Commun. Korean Math. Soc. **34** (2019), No. 1, pp. 267–277
https://doi.org/10.4134/CKMS.c170396
pISSN: 1225-1763 / eISSN: 2234-3024

DUAL SURFACES DEFINED BY $z = f(u) + g(v)$ IN SIMPLY ISOTROPIC 3-SPACE $\mathbb{I}_3^1$

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Abstract. In this study, we define the dual surfaces by $z = f(u) + g(v)$ and also classify these surfaces in $\mathbb{I}_3^1$ satisfying some algebraic equations in terms of the coordinate functions and the Laplace operators according to fundamental forms of the surface.

1. Introduction

A surface obtained by translating a curve $\alpha(u)$ over another curve $\beta(v)$ is called a translation surface. A translation surface can be defined as the sum of the two generating curves $\alpha(u)$ and $\beta(v)$. Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures. A translation surface in a Euclidean 3-space $\mathbb{E}^3$ formed by translating two curves lying in orthogonal planes is the graph of a function $z(u, v) = f(u) + g(v)$, where $f(u)$ and $g(v)$ are smooth functions on some interval of $\mathbb{R}$ ([1, 9]).

In 1835, H. F. Scherk studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z(u, v) = f(u) + g(v)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z(u, v) = \frac{1}{a} \log \left| \frac{\cos(au)}{\cos(av)} \right| = \frac{1}{a} \log |\cos(au)| - \frac{1}{a} \log |\cos(av)|,$$

where a is a non-zero constant. These surfaces are now referred as Scherk's minimal surfaces ([21]).

Translation surfaces have been investigated from various viewpoints by many differential geometers. Liu described translation surfaces having constant Gaussian and mean curvature in the Euclidean and Minkowski space ([12]). Goemans proved classification theorems of Weingarten translation surfaces

Received September 22, 2017; Accepted December 21, 2017.
2010 *Mathematics Subject Classification.* Primary 53A35, 53A40.
Key words and phrases. dual surfaces, simply isotropic space, Monge patch, Laplace operator.

([9]). Baba-Hamed, Bekkar and Zoubir studied coordinate finite type translation surfaces in a 3-dimensional Minkowski space ([3]). Yoon classified coordinate finite type translation surfaces in a 3-dimensional Galilean space ([20]). Bekkar and Senoussi researched the translation surfaces in the 3-dimensional space satisfying the equation

$$\Delta^{\mathbf{III}}\mathbf{r}_i = \mu_i \mathbf{r}_i,$$

where $\mathbf{r}_i$ is the coordinate functions of the position vector and the Laplace operator $\Delta^{\mathbf{III}}$ with respect to the third fundamental form, respectively ([4]). Cakmak, Karacan, Kiziltug and Yoon studied the translation surfaces in the 3-dimensional Galilean space satisfying the equation

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

([8]). Sipus described translation surfaces in a simply isotropic space having constant isotropic Gaussian or mean curvature ([17]). Aydin studied the translation surfaces generated by a space curve and a planar curve in the isotropic 3-space $\mathbb{I}_3$ ([2]). Bukcu, Karacan and Yoon classified translation surfaces of Type 1 and Type 2 that satisfy the condition

$$\Delta^{\mathbf{I},\mathbf{II},\mathbf{III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

in $\mathbb{I}_3^1$ ([6, 7, 11]).

In this study, we examine the dual surfaces defined by $z = f(u) + g(v)$ in $\mathbb{I}_3^1$ satisfying the condition $\Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $J = \mathbf{I}, \mathbf{II}, \mathbf{III}$, where $\lambda_i \in \mathbb{R}$ and Δ^J indicates the Laplace operator according to the first, second and third fundamental forms, respectively.

2. Preliminaries

The simply isotropic space $\mathbb{I}_3^1$ is a Cayley–Klein space described from the projective 3-space $\mathcal{P}(\mathbb{R}^3)$ with an absolute figure consisting of a plane w and two complex-conjugate straight lines f_1, f_2 in w. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane w is given by $x_0 = 0$ and the absolute lines f_1, f_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. $\mathbb{F}(0:0:0:1)$ is described as intersection point of these two lines and called as the absolute point. The group of motions of $\mathbb{I}_3^1$ is a six-parameter group given in the affine coordinates $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ by

$$\begin{cases} x' = c_1 + x\cos\alpha - y\sin\alpha, \\ y' = c_2 + x\sin\alpha + y\cos\alpha, \\ z' = c_3 + c_4 x + c_5 y + z, \end{cases} \tag{2.1}$$

where $c_1, c_2, c_3, c_4, c_5, \alpha \in \mathbb{R}$. These affine transformations are called isotropic congruence transformations ([13, 14]). The metric of $\mathbb{I}_3^1$ is given by

$$ds^2 = dx^2 + dy^2.$$

This metric is induced by the absolute figure. The lines parallel to the z-direction are define as isotropic lines. In addition, if the planes containing an isotropic line are define as isotropic planes. Otherwise, they are non-isotropic.

