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YAMABE SOLITONS ON KENMOTSU MANIFOLDS

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Abstract. The present paper deals with a study of infinitesimal CL-transformations on Kenmotsu manifolds, whose metric is Yamabe soliton and obtained sufficient conditions for such solitons to be expanding, steady and shrinking. Among others, we find a necessary and sufficient condition of a Yamabe soliton on Kenmotsu manifold with respect to CL-connection to be Yamabe soliton on Kenmotsu manifold with respect to Levi-Civita connection. We found the necessary and sufficient condition for the Yamabe soliton structure to be invariant under Schouten-Van Kampen connection. Finally, we constructed an example of steady Yamabe soliton on 3-dimensional Kenmotsu manifolds with respect to Schouten-Van Kampen connection.

1. Introduction and backgrounds

In [9], Kenmotsu characterized the geometric properties of class (iii) of Tanno's classification [17], which are nowadays called Kenmotsu manifolds. It may be noted that it is the $(0, 1)$ type trans-Sasakian manifolds introduced by Oubiña [12].

It is known that loxodrome is a curve on the unit sphere that intersects the meridians at a fixed angle and C-loxodrome is a loxodrome cutting geodesic trajectories of the characteristic vector field ξ of the Sasakian manifold with constant angle. In 1963, Tashiro and Tachibana [18] introduced a transformation, called CL-transformation, on a Sasakian manifold under which C-loxodrome remains invariant. Here 'CL' stands for C-loxodrome. CL-transformation have been studied by various authors in different context such as Koto and Nagao [10], Takamatsu and Mizusawa [16], Shaikh et al. [14] and many others.

The notion of Yamabe flow was introduced by Hamilton ([6], [7]) as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold (M^n, g), $n \geq 3$. The Yamabe

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flow is an evolution equation for metrics on a Riemannian manifolds as follows:

$$\frac{\partial}{\partial t} g = -rg,$$

where r is the scalar curvature corresponding to g. In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n > 2$, the Yamabe and Ricci flows do not agree as the first one preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms generated by a fixed (time-independent) vector field V on M and homothetic.

A Yamabe soliton on a Riemannian manifold (M, g) is a triplet (g, V, σ) such that

$$\frac{1}{2}\pounds_V g = (r - \sigma)g, \tag{1.1}$$

where $\pounds_V$ denotes the Lie derivative in the direction of the vector field V and σ is a constant. The Yamabe soliton is said to be shrinking, steady and expanding according as $\sigma < 0, = 0$ and > 0 respectively. If σ is a smooth function on M then the metric satisfying (1.1) is called almost Yamabe soliton [1]. It may be noted that Yamabe solitons coincide with the Ricci solitons in dimension $n = 2$ and for $n > 2$, the Ricci solitons and Yamabe solitons have different behaviours. In this connection, it is mentioned that Hui and Chakraborty [8] recently studied infinitesimal CL-transformations on Kenmotsu manifolds whose metric tensor is Ricci soliton.

Motivated by the above studies, the present paper deals with the study of infinitesimal CL-transformations on Kenmotsu manifolds whose metric is Yamabe soliton. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of infinitesimal CL-transformations and Yamabe solitons on Kenmotsu manifolds. It is proved that if (g, V, σ) is a Yamabe soliton on a Kenmotsu manifold M such that V is an infinitesimal CL-transformation, then V is a projective Killing vector field. In [10] Koto and Nagao introduced a new type of an affine connection, called CL-connection. In this section we have studied Yamabe solitons on Kenmotsu manifolds with respect to CL-connection and obtain a necessary and sufficient condition of a Yamabe soliton on Kenmotsu manifold with respect to CL-connection to be a Yamabe soliton on Kenmotsu manifold with respect to Levi-Civita connection. Among others, Yamabe soliton on CL-flat (respectively CL-symmetric and CL-semisymmetric) Kenmotsu manifolds are also investigated.

