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ON AN INVOLUTION ON PARTITIONS WITH CRANK 0

Byungchan Kim

Abstract. Kaavya introduce an involution on the set of partitions with crank 0 and studied the number of partitions of n which are invariant under Kaavya's involution. If a partition λ with crank 0 is invariant under her involution, we say λ is a self- conjugate partition with crank 0. We prove that the number of such partitions of n is equal to the number of partitions with rank 0 which are invariant under the usual partition conjugation. We also study arithmetic properties of such partitions and their q-theoretic implication.

1. Introduction

A partition of a non-negative integer n is a non-increasing sequence of positive integers whose sum is n. We define $p(n)$ to be the number of partitions of n. To explain Ramanujan's famous three partition congruences,

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11},$$

several partition statistics have been introduced. Among them, the partition rank introduced by Dyson [5] and the partition crank introduced by Andrews and Garvan [2] have been extensively studied. While studying partitions with crank 0, Kaavya [7] introduced an involution on the set of partitions with crank 0 (for the exact definition, see Section 2.1). We denote $sc(n)$ to be the number of partitions of n with crank 0, which are invariants under Kaavya's involution. After Kaavya, we call a partition of n counted by $sc(n)$ as a self-conjugate partition with crank 0. Kaavya obtained a generating function

$$\sum_{n\geq 1} sc(n)q^n = (1+q)\sum_{n\geq 1} \frac{q^{n^2+2n}}{(q^2;q^2)_n},$$

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where here and in the sequel, we adopt a standard q-product notation

$$(a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1})$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$. From the combinatorial reason, it is clear that $p(n) \equiv sc(n) \pmod 2$ and this was a motivation to study the function $sc(n)$. In this note, for the convenience, we define $sc(0) = sc(1) = 1$. Then, with this convention, the generating function for $sc(n)$ is now

$$\sum_{n\geq 0} sc(n)q^n = (1+q)\sum_{n\geq 0}\frac{q^{n^2+2n}}{(q^2;q^2)_n}.$$

It is natural to compare this with the number of partitions of n with rank 0, which are invariant under the partition conjugation. We define $sr(n)$ by the number of such partitions of n, i.e. the number of self-conjugate rank 0 partitions. It is straight-forward to find that

$$\sum_{n\geq 0} sr(n)q^n = \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n} = (-q;q^2)_\infty,$$

where we use a convention that $sr(0) = 1$. Our first result is that $sr(n)$ equals to $sc(n)$.

Theorem 1.1. *For all non-negative integers n,*

$$sc(n) = sr(n).$$

In Section 2.1, we will give two proofs. The first proof is based on the generating function manipulation and the other proof is based on the bijection between two partition classes.

We define $M_0(n)$ (resp. $N_0(n)$) to be the number of partitions of n with crank (resp. rank) 0. By the definition, it is immediate that $M_0(n) \geq sc(n)$ for $n \geq 2$ and $N_0(n) \geq sc(n)$ for $n \geq 1$. Using the generating function for $M_0(n)$, $N_0(n)$, and $sc(n)$, we find two curious non-negativities.

Proposition 1.2. *For integers $N > 1$, the N-th coefficients of q-expansions of*

$$\frac{1}{(q;q)_\infty}\left(-1+\sum_{n\geq 1}(-1)^{n-1}q^{n(n-1)/2}(1-q^n+2q^{n(3n+1)/2})\right)$$

and

$$\frac{1}{(q;q)_\infty}\left(-1+\sum_{n\geq 1}(-1)^{n-1}q^{n(3n-1)/2}(1-q^n+2q^{n(n+1)/2})\right)$$

are non-negative.

Without referring combinatorial meaning, the above proposition is not trivial at all. In general, the positivity or non-negativity of the coefficients in q-expansion of the form

$$\frac{1}{(q;q)_\infty} \times \text{ a linear combination of partial theta functions}$$

is not clear at all. We leave proving the non-negativity of two expression via q-series manipulations to interested readers.

It is also natural to ask whether there are more non self-conjugate crank 0 partitions compared to self-conjugate crank 0 partitions. From the combinatorial arguments and careful analysis on generating function, we find the following inequality.

Theorem 1.3. *For all integers $n > 11$,*

$$M_0(n) - sc(n) > sc(n),$$

i.e. there are more non-self-conjugate partitions of n with crank 0 than self-conjugate partitions of n with crank 0 in the partitions of n with crank 0.

Remark 1. Since Chen, Ji, and Zang [4] proved $N_0(n) \geq M_0(n)$, we also see that $N_0(n) - sr(n) > sr(n)$ holds for large enough integers n.