Let $\mathbf{M}$ be a surface immersed in $\mathbb{I}_3^1$. This surface is described as admissible if it has no isotropic tangent planes. In this case, the coefficients E, F, G of the first fundamental form I of $\mathbf{M}$ and the coefficients e, f, g of the second fundamental form II of $\mathbf{M}$ are easily determined according to the induced metric. Hence, the (isotropic) Gaussian curvature K and (isotropic) mean curvature H are described as

$$\mathbf{K} = k_1 k_2 = \frac{eg - f^2}{EG - F^2}, \quad 2\mathbf{H} = k_1 + k_2 = \frac{Eg - 2Ff + Ge}{EG - F^2}, \tag{2.2}$$

where k_1, k_2 are principal curvatures. In other words, k_1, k_2 are extrema of the normal curvature determined by the normal section of a surface. Here, if $K = 0$, the surface M is isotropic flat. If $H = 0$, the surface M is isotropic minimal [2, 15, 17, 18]. The Laplacian operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$, $\Delta^{\mathbf{III}}$ of the I, the II and the III fundamental forms on $\mathbf{M}$ according to local coordinates $\{u, v\}$ of $\mathbf{M}$ are defined by ([3–5, 7, 8, 10, 11, 16])

$$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{|EG - F^2|}}\left[\frac{\partial}{\partial u}\left(\frac{G\mathbf{x}_u - F\mathbf{x}_v}{\sqrt{|EG - F^2|}}\right) - \frac{\partial}{\partial v}\left(\frac{F\mathbf{x}_u - E\mathbf{x}_v}{\sqrt{|EG - F^2|}}\right)\right], \tag{2.3}$$

and

$$\Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{|eg - f^2|}}\left[\frac{\partial}{\partial u}\left(\frac{g\mathbf{x}_u - f\mathbf{x}_v}{\sqrt{|eg - f^2|}}\right) - \frac{\partial}{\partial v}\left(\frac{f\mathbf{x}_u - e\mathbf{x}_v}{\sqrt{|eg - f^2|}}\right)\right], \tag{2.4}$$

$$\Delta^{\mathbf{III}}\mathbf{x} = -\frac{\sqrt{|EG - F^2|}}{eg - f^2}\left[\begin{array}{c} \frac{\partial}{\partial u}\left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{(eg - f^2)\sqrt{|EG - F^2|}}\right) - \\ \frac{\partial}{\partial v}\left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{(eg - f^2)\sqrt{|EG - F^2|}}\right) \end{array}\right], \tag{2.5}$$

where

$$\begin{aligned} X &= Ef^2 - 2Fef + Ge^2, \\ Y &= Efg - Feg + Gef - Ff^2, \\ Z &= Gf^2 - 2Fgf + Eg^2. \end{aligned}$$

3. Curvatures of the dual surfaces defined by $z = f(u) + g(v)$ in $\mathbb{I}_3^1$

In this chapter, we define the dual surfaces defined by $z = f(u) + g(v)$ in the three dimensional simply isotropic space. Consider a surface in $\mathbb{I}_3^1$ as a the graph of a function $z = h(u, v)$ of two variables, which is itself the sum of two functions f and g of one variable. A surface can be defined via the surface patch $z = f(u) + g(v)$.

Here, we restrict our topic to regular surfaces $\mathbf{x}$ without isotropic tangent planes. Thus, we can express in open form as

$$\mathbf{x} : z = h(u,v). \tag{3.1}$$

A surface $\mathbf{x} : z = h(u,v)$ is a set of contact elements. This surface correspond to a surface $\mathbf{x}^*$, given by

$$\begin{cases} x^* = h_u(u,v), \\ y^* = h_v(u,v), \\ z^* = uh_u(u,v) + vh_v(u,v) - h(u,v). \end{cases} \tag{3.2}$$

So, using the equations (3.1) and (3.2), we can define the dual surfaces defined by $z = f(u) + g(v)$ as

$$\mathbf{x}^*(u,v) = \left(f'(u), g'(v), uf'(u) + vg'(v) - f(u) - g(v)\right). \tag{3.3}$$

Let $(\mathbf{M}, \mathbf{M}^*)$ be a dual surface pairs. In this case, the relationship between the curvatures of these surfaces is as follows:

$$\mathbf{K}^* = \frac{1}{\mathbf{K}}, \quad \mathbf{H}^* = \frac{\mathbf{H}}{\mathbf{K}}. \tag{3.4}$$

