for the case where $r = 3$. The result is

Theorem 1.1 (Theorem 1.1 in [8])**.** *Let $C_d \subset \mathbb{P}^3$ be a rational curve defined as the parametrization*

$$C_d = \{[s^d(P) : s^{d-1}t(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}$$

where $d \geq 3$. Then the defining ideal I_{C_d} of C_d is minimally generated as following:

$$I_{C_d} = \langle X_0X_3 - X_1X_2, F_1, F_2, \ldots, F_{d-1}\rangle$$

where $F_i = X_0^{d-i-1}X_2^i - X_1^{d-i}X_3^{i-1}$ for $1 \leq i \leq d-1$.

As a next case, we study the set of minimal generators of an ideal defining rational curves in $\mathbb{P}^4$ parameterized by

$$C_d = \{[s^d(P) : s^{d-1}t(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\}.$$

First we show that C_d is a smooth rational curve of degree d which is contained in the rational normal surface scroll $S(1,2)$ as a divisor $H + (d-3)F$. Here H and F are respectively the hyperplane divisor and a ruling line (see Lemma 3.3 and Proposition 3.4). This observation enables us to obtain the exact number of minimal generators of I_{C_d} thanks to [5, Theorem 1.2]. We also compute several examples by means of the Computer Algebra System SINGULAR [1](see Example 3.1) which pose the concrete expressions of the generators of I_{C_d}. In our main result, Theorem 3.2 provides an explicit description of a set of minimal generators of the ideal I_{C_d} according to the degree $d = 2n$ and $d = 2n+1$ for $n \geq 2$.

2. preliminaries

We begin with the concept of a minimal generating set of the defining ideal I_X for a nondegenerate irreducible projective variety $X \subset \mathbb{P}^r$. Let

$$M = \{F_{i,j} \in \mathbb{K}[X_0, X_1, \ldots, X_r] \quad \mid \quad F_{i,j} \in I_X \quad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq \ell_i\}$$

be the set of homogeneous polynomials $F_{i,j}$ of degree $\deg(F_{i,j}) = i$. Let $(I_X)_{\leq t}$ be the ideal generated by the homogeneous polynomials in I_X of degree at most t.

Definition 2.1. M is a minimal set of generators of I_X if the following three conditions hold:

(*i*) I_X is generated by the polynomials in M (i.e., $I_X = \langle M\rangle$).

(ii) $F_{i,1}, F_{i,2}, \ldots, F_{i,\ell_i}$ are $\mathbb{K}$-linearly independent forms of degree i for each $1 \le i \le m$.

(iii) $F_{i,j} \notin \langle (I_X)_{\le i-1}, F_{i,1}, \ldots, F_{i,j-1} \rangle$ for each $1 \le i \le m$ and $1 \le j \le \ell_i$.

Notation and Remarks 2.2. (a) Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ be a vector bundle on $\mathbb{P}^1$ where $0 < a_1 \le a_2$. Then the smooth rational normal surface scroll $S(a_1, a_2)$ is the image of the map defined by the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of $\mathbb{P}(\mathcal{E})$.
(b) The divisor class group of $S(a_1, a_2)$ is freely generated by the hyperplane divisor H and a ruling line F of $S(a_1, a_2)$. That is, a divisor of $S(a_1, a_2)$ is written by $aH + bF$ for $a, b \in \mathbb{Z}$.
(c) The rational normal surface scroll $S := S(1,2) \subset \mathbb{P}^4$ of degree 3 can be described as

$$S = \{[su : tu : s^2v : stv : t^2v] \mid (s,t), (u,v) \in \mathbb{K}^2 \setminus (0,0)\} \subset \mathbb{P}^4. \qquad (2)$$