The Schouten-Van Kampen connection is one of the most natural connections adapted to a pair of complementary distributions on a smooth manifold endowed with an Affine connection [13]. Olszak [11] studied Schouten-Van Kampen connection in an almost contact metric structure. In [20], Yildiz et al. studied 3-dimensional f-Kenmotsu manifolds with respect to Schouten-Van Kampen connection and Chakrabarty et al. [3] recently studied Ricci solitons on 3-dimensional β-Kenmotsu manifolds with respect to Schouten-Van Kampen connection. Wang [19] studied Yamabe solitons on three-dimensional Kenmotsu manifolds. In this connection it may be mentioned that Erken [5] studied Yamabe solitons on three-dimensional normal almost para-contact metric manifolds. In Section 4, we have studied Yamabe soliton on 3-dimensional Kenmotsu manifolds with respect to Schouten-Van Kampen connection. We obtained the necessary and sufficient condition of Yamabe soliton on 3-dimensional Kenmotsu manifold to be invariant under Schouten-Van Kampen connection. We showed that the example which is given by Shukla and Shukla [15] is a steady Yamabe soliton on 3-dimensional Kenmotsu manifold with respect to Schouten-Van Kampen connection.

2. Preliminaries

A smooth manifold (M^n, g) $(n = 2m+1 \geq 3)$ is said to be an almost contact metric manifold [2] if it admits a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following conditions hold [9]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$, the following relations hold [9]:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$S(X, \xi) = -(n-1)\eta(X) \tag{2.7}$$

for any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2) such that $g(QX, Y) = S(X, Y)$.

Definition 2.1. A vector field V on a Kenmotsu manifold M is said to be an infinitesimal CL-transformation ([14], [16]) if it satisfies

$$\pounds_V \left\{ {h \atop ij} \right\} = \rho_j \delta_i^h + \rho_i \delta_j^h + \alpha(\eta_j \phi_i^h + \eta_i \phi_j^h) \tag{2.8}$$

for a certain constant α, where ρ_i are the components of the 1-form ρ, $\pounds_V$ denotes the Lie derivative with respect to V and $\left\{ {h \atop ij} \right\}$ is the Christoffel symbol of the Riemannian metric g.

We recall the followings:

Proposition 2.1 ([14]). *If V is an infinitesimal CL-transformation on a Kenmotsu manifold M, then the 1-form ρ is closed.*

Theorem 2.1 ([14]). *If V is an infinitesimal CL-transformation on a Kenmotsu manifold M, then the relation*

$$(\pounds_V g)(Y,Z) = (\nabla_Y \rho)(Z) - \alpha g(Y, \phi Z) \tag{2.9}$$

holds for any vector fields Y and Z on M.

Definition 2.2 ([10]). A transformation f on an $n(= 2m+1)$-dimensional Kenmotsu manifold M with structure (ϕ, ξ, η, g) is said to be a CL-transformation if the Levi-Civita connection ∇ and a symmetric affine connection ∇^f, called CL-connection, induced from ∇ by f are related by

$$\nabla^f_X Y = \nabla_X Y + \rho(X)Y + \rho(Y)X + \alpha\{\eta(X)\phi Y + \eta(Y)\phi X\}, \tag{2.10}$$

where ρ is a 1-form and α is a constant.

If R and R^f are the curvature tensor with respect to Levi-Civita connection ∇ and CL-connection ∇^f, respectively, in a Kenmotsu manifold, then we have [14]

$$\begin{aligned} R^f(X,Y)Z = R(X,Y)Z &+ B(X,Z)Y - B(Y,Z)X \\ &- \alpha\Big[\{\eta(Y)\phi X - \eta(X)\phi Y\}\eta(Z) \\ &+ \{g(Y,Z)\phi X - g(X,Z)\phi Y\} - \{g(Y,\phi Z)\eta(X) \\ &- g(X,\phi Z)\eta(Y) - 2g(X,\phi Y)\eta(Z)\}\xi\Big] \end{aligned} \tag{2.11}$$

for all vector fields X, Y, Z on M, where the symmetric tensor field B is given by

$$\begin{aligned} B(X,Y) = (\nabla_X \rho)(Y) &- \rho(X)\rho(Y) - \alpha^2\eta(X)\eta(Y) \\ &- \alpha\big[\eta(X)\rho(\phi Y) + \eta(Y)\rho(\phi X)\big]. \end{aligned} \tag{2.12}$$

From (2.11) we get

$$S^f(Y,Z) = S(Y,Z) - (n-1)B(Y,Z), \tag{2.13}$$

where S^f and S are respectively the Ricci tensor of a Kenmotsu manifold with respect to the CL-connection ∇^f and Levi-Civita connection ∇.