2. Proofs of Results

2.1. Proof of Theorem 1.1

We start with a proof using generating function manipulations.

q-theoretic proof of Theorem 1.1. We find that

$$\begin{aligned}\sum_{n\geq 0} sc(n)q^n &= (1+q)\sum_{n\geq 0}\frac{q^{n^2+2n}}{(q^2;q^2)_n}\\ &= \sum_{n\geq 0}\frac{q^{n^2+2n}}{(q^2;q^2)_n} + \sum_{n\geq 1}\frac{q^{n^2}}{(q^2;q^2)_{n-1}}\\ &= 1+\sum_{n\geq 1}\frac{q^{n^2+2n}+q^{n^2}(1-q^{2n})}{(q^2;q^2)_n}\\ &= \sum_{n\geq 0}\frac{q^{n^2}}{(q^2;q^2)_n}\\ &= \sum_{n\geq 0} sr(n)q^n.\end{aligned}$$

□

Before giving a combinatorial proof of Theorem 1.1, we introduce some notations. A partition λ of n is a non-increasing sequences of which sum equal to

n. We denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. We define $e(\lambda)$ to be the first excess, i.e. $\lambda_1 - \lambda_2$. First we recall the crank $c(\lambda)$ of a partition λ [2] is defined as

$$c(\lambda) := \begin{cases} \lambda_1, & \text{if } r = 0, \\ \omega(\lambda) - r, & \text{if } r \geq 1, \end{cases}$$

where r is the number of 1's in λ, $\omega(\lambda)$ is the number of parts in λ that are strictly larger than r. Therefore, to be $c(\lambda) = 0$, r should be positive. For a partition λ with $c(\lambda) = 0$, there are r parts larger than r. Therefore, there is a Durfee rectangle of size $(r+1) \times r$. We define λ_r to be the partition in the right side of Durfee rectangle after subtracting the first excess $e(\lambda)$ from the largest part. We also define λ_b to be the partition below the Durfee rectangle with parts > 1. Then an involution on the set of partitions of n with crank 0 is defined by switching the conjugations of λ_r and λ_b. We define SC_n is the set of the partitions of n with crank 0, which are invariant under this involution. Then, the number of elements in SC_n is $sc(n)$. We also define SR_n to be the set of partitions of n counted by $sr(n)$. If $\lambda \in SC_n$, we have $\lambda_r = \lambda_b$. Now we are ready to prove Theorem 1.1 combinatorially.

Combinatorial proof of Theorem 1.1. We define a map from SC_n to SR_n as follows.
Case 1. $e(\lambda)$ is even.

We delete r ones from the partition λ and add the part r below Durfee rectangle. Delete $e(\lambda)/2$ from the largest part of λ and add $e(\lambda)/2$ copies of one. Since $\lambda_r = \lambda_b$ and the first excess of the resulting partition and the number of ones are the same, the resulting partition is invariant under the partition conjugation. Therefore, the resulting partition is in SR_n.
Case 2. $e(\lambda)$ is odd.

We delete r ones from the partition λ, subtract 1 from the largest part (for this case $\lambda_1 > \lambda_2$.), and add the part $r+1$ below Durfee rectangle. We subtract $(e(\lambda)-1)/2$ from the largest part, and add the same number of 1s in the bottom. The resulting partition is symmetric with the Durfee square of side $r+1$ and thus is in SR_n. Note that for this case, the largest part of λ_b is strictly smaller than $r+1$.

This map is reversible as we can discriminate the above two cases by looking at whether the partition has the Durfee rectangle of size $(k+1) \times k$ or not. □

2.2. Proof of Proposition 1.2

It is well known that the following equalities hold (see [6] for example)

$$\sum_{n\geq 0} M_0(n) = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(n-1)/2}(1-q^n), \tag{1}$$

$$\sum_{n\geq 0} N_0(n) = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} (-1)^{n-1} q^{n(3n-1)/2}(1-q^n). \tag{2}$$

From Jacobi triple product identity (see [3] for example), we find that

$$\sum_{n\geq 0} sc(n)q^n = \frac{(q;q)_\infty(-q;q^2)_\infty}{(q;q)_\infty} = \frac{1}{(q;q)_\infty}\left(1+2\sum_{n\geq 1}(-1)^n q^{2n^2}\right). \quad (3)$$

By subtracting (3) from (1), we observe that the first q-expansion in Theorem 1.2 is non-negative as $M_0(n) \geq sc(n)$. Moreover, as we can pair two partitions with crank 0 under the involution if it is not fixed by the involution, we find that $M_0(n) - sc(n)$ should be even. The statement on the second q-expansion follows from subtracting (3) from (2).

2.3. Proof of Theorem 1.3

Before starting the proof, we introduce another generating function for $M_0(n)$. For a given partition λ of $N > 1$, if the crank of λ is 0, then there are parts of size 1. This is because if there is no 1 in the partition λ, then $c(\lambda)$ is at least 2. Now assume that there are k copies of 1 in the partition λ, then there must be exactly k parts larger than k. These parts are generated by $\frac{q^{k^2+k}}{(q;q)_k}$ and $(1-q)\frac{q^k}{(q;q)_k}$ generates parts $\leq k$ with exactly k copies of 1. With the usual convention $M_0(0) = 1$ and $M_0(1) = -1$, we arrive at

$$\sum_{n\geq 0} M_0(n)q^n = (1-q)\sum_{k\geq 0}\frac{q^{k^2+2k}}{(q;q)_k^2}.$$

Now we prove Theorem 1.3.