Then S is defined by (2×2)-minors of the matrix

$$\begin{bmatrix} X_0 & X_2 & X_3 \\ X_1 & X_3 & X_4 \end{bmatrix}.$$

Thus the ideal I_S of S is generated by $X_0X_3 - X_1X_2$, $X_0X_4 - X_1X_3$ and $X_2X_4 - X_3^2$.
(d) Let $\widetilde{C} \subset \mathbb{P}^4$ be a rational normal curve of degree 4 parameterized as in (1). Then the ideal $I_{\widetilde{C}}$ of $\widetilde{C}$ is minimally generated by the set of six quadratic equations:

$$\{X_0X_3 - X_1X_2, X_0X_4 - X_1X_3, X_2X_4 - X_3^2, X_0X_2 - X_1^2, X_1X_3 - X_2^2, X_1X_4 - X_2X_3\}.$$

Thus $\widetilde{C}$ is contained the rational normal surface scroll $S(1,2)$. Furthermore, $\widetilde{C}$ is linearly equivalent to a divisor $H + F$. (For details, see [10, Theorem 5.10]).
(e) For a smooth curve $Z \subset \mathbb{P}^r$ and an integer $s \ge 2$, we defined the closure Z^s, say the s-th join of Z with itself, of the set of points lying in $(s-1)$-dimensional linear subspaces spanned by general collections of s points in Z. Then there is a strictly ascending filtration

$$Z \subsetneqq Z^2 \subsetneqq Z^3 \subsetneqq \cdots \subsetneqq Z^{\lceil \frac{d+1}{2} \rceil - 1} \subsetneqq Z^{\lceil \frac{d+1}{2} \rceil} = \mathbb{P}^r$$

Then it is well known that the linear projection map $\pi_\Lambda : Z \to \mathbb{P}^{r-n-1}$ of Z from an n-dimensional linear subspace Λ of $\mathbb{P}^r$ with the condition $\Lambda \cap C^2 = \varnothing$ is an isomorphism. For details, we refer to the reader to [11].
(f) Let $Z \subset \mathbb{P}^r$ be a nondegenerate irreducible projective curve of degree d. Z is said to be m-regular if its sheaf of ideal $\mathcal{I}_Z$ satisfies the vanishing

$$H^i(\mathbb{P}^r, \mathcal{I}_Z(m-i)) = 0 \quad \text{for all } i \ge 1.$$

The Castelnuovo-Mumford regularity (or simply the regularity) of Z, denoted by $\operatorname{reg}(Z)$, is defined as the least integer m such that Z is m-regular(cf. [9]). Another interest of this notion stems partly from the fact that Z is m-regular if and only if for every $j \geq 0$ the minimal generators of the j-th syzygy module of the homogeneous ideal $I(Z)$ of Z occur in degree $\leq m + j$. In particular, $I(Z)$ is generated by forms of degree $\leq m$. Thus the existence of an ℓ-secant line guarantees that $\mathrm{reg}(Z) \geq \ell$. By a well-known result of Gruson-Lazarsfeld-Peskine [2], $\mathrm{reg}(Z)$ is bounded by $\mathrm{reg}(Z) \leq d-r+2$. They further classified the extremal curves which fail to be $(d-r+1)$-regular, showing in particular that if $d \geq r+2$ then Z is a smooth rational curve with a unique $(d-r+2)$-secant line.

3. Main Theorem

In this section, we provide a complete description of equations which generate the defining ideals of some rational curves of maximal regularity in $\mathbb{P}^4$. Let $C_d \subset \mathbb{P}^4$ be a curve parameterized as

$$C_d = \{[s^d(P) : s^{d-1}t(P) : s^2t^{d-2}(P) : st^{d-1}(P) : t^d(P)] \mid P \in \mathbb{P}^1\} \qquad (3)$$

for $d \geq 4$. Let $R := \mathbb{K}[X_0, X_1, X_2, X_3, X_4]$ be the homogeneous coordinate ring of $\mathbb{P}^4$. First we fix some notations for $n \geq 2$,

$(*)$ $d = 2n$ and

$$\begin{cases} G_{[n,i]} = X_1X_3^{i-1}X_4^{n-i} - X_2^{n+i-2}X_3^{2-i} & \text{for } i = 1,2 \\ H_{n+j-1} = X_0^{2j-1}X_2^{n-j} - X_1^{2j}X_4^{n-j-1} & \text{for } 1 \leq j \leq n-1 \end{cases}, \textit{and}$$