Also from (2.13), we get

$$r^f = r - (n-1)Tr.B, \tag{2.14}$$

where r^f is the scalar curvature of Kenmotsu manifold M with respect to the CL-connection ∇^f.

The Schouten-Van Kampen connection $\widetilde{\nabla}$ and Levi-Civita connection ∇ on a Kenmotsu manifold M are related by [13]

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(X,Y)\xi - \eta(Y)X. \tag{2.15}$$

If $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{r}$ are the curvature tensor, Ricci tensor and scalar curvature on a 3-dimensional Kenmotsu manifold M with respect to Schouten-Van Kampen connection then we have

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y, \tag{2.16}$$

$$\widetilde{S}(X,Y) = S(X,Y) + 2g(X,Y), \tag{2.17}$$

$$\widetilde{r} = r + 6 \tag{2.18}$$

for all $X, Y, Z \in \chi(M)$.

3. Infinitesimal CL-transformations and Yamabe solitons

This section deals with the infinitesimal CL-transformations on Kenmotsu manifolds whose metric tensor is Yamabe soliton.

Let us take a Yamabe soliton (g, V, σ) on a Kenmotsu manifold M. Then we get the relation (1.1). From (1.1) and (2.9), we obtain

$$(r - \sigma)g(Y,Z) = \frac{1}{2}(\nabla_Y \rho)(Z) - \frac{\alpha}{2} g(Y, \phi Z). \tag{3.1}$$

Since the metric tensor g is symmetric and the 1-form ρ is closed by Proposition 2.1, so interchanging Y and Z in (3.1) and subtracting the obtained result from (3.1), we get by virtue of (2.2) that $\alpha g(Y, \phi Z) = 0$, which implies that $\alpha = 0$ and hence the infinitesimal CL-transformation V is a projective Killing vector field. Also (3.1) yields

$$(r - \sigma)g(Y,Z) = \frac{1}{2}(\nabla_Y \rho)(Z). \tag{3.2}$$

This leads to the following:

Theorem 3.1. *If (g, V, σ) is a Yamabe soliton on a Kenmotsu manifold M such that V is an infinitesimal CL-transformation, then V is a projective Killing vector field and* (3.2) *holds.*

We now consider a Yamabe soliton (g, V, σ) on a Kenmotsu manifold M with respect to CL-connection ∇^f. Then we have

$$\frac{1}{2}(\pounds^f_V g)(Y,Z) = (r^f - \sigma)g(Y,Z), \tag{3.3}$$

where $\pounds^f_V$ is the Lie derivative along the vector field V on M with respect to CL-connection ∇^f.

By virtue of (2.10) we have

$$\begin{aligned}(\pounds^f_V g)(Y,Z) &= g(\nabla^f_Y V, Z) + g(Y, \nabla^f_Z V)\\ &= g\big(\nabla_Y V + \rho(Y)V + \rho(V)Y + \alpha\{\eta(Y)\phi V + \eta(V)\phi Y\}, Z\big)\\ &\quad + g\big(Y, \nabla_Z V + \rho(Z)V + \rho(V)Z + \alpha\{\eta(Z)\phi V + \eta(V)\phi Z\}\big)\\ &= (\pounds_V g)(Y,Z) + \rho(Y)g(V,Z) + \rho(Z)g(Y,V)\end{aligned} \tag{3.4}$$