As it can be seen, if $\mathbf{K} = 0$, $\mathbf{M}^*$ may have singularities. In addition, the dual isotropic minimal surface is also isotropic minimal ([13,14,19]). Using the equation (3.3), the coefficients of the first and the second fundamental forms are given by

$$E = f''^2(u), \; G = g''^2(v), \; F = 0, \tag{3.5}$$

and

$$e = f''(u), \; g = g''(v), \; f = 0, \tag{3.6}$$

respectively. The dual Gaussian curvature $\mathbf{K}^*$ and the mean curvature $\mathbf{H}^*$ of the dual surfaces defined by $z = f(u) + g(v)$ are given by

$$\mathbf{K}^* = \frac{1}{f''(u)g''(v)} \tag{3.7}$$

and

$$\mathbf{H}^* = \frac{f''(u) + g''(v)}{2f''(u)g''(v)}, \tag{3.8}$$

respectively.

Let's assume that the dual surface has the constant Gaussian curvature. Then

$$\frac{1}{f''(u)g''(v)} = A, \tag{3.9}$$

where $A \in \mathbb{R}$. If we use separation of variables method, the Gaussian curvature $\mathbf{K}^* = const. \neq 0$ if and only if

$$\frac{1}{f''(u)} = const. = A_1 \neq 0 \tag{3.10}$$

and

$$\frac{1}{g''(v)} = const. = A_2 \neq 0. \tag{3.11}$$

We can get easily

$$\begin{cases} f(u) = c_1 + uc_2 + \frac{u^2}{2A_1}, \\ g(v) = c_3 + vc_4 + \frac{v^2}{2A_2}, \end{cases} \tag{3.12}$$

where $c_i, A_1, A_2 \in \mathbb{R}$. Thus, we have the following results:

Corollary 3.1. *Let $\mathbf{M}^*$ be the dual surface by $z = f(u)+g(v)$ with the constant Gaussian curvature $\mathbf{K}^* \neq 0$ in $\mathbb{I}_3^1$. Then, z can be written as* (3.12).

Corollary 3.2. *There is no dual surface $\mathbf{M}^*$ defined by $z = f(u) + g(v)$ with the zero Gaussian curvature $\mathbf{K}^* = 0$ (flat) in $\mathbb{I}_3^1$.*

Let's assume that the dual surface has the constant mean curvature, so

$$\frac{f''(u) + g''(v)}{f''(u)g''(v)} = 2C, \tag{3.13}$$

where $C \in \mathbb{R}$. If we use separation of variables method, the mean curvature $\mathbf{H}^* = const. \neq 0$ if and only if

$$\frac{1}{f''(u)} = C_1, \tag{3.14}$$

and

$$\frac{1}{g''(v)} = C_2, \tag{3.15}$$

where $C_1, C_2 \in \mathbb{R}$. Thus, we get

$$\begin{cases} f(u) = c_1 + uc_2 + \frac{u^2}{2C_1}, \\ g(v) = c_3 + vc_4 + \frac{v^2}{2C_2}, \end{cases} \tag{3.16}$$

where $c_i \in \mathbb{R}$.

Theorem 3.3. *Let $\mathbf{M}^*$ be the dual surface by $z = f(u)+g(v)$ with the constant mean curvature $\mathbf{H}^* = C \neq 0$ in $\mathbb{I}_3^1$. Then z can be written as* (3.16).

Suppose that $\mathbf{H}^*$ satisfies the condition $\mathbf{H}^* = 0$. In this case, we define as a surface satisfying that condition dual isotropic minimal. Then, from the equation (3.8) we can write

$$f''(u) + g''(v) = 0, \tag{3.17}$$

where u, v are independent variables and both sides of the equation (3.17) are constant. If we show that this constant is equal to p, we get

$$f''(u) = p = -g''(v). \tag{3.18}$$

Hence, we can write

$$\left\{ \begin{array}{l} f(u) = c_1 + uc_2 + \frac{pu^2}{2}, \\ g(v) = c_3 + vc_4 - \frac{pv^2}{2}, \end{array} \right. \tag{3.19}$$

where $p, c_i \in \mathbb{R}$. Here, if $p = 0$, we obtain

$$\left\{ \begin{array}{l} f(u) = c_1 + uc_2, \\ g(v) = c_3 + vc_4, \end{array} \right. \tag{3.20}$$

where $c_i \in \mathbb{R}$.

Theorem 3.4. *Let* $\mathbf{M}^*$ *be the dual surface by* $z = f(u) + g(v)$ *with zero mean curvature* (*dual isotropic minimal,* $\mathbf{H}^* = 0$) *in* $\mathbb{I}_3^1$. *Then* z *can be written as* (3.19) *or* (3.20).