$(**)$ $d = 2n+1$ and

$$F_{n+i-1} = X_0^{2i-2}X_2^{n-i+1} - X_1^{2i-1}X_4^{n-i} \quad \text{for } 1 \leq i \leq n.$$

Note that the following three quadratic polynomials

$$Q_{[2,1]} = X_0X_3 - X_1X_2, \quad Q_{[2,2]} = X_0X_4 - X_1X_3 \quad \text{and} \quad Q_{[2,3]} = X_2X_4 - X_3^2$$

are the minimal generators of the defining ideal $I_{S(1,2)}$ of the rational normal surface scroll $S(1,2)$. Then the Computer Algebra System SINGULAR provides

Example 3.1. For $d = 5, 6, 7, 8, 9, 10$, let $C_d \subset \mathbb{P}^4$ be rational curves defined as above. Then the minimal sets of generators defining the ideal I_{C_d} are

(i) $I_{C_5} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, F_2, F_3 \rangle$,

(ii) $I_{C_6} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[3,1]}, G_{[3,2]}, H_3, H_4 \rangle$,

(iii) $I_{C_7} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, F_3, F_4, F_5 \rangle$,

(iv) $I_{C_8} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[4,1]}, G_{[4,2]}, H_4, H_5, H_6 \rangle$,

(v) $I_{C_9} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, F_4, F_5, F_6, F_7 \rangle$ and

(vi) $I_{C_{10}} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[5,1]}, G_{[5,2]}, H_5, H_6, H_7, H_8 \rangle$.

This example enables us to pose the theorem:

Theorem 3.2. *Let $C_d \subset \mathbb{P}^4$, $d \geq 4$ be a curve stated as in (3). Then C_d is a smooth rational curve of degree d and of maximal regularity $d-2$. In particular, the defining ideal I_{C_d} of C_d is minimally generated as followings: For $n \geq 2$,*

(1) *If $d = 2n$, then*

$$I_{C_d} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[n,1]}, G_{[n,2]}, H_n, H_{n+1}, \cdots, H_{2n-2} \rangle.$$

(2) *If $d = 2n+1$, then*

$$I_{C_d} = \langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, F_n, F_{n+1}, \cdots, F_{2n-1} \rangle.$$

Before proving this theorem, we investigate several properties of the curve C_d.

Lemma 3.3. *Let C_d be as in Theorem 3.2. Then C_d is smooth rational and of degree d.*

Proof. The case where $d = 4$ follows immediate from (1). Suppose that $d > 4$. Let Λ be a $(d-5)$-dimensional linear subspace of $\mathbb{P}^d$ spanned by $(d-4)$ standard coordinate points

$$\{[0,0,1,0,\ldots,0,0], [0,0,0,1,0,\ldots,0,0], \ldots, [0,0,\cdots,0,1,0,0,0]\}.$$

Then we can see that the curve C_d is obtained by the linear projection map $\pi_\Lambda : \widetilde{C} \to \mathbb{P}^4$ of the rational normal curve $\widetilde{C}$ in(1) from the center Λ. Since $\Lambda \subset \mathbb{P}^d \setminus C_d^2$, the map π_Λ is an isomorphism by Notation and Remarks 2.2.(e). This competes the proof. ☐

Proposition 3.4. *Let C_d be as in Theorem 3.2. Then,*

(1) *The curve C_d is contained in the rational normal surface scroll $S(1,2)$ as a divisor linearly equivalent to $H + (d-3)F$ where H and F are respectively the hyperplane divisor and a ruling line.*
(2) *The curve C_d is of maximal regularity $d-2$ and the minimal section $S(1)$ of $S(1,2)$ is a unique $d-2$ secant line to C_d.*