$$+ 2\rho(V)g(Y,Z) + \alpha\{\eta(Y)g(\phi V,Z) + \eta(Z)g(Y,\phi V)\}.$$

In view of (2.14) and (3.4), (3.3) yields

$$\begin{aligned} &\frac{1}{2}(\pounds_V g)(Y,Z) - (r-\sigma)g(Y,Z) - (n-1)Tr.B\ g(Y,Z) \\ &- \frac{1}{2}\{\rho(Y)g(V,Z) + \rho(Z)g(Y,V)\} - \rho(V)g(Y,Z) \\ &- \frac{\alpha}{2}\{\eta(Y)g(\phi V,Z) + \eta(Z)g(Y,\phi V)\} = 0. \end{aligned} \tag{3.5}$$

If (g,V,σ) is a Yamabe soliton on a Kenmotsu manifold with respect to Levi-Civita connection then (1.1) holds. Thus from (1.1) and (3.5), we can state the following:

Theorem 3.2. *A Yamabe soliton (g,V,σ) on a Kenmotsu manifold is invariant under CL-connection if and only if the relation*

$$\begin{aligned} &\rho(Y)g(V,Z) + \rho(Z)g(Y,V) + 2\rho(V)g(Y,Z) \\ &+ \alpha\{\eta(Y)g(\phi V,Z) + \eta(Z)g(Y,\phi V)\} + 2(n-1)Tr.B\ g(Y,Z) = 0 \end{aligned}$$

holds for arbitrary vector fields Y and Z.

Now, let (g,ξ,σ) be a Yamabe soliton on a Kenmotsu manifold with respect to CL-connection. Then we have

$$\frac{1}{2}(\pounds^f_\xi g)(Y,Z) = (r^f - \sigma)g(Y,Z). \tag{3.6}$$

From (2.1), (2.2), (2.4) and (2.10), we have

$$\begin{aligned} (\pounds^f_\xi g)(Y,Z) &= g(\nabla^f_Y\xi, Z) + g(Y, \nabla^f_Z\xi) \\ &= g(Y - \eta(Y)\xi + \rho(Y)\xi + \rho(\xi)Y + \alpha\phi Y, Z) \\ &\quad + g(Y, Z - \eta(Z)\xi + \rho(Z)\xi + \rho(\xi)Z + \alpha\phi Z) \\ &= 2[\{1+\rho(\xi)\}g(Y,Z) - \eta(Y)\eta(Z)] + \rho(Y)\eta(Z) + \rho(Z)\eta(Y). \end{aligned} \tag{3.7}$$

Using (2.14) and (3.7) in (3.6), we get

$$\begin{aligned} &\{r - \sigma - 1 - \rho(\xi)\}g(Y,Z) + \eta(Y)\eta(Z) \\ &- (n-1)Tr.B\ g(Y,Z) - \frac{1}{2}\{\rho(Y)\eta(Z) + \rho(Z)\eta(Y)\} = 0. \end{aligned} \tag{3.8}$$

This leads to the following:

Theorem 3.3. *If (g,ξ,σ) is a Yamabe soliton on a Kenmotsu manifold M with respect to CL-connection, then* (3.8) *holds.*

Putting $Y = Z = \xi$ in (3.8) and using (2.1) and (2.2), we get

$$\sigma = r - 2\rho(\xi) - (n-1)Tr.B. \tag{3.9}$$

This leads to the following:

Theorem 3.4. *A Yamabe soliton (g, ξ, σ) on a Kenmotsu manifold M with respect to CL-connection is shrinking, steady and expanding according as $r \lesseqgtr 2\rho(\xi) + (n-1)Tr.B$ respectively.*

Also, Shaikh et al. [14] proved the tensor field

$$
\begin{aligned}
A(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\Big[&\{S(Y,Z)X - S(X,Z)Y\} \\
&- \{g(Y,Z) + \eta(Y)\eta(Z)\}QX + \{g(X,Z) + \eta(X)\eta(Z)\}QY \\
&+ \{S(X,Z) + (n-1)g(X,Z)\}\eta(Y)\xi \\
&- \{S(Y,Z) + (n-1)g(Y,Z)\}\eta(X)\xi \\
&+ 2\{S(X,Y) + (n-1)g(X,Y)\}\eta(Z)\xi\Big] \\
&+ \{g(Y,Z) + \eta(Y)\eta(Z)\}X - \{g(X,Z) + \eta(X)\eta(Z)\}Y
\end{aligned}
$$

is invariant on a Kenmotsu manifold M under a CL-transformation, and it is called the CL-curvature tensor field on M.