Proof of Theorem 1.3. We first note that

$$\begin{aligned}
&\sum_{n\geq 0}(M_0(n) - 2sc(n))q^n \\
&= (1-q)\sum_{k\geq 0}\frac{q^{k^2+2k}}{(q;q)_k^2} - 2(1+q)\sum_{k\geq 0}\frac{q^{k^2+2k}}{(q^2;q^2)_k} \\
&= -1 - 3q - \frac{q^3}{1-q} + \sum_{k\geq 2} q^{k^2+2k}\left(\frac{1}{(q;q)_k(q^2;q)_{k-1}} - \frac{2}{(1-q)(q^4;q^2)_{k-1}}\right).
\end{aligned}$$

We observe that

$$\frac{1}{(q)_k(q^2;q)_{k-1}} \quad (4)$$

is a generating function for the number of two-color (say blue and red) partitions of n into parts $\leq k$ such that the red color is not available for the part 1. On the other hand,

$$\frac{1}{(1-q)(q^4;q^2)_{k-1}} \quad (5)$$

is a generating function for the number of partitions of n into 1 and even parts between 4 and $2k$. For convenience, we think all parts for this partition are all blue. We define $\mathcal{P}_{k,1}(n)$ (resp. $\mathcal{P}_{k,2}(n)$) is the set of partitions of n generated by (4) (resp. (5)). Now we build an one-to-multiple map from $P_{k,2}(n)$ to $P_{k,1}(n)$ for $n > 1$.

For each partition $\lambda \in \mathcal{P}_{k,2}(n)$ with $n > 1$, we consider two cases; all parts of λ are 1 or there is a part other than 1. For the first case, the partition $1_b + 1_b + \cdots + 1_b \in \mathcal{P}_{k,1}(n)$ and the partition replacing two 1_b by one 2_b is in $\mathcal{P}_{k,1}(n)$. For the second case, there must be an even part $2\ell > 2$ in λ, and for each even part $2\ell_b$, the partition replacing $2\ell_b$ by $\ell_r + \ell_r$, $\ell_r + \ell_b$, or $\ell_b + \ell_b$, we can find that there are at least three partitions in $\mathcal{P}_{k,1}(n)$ corresponding to the partition $\lambda \in \mathcal{P}_{k,2}(n)$. Note that there is no overlap between the first case and the second case as there is only one part of size 2 in the resulting partition in the first case. Moreover, if λ and π are the partitions in $\mathcal{P}_{k,2}(n)$, then the resulting partitions in the above correspondence are all distinct. Moreover, the partitions in $\mathcal{P}_{k,1}(n)$ having exactly one 2_r and the other parts are all 1_b does not corresponds in the above process. Therefore, we can conclude that the n-th coefficient of q-expansion of

$$\frac{1}{(q)_k(q^2;q)_{k-1}} - \frac{2}{(1-q)(q^4;q^2)_{k-1}}$$

is at least the n-th coefficients of q-expansion of

$$-1 - q + \frac{q^2}{1-q}.$$

We define $a(n)$ by

$$-1 - 3q - \frac{q^3}{1-q} + \sum_{k\geq 2} q^{k^2+2k}\left(-1 - q + \frac{q^2}{1-q}\right) =: \sum_{n\geq 0} a(n)q^n,$$

then we find that $a(n)$ is positive for $n > 25$. This is because for $n > 3$

$$a(n) = \begin{cases} |\{k^2 < n : k \geq 3\}| - 2, & \text{if } n \text{ or } n-1 \text{ is a square,} \\ |\{k^2 < n : k \geq 3\}| - 1, & \text{otherwise.} \end{cases}$$

Now by checking $M_0(n) - 2sc(n)$ up to $n = 25$, we can conclude the desirable result. □

3. Concluding remarks

By adopting the same method to obtain a generating function for $M_0(n)$ in Eulerian form, we can also find that for non-negative integers m,

$$\sum_{n\geq 0} M_m(n)q^n = (1-q)\chi(m=0) + \frac{\chi(m>0)q^m}{(q^2)_{m-1}} + (1-q)\sum_{n\geq 1}\frac{q^{(n+1)(n+m)+n}}{(q)_n(q)_{n+m}},$$

where $M_m(n)$ is the number of partitions of n with crank m, and χ (a statement) is defined to be 1 if the statement is true and to be 0 otherwise. From this, we obtain a curious identity:

$$\frac{1}{(q;q)_\infty}\sum_{n\geq 1}(-1)^{n-1}q^{n(n-1)/2+mn}(1-q^n) = (1-q)\chi(m=0) + \frac{\chi(m>0)q^m}{(q^2)_{m-1}} + (1-q)\sum_{n\geq 1}\frac{q^{(n+1)(n+m)+n}}{(q)_n(q)_{n+m}}.$$

A q-theoretic proof for the above identity would be very interesting.

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Byungchan Kim
School of Liberal Arts
Seoul National University of Science and Technology
232 Gongneung-ro, Nowon-gu, Seoul 01811, Korea
E-mail address: bkim4@seoultech.ac.kr