Proof. (1) We denote S the rational normal surface scroll $S(1,2)$. Then it is easy to see that the curve C_d satisfies the three quadratic equations

$$\{X_0X_3 - X_1X_2, X_0X_4 - X_1X_3, X_2X_4 - X_3^2\}$$

which are generators of the defining ideal I_S of S (see Notation and Remarks 2.2.(c)). Thus C_d is linearly equivalent to a divisor $aH + bF$ of S for some integers a and b. Indeed, we may assume that $a \geq 1$ since C_d is an irreducible and effective divisor. Now suppose that $a \geq 2$ and consider the exact sequence

$$0 \to \mathcal{I}_S \to \mathcal{I}_{C_d} \to \mathcal{O}_S(-aH - bF) \to 0.$$

Then we have the following long exact sequence

$$\to H^1(\mathbb{P}^r, \mathcal{I}_S(1)) \to H^1(\mathbb{P}^r, \mathcal{I}_{C_d}(1)) \to H^1(S, \mathcal{O}_S((1-a)H - bF)) \to H^2(\mathbb{P}^r, \mathcal{I}_S(1)) \to \cdots.$$

Since S is arithmetically Cohen-Macaulay (i.e., $H^i(\mathbb{P}^r, \mathcal{I}_S(j)) = 0$ for $1 \le i \le 3$ and $j \in \mathbb{Z}$), we get an isomorphism

$$H^1(\mathbb{P}^r, \mathcal{I}_{C_d}(1)) \cong H^1(S, \mathcal{O}_S((1-a)H - bF)). \tag{4}$$

Remind that the curve C_d is not linearly normal because it is the image of an isomorphic projection of a rational normal curve of degree d by Lemma 3.3. Thus we have a contradiction by applying the vanishing $H^1(S, \mathcal{O}_S((1-a)H - bF)) = 0$ for $a \ge 2$ to the isomophism (4). It can be easily shown that $b = d-2$ by degree calculation of the divisor $H + bF$.
(2) First note that the regularity of C_d is bounded by $d-2$ by Notation and Remarks 2.2.(f). Also every ℓ-secant line for $\ell \ge 3$ should be contained in S since S is cut out by quadratic equations. Now we consider the lines contained in S. They are precisely the minimal section $S(1) \equiv H - 2F$ and the ruling lines F. Then the intersection numbers of C_d with those lines are given by

$$\begin{cases} \sharp(C_d \cap S(1)) = (H + (d-3)F).(H - 2F) = d-2 \\ \sharp(C_d \cap F) = (H + (d-3)F).F = 1 \end{cases}.$$

Thus the minimal section $S(1)$ is a unique $(d-2)$-secant line to C_d and hence C_d attains the maximal regularity $d-2$. □

Proof of Theorem 3.2. First we have our assertions by Lemma 3.3 and Proposition 3.4. In the remaining parts of the proof, we will describe the minimal set of generators defining the ideal I_{C_d} for $d = 2n$ and $d = 2n+1$ with $n \ge 2$, in turn. We denote by M_{2n} and M_{2n+1} respectively the sets

$$M_{2n} = \{Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[n,1]}, G_{[n,2]}, H_n, H_{n+1}, \cdots, H_{2n-2}\} \quad \text{and}$$
$$M_{2n+1} = \{Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, F_n, F_{n+1}, \cdots, F_{2n-1}\}.$$

And we also denote $I_{M_d} := \langle M_d \rangle$ as an ideal generated by the set M_d. Then it is easy to see that the equations in M_d satisfy C_d in (3) for each case. That is, $I_{M_d} \subseteq I_{C_d}$. To show the equality $I_{M_d} = I_{C_d}$, it suffices to verify that

1. M_d is a minimal set generating the ideal I_{M_d} and
2. the number of elements in M_d is equal to that of the minimal set of generator of I_{C_d}.