Definition 3.1 ([14])**.** A Kenmotsu manifold M is said to be CL-flat if the CL-curvature tensor field A of the type $(1,3)$ vanishes identically on M.

Definition 3.2 ([14])**.** A Kenmotsu manifold M is said to be CL-symmetric if $\nabla A = 0$.

Definition 3.3 ([14])**.** A Kenmotsu manifold M is said to be CL-semisymmetric if the $R(X,Y) \cdot A = 0$.

In [14], Shaikh et al. proved that in a Kenmotsu manifold M, the concept of CL-semisymmetry, CL-symmetry, CL-flatness and manifold of constant curvature -1, i.e., manifold is Einsteinnian are equivalent and its Ricci tensor is of the form

$$S(Y,Z) = -(n-1)g(Y,Z). \tag{3.10}$$

From (3.10), we get

$$r = -n(n-1). \tag{3.11}$$

Again in [4], Debnath and Bhattacharyya studied second order parallel tensor in trans-Sasakian manifolds and as a corollary of their result we have the following:

Theorem 3.5. *In a Kenmotsu manifold M, every second order parallel symmetric tensor is a constant multiple of the metric tensor.*

Suppose that the (0,2) type symmetric tensor field $\pounds_V g - 2rg$ is parallel for any vector field V on a Kenmotsu manifold M. Then Theorem 3.5 yields $\pounds_V g - 2rg$ is a constant multiple of the metric tensor g, i.e., $(\pounds_V g)(X,Y) - 2rg(X,Y) = -2\sigma g(X,Y)$ for all X, Y on M, where σ is a constant. Hence the relation (1.1) holds. This implies that (g, V, σ) yields a Yamabe soliton. Hence we can state the following:

Theorem 3.6. *If the tensor field $\pounds_V g - 2rg$ on a Kenmotsu manifold is parallel for any vector field V, then (g, V, σ) is a Yamabe soliton.*

Let us consider h be a $(0,2)$ symmetric parallel tensor field on a Kenmotsu manifold such that

$$h(X,Y) = (\pounds_\xi g)(X,Y) - 2rg(X,Y). \tag{3.12}$$

From (2.4) we have

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 2[g(X,Y) - \eta(X)\eta(Y)]. \tag{3.13}$$

Using (3.11) and (3.13) in (3.12), we get

$$h(X,Y) = -2(n^2 - n - 1)g(X,Y) - 2\eta(X)\eta(Y). \tag{3.14}$$

Putting $X = Y = \xi$ in (3.14), we obtain

$$h(\xi,\xi) = -2n(n-1). \tag{3.15}$$

If (g, ξ, σ) is a Yamabe soliton on a Kenmotsu manifold M, then from (1.1) we have

$$h(X,Y) = -2\sigma g(X,Y) \tag{3.16}$$

and hence

$$h(\xi,\xi) = -2\sigma. \tag{3.17}$$

From (3.15) and (3.17) we get $\sigma = n(n-1) > 0$ and consequently the Yamabe soliton (g, ξ, σ) is expanding. Thus we can state the following:

Theorem 3.7. *If the tensor field $\pounds_\xi g - 2rg$ on a CL-flat (respectively CL-symmetric, CL-semisymmetric) Kenmotsu manifold is parallel, then the Yamabe soliton (g, ξ, σ) is always expanding.*

4. Schouten-Van Kampen connection and Yamabe solitons

This section deals with the Yamabe solitons on 3-dimensional Kenmotsu manifolds with respect to Schouten-Van Kampen connection $\widetilde{\nabla}$.

Let (g, V, σ) be a Yamabe soliton on a 3-dimensional Kenmotsu manifold M with respect to $\widetilde{\nabla}$. Then we have

$$\frac{1}{2}(\widetilde{\pounds}_V g)(Y,Z) = (\widetilde{r} - \sigma)g(Y,Z), \tag{4.1}$$

where $\widetilde{\pounds}_V$ is the Lie derivative along the vector field V on M with respect to $\widetilde{\nabla}$. By virtue (2.15) we obtain

$$\begin{aligned}(\widetilde{\pounds}_V g)(Y,Z) &= g(\widetilde{\nabla}_Y V, Z) + g(Y, \widetilde{\nabla}_Z V) \\ &= g(\nabla_Y V + g(Y,V)\xi - \eta(V)Y, Z) \\ &\quad + g(Y, \nabla_Z V + g(Z,V)\xi - \eta(V)Z) \\ &= (\pounds_V g)(Y,Z) + g(Y,V)\eta(Z) \\ &\quad + g(Z,V)\eta(Y) - 2\eta(V)g(Y,Z).\end{aligned} \tag{4.2}$$

Using (4.2) and (2.18) in (4.1) we get

$$\frac{1}{2}(\pounds_V g)(Y,Z) = (r-\sigma)g(Y,Z) + \{6+\eta(V)\}g(Y,Z) - \frac{1}{2}\{g(Y,V)\eta(Z) + g(Z,V)\eta(Y)\}. \tag{4.3}$$

So, from (1.1) and (4.3), we can state the following:

Theorem 4.1. *A Yamabe soliton (g, V, σ) on a 3-dimensional Kenmotsu manifold is invariant under Schouten-Van Kampen connection if and only if the relation*

$$\{6+\eta(V)\}g(Y,Z) = \frac{1}{2}\{g(Y,V)\eta(Z) + g(Z,V)\eta(Y)\}$$

holds for arbitrary vector fields Y and Z.

Corollary 4.1. *If (g, ξ, σ) is a Yamabe soliton on a 3-dimensional Kenmotsu manifold with respect to Schouten-Van Kampen connection, then its scalar curvature with respect to Levi-Civita connection is $\sigma - 6$ and that Yamabe soliton (g, ξ, σ) is shrinking, steady and expanding according as $r \lesseqgtr -6$ respectively.*

Proof. From (4.2), we obtain $(\widetilde{\pounds}_\xi g)(Y,Z) = 0$ and using (2.18) in (4.1) we get the requested result. □

Example 4.1. We refer the example of Kenmotsu manifold constructed by Shukla and Shukla [15].

Consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : x \neq 0\}$ and choose the vector fields are $e_1 = x\frac{\partial}{\partial z}$, $e_2 = x\frac{\partial}{\partial y}$, $e_3 = -x\frac{\partial}{\partial z}$, which are linearly independent at each point of M. Let g be the Riemannian metric defined by $g(e_i, e_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$ for $i, j = 1, 2, 3$. The 1-form η is defined by $\eta(Z) = g(Z, e_3)$ for any vector field Z on M and the $(1,1)$ tensor field ϕ is defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$.

For $e_3 = \xi$, and using Koszul's formula, we have

$$\begin{aligned} &\nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1, \quad \nabla_{e_1} e_1 = -e_3, \\ &\nabla_{e_2} e_3 = e_2, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_1 = 0, \\ &\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0. \end{aligned} \tag{4.4}$$

Consequently for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ is a 3-dimensional Kenmotsu manifold with its scalar curvature r is equal to -6 [15].