Remind that $I_{M_4} = I_{C_4}$ since M_4 is just the minimal set of generators of a rational normal curve of degree 4 (see Notation and Remarks 2.2.(d)). To prove statement 1, we show the three conditions in Definition 2.1. (i) It is clear by the definition of I_{M_d}. (ii) It follows that $Q_{[2,1]}$, $Q_{[2,2]}$ and $Q_{[2,3]}$ are $\mathbb{K}$-linearly independent quadratic equations since those are the minimal generators of $I_{S(1,2)}$. Now suppose that $d = 2n$ for $n \ge 3$. Then the degrees of $G_{[n,1]}$, $G_{[n,2]}$, $H_n, \ldots, H_{2n-2}$ are at least 3 and the degree of H_{n+j-1} is strictly increasing. Thus it is enough to show that $G_{[n,0]}$, $G_{[n,1]}$ and H_n are $\mathbb{K}$-linearly independent. This comes immediately from the exclusive monomials of each polynomials. Suppose that $d = 2n+1$. By the similar argument, it is enough to show that $n = 2$ and $Q_{[2,1]}$, $Q_{[2,2]}$, $Q_{[2,3]}$ and F_2 are $\mathbb{K}$-linearly independent which follows from the exclusive monomials. (iii) First we can see that $G_{[n,i]}, F_j, H_k$ for all i,j,k are not contained in the ideal $\langle Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}\rangle$ by using the parametrization of $S(1,2)$ in (2). Suppose that $d = 2n$ and consider a combination

$$\begin{aligned}(X_1X_3X_4^{n-2} - X_2^n) = \quad & A_{[n-2,1]}(X_0X_3 - X_1X_2) + A_{[n-2,2]}(X_0X_4 - X_1X_3) \\ & + A_{[n-2,3]}(X_2X_4 - X_3^2) + b(X_1X_4^{n-1} - X_2^{n-1}X_3)\end{aligned} \tag{5}$$

of $Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[n,1]}$ and $G_{[n,2]}$ where $A_{[n-2,i]}$ for $i = 1,2,3$ are the homogeneous polynomials of degree $n-2$ in R and b is a constant. On the other hand, the equality in (5) fails to hold at the points $p = [0,0,1,0,0] \in \mathbb{P}^4$. This guarantees that (iii) holds for the polynomials

$$Q_{[2,1]}, Q_{[2,2]}, Q_{[2,3]}, G_{[n,1]}, \text{ and } G_{[n,2]}.$$

Similarly for $1 \le \ell \le n-1$, consider a combination

$$H_{n+\ell-1} = \sum_{i=1}^{3} A_{[n+\ell-3,i]}Q_{[2,i]} + \sum_{j=1}^{2} B_{[\ell-1,j]}G_{[n,j]} + \sum_{k=1}^{\ell-1} C_{\ell-k}H_{n+k-1} \tag{6}$$

where $A_{[n+\ell-3,i]}$, $B_{[\ell-1,j]}$ and $C_{\ell-k}$ are respectively the homogeneous polynomials of degree $n+\ell-3$, $\ell-1$ and $\ell-k$ in R. Then the combination (6) is written by

$$X_2^{n-\ell} = -B_{[\ell-1,2]}(p)X_2^n + \sum_{k=1}^{\ell-1} C_{\ell-k}(p)X_2^{n-k}$$

on the set of points $\{p = [1,0,X_2,0,0]\} \subset \mathbb{P}^4$ which can not occur. This finishes the proof of (iii) for $d = 2n$. Suppose that $d = 2n+1$ and write

$$F_{n+\ell-1} = \sum_{i=1}^{3} A_{[n+\ell-3,i]}Q_{[2,i]} + \sum_{k=1}^{\ell-1} B_{\ell-k}F_{n+k-1} \tag{7}$$

for $1 \le \ell \le n$ where $A_{[n+\ell-3,i]}$ and $B_{\ell-k}$ are respectively the homogeneous polynomials of degree $n+\ell-3$, $\ell-k$ in R. On the set of points $\{p = [1,0,X_2,0,0]\} \subset \mathbb{P}^4$, (7) is represented as $X_2^{n-\ell+1} = \sum_{k=1}^{\ell-1} B_{\ell-k}(p)X_2^{n-k+1}$ which also can not happen.

For the proof of statement 2, we apply [5, Theorem1.2] to the curve C_d. Indeed C_d is contained in the rational normal surface scroll $S(1,2)$ as a divisor $H+(d-3)F$ by Proposition 3.4.(1). Thus for $\delta = \lceil \frac{d-4}{2} \rceil$, the number of minimal generators of the ideal I_{C_d} is $n+4$ if $d = 2n(\ge 6)$ and $n+3$ if $d = 2n+1(\ge 5)$. This finishes the proof. □

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