By virtue of (2.15) and (4.4) we obtain that

$$\widetilde{\nabla}_{e_i} e_j = 0 \quad \text{for} \quad i, j = 1, 2, 3. \tag{4.5}$$

Since $\{e_1, e_2, e_3\}$ form a basis of M, any vector field $Y, Z \in \chi(M)$ can be written as $Y = a_1 e_1 + b_1 e_2 + c_1 e_3$, $Z = a_2 e_1 + b_2 e_2 + c_2 e_3$, where $a_i, b_i, c_i \in \mathbb{R}^+$, $i = 1, 2, 3$. Then for $e_3 = \xi$, we have $\widetilde{\nabla}_Y \xi = 0$ and $\widetilde{\nabla}_Z \xi = 0$ and hence $(\widetilde{\pounds}_\xi g)(Y,Z) = 0$. Also we can calculate that $\widetilde{r} = 0$. Hence the relation (4.1) holds for $V = \xi$

and $\sigma = 0$. Thus it is a steady Yamabe soliton with respect to Schouten-Van Kampen connection with potential vector field $e_3 = \xi$ and Corollary 4.1 is verified.

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References

[1] E. Barbosa and E. Ribeiro, Jr., *On conformal solutions of the Yamabe flow*, Arch. Math. (Basel) **101** (2013), no. 1, 79–89.
[2] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin, 1976.
[3] D. Chakraborty, V. N. Mishra, and S. K. Hui, *Ricci solitons on three dimensional β-Kenmotsu manifolds with respect to Schouten-Van Kampen connection*, J. Ultra Scientist of Physical Sciences **30** (2018), no. 1, 86–91.
[4] S. Debnath and A. Bhattacharyya, *Second order parallel tensor in trans-Sasakian manifolds and connection with Ricci soliton*, Lobachevskii J. Math. **33** (2012), no. 4, 312–316.
[5] K. Erken, *Yamabe solitons on three-dimensional normal almost para-contact metric manifolds*, arXiv: 1708. 04882v2. [math. DG] (2017).
[6] R. S. Hamilton, *The Ricci flow on surfaces*, in Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.
[7] ———, *Lectures on geometric flows*, unpublished manuscript, 1989.
[8] S. K. Hui and D. Chakraborty, *Infinitesimal CL-transformations on Kenmotsu manifolds*, Bangmod Int. J. Math. and Comp. Sci. **3** (2017), 1–9.
[9] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tôhoku Math. J. (2) **24** (1972), 93–103.
[10] S. Koto and M. Nagao, *On an invariant tensor under a CL-transformation*, Kōdai Math. Sem. Rep. **18** (1966), 87–95.
[11] Z. Olszak, *The Schouten-van Kampen affine connection adapted to an almost (para) contact metric structure*, Publ. Inst. Math. (Beograd) (N.S.) **94(108)** (2013), 31–42.
[12] J. A. Oubiña *New classes of almost contact metric structures*, Publ. Math. Debrecen **32** (1985), no. 3-4, 187–193.
[13] J. A. Schouten and E. R. van Kampen, *Zur Einbettungs und Krümmungstheorie nichtholonomer Gebilde*, Math. Ann. **103** (1930), no. 1, 752–783.
[14] A. A. Shaikh, F. R. Al-Solamy, and H. Ahmad, *Some transformations on Kenmotsu manifolds*, SUT J. Math. **49** (2013), no. 2, 109–119.
[15] S. S. Shukla and M. K. Shukla, *On ϕ-Ricci symmetric Kenmotsu manifolds*, Novi Sad J. Math. **39** (2009), no. 2, 89–95.
[16] K. Takamatsu and H. Mizusawa, *On infinitesimal CL-transformations of compact normal contact metric spaces*, Sci. Rep. Niigata Univ. Ser. A No. **3** (1966), 31–39.
[17] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Tôhoku Math. J. (2) **21** (1969), 21–38.
[18] Y. Tashiro and S. Tachibana, *On Fubinian and C-Fubinian manifolds*, Kōdai Math. Sem. Rep. **15** (1963), 176–183.

[19] Y. Wang, *Yamabe solitons on three-dimensional Kenmotsu manifolds*, Bull. Belg. Math. Soc. Simon Stevin **23** (2016), no. 3, 345–355.

[20] A. Yildiz and A. Sazak, *f-Kenmotsu manifolds with the Schouten–van Kampen connection*, Publ. Inst. Math. (Beograd) (N.S.) **102(116)** (2017), 93–105.

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