4. The dual surfaces defined by $z = f(u) + g(v)$ satisfying $\Delta^{\mathbf{I}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*$

In this section, we classify dual surface defined by $z = f(u) + g(v)$ in $\mathbb{I}_3^1$ under the condition

$$\Delta^{\mathbf{I}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*, \tag{4.1}$$

where $\lambda_i \in \mathbb{R}$, i=1, 2, 3 and

$$\Delta^{\mathbf{I}}\mathbf{x}^* = \left(\Delta^{\mathbf{I}}\mathbf{x}_1^*, \Delta^{\mathbf{I}}\mathbf{x}_2^*, \Delta^{\mathbf{I}}\mathbf{x}_3^*\right), \tag{4.2}$$

where

$$\mathbf{x}_1^* = f'(u),\ \mathbf{x}_2^* = g'(v),\ \mathbf{x}_3^* = uf'(u) + vg'(v) - f(u) - g(v). \tag{4.3}$$

From the equations (4.2), (4.3) and (2.7), we obtain

$$\Delta^{\mathbf{I}}\mathbf{x}_i^* = \left(0, 0, \frac{-f''(u) - g''(v)}{f''(u)g''(v)}\right). \tag{4.4}$$

If $\mathbf{M}^*$ satisfies the equation (4.1), from the equations (4.3) and (4.4), we get

$$\frac{-f''(u) - g''(v)}{f''(u)g''(v)} = \lambda \left(uf'(u) + vg'(v) - f(u) - g(v)\right), \tag{4.5}$$

where $\lambda \in \mathbb{R}$. Then, $\mathbf{M}^*$ is of 1-type. In this case, if $\mathbf{M}^*$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x}_i^* = 0$, this surface is defined as a harmonic surface or dual isotropic minimal. As a result of the equation (4.5), we obtain

$$f''(u) + g''(v) = 0. \tag{4.6}$$

Thus we have the solutions of the equations (3.19) and (3.20).

Theorem 4.1. *Suppose that* $\mathbf{M}^*$ *is a dual surface which satisfies the condition* (3.1) *in* $\mathbb{I}_3^1$. *If* $\mathbf{M}^*$ *is harmonic or dual isotropic minimal, then* z *can be written as* (3.19) *or* (3.20).

If $\lambda \neq 0$, from the equation (4.5), we get

$$-\frac{1}{f''(u)} - \lambda u f'(u) + \lambda f(u) = \frac{1}{g''(v)} + \lambda v g'(v) - \lambda g(v), \tag{4.7}$$

which implies there exists a real number p such that

$$-\frac{1}{f''(u)} - \lambda u f'(u) + \lambda f(u) = p = \frac{1}{g''(v)} + \lambda v g'(v) - \lambda g(v). \tag{4.8}$$

The second order nonlinear differential the equation (4.8) can not be solved analytically. If we differentiate both sides of the equation (4.8) according to u and v, we obtain the following:

$$-\lambda u f'' + \frac{f'''}{f''^2} = 0, \tag{4.9}$$

$$\lambda v g'' - \frac{g'''}{g''^2} = 0. \tag{4.10}$$

We deal with two cases with respect to constant λ.

Case 1: If $\lambda > 0$, the general solutions of the equations (4.9) and (4.10) are given by

$$\begin{cases} f(u) = c_1 + uc_2 \pm \dfrac{u \arctan\left(\frac{u\sqrt{\lambda}}{\sqrt{-\lambda u^2 - 2c_3}}\right) + \frac{\sqrt{-\lambda u^2 - 2c_3}}{\sqrt{\lambda}}}{\sqrt{\lambda}}, \\ g(v) = c_3 + vc_4 \pm \dfrac{v \arctan\left(\frac{v\sqrt{\lambda}}{\sqrt{-\lambda v^2 - 2c_5}}\right) + \frac{\sqrt{-\lambda v^2 - 2c_5}}{\sqrt{\lambda}}}{\sqrt{\lambda}}, \end{cases} \tag{4.11}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$.

Case 2: If $\lambda < 0$, general solutions of the equations (4.9) and (4.10) are given by

$$\begin{cases} f(u) = c_1 + uc_2 \pm \dfrac{-\frac{\sqrt{\lambda u^2 - 2c_3}}{\sqrt{\lambda}} + u \log\left(u\lambda + \sqrt{\lambda^2 u^2 - 2\lambda c_3}\right)}{\sqrt{\lambda}}, \\ g(v) = c_3 + vc_4 \pm \dfrac{-\frac{\sqrt{\lambda v^2 - 2c_5}}{\sqrt{\lambda}} + v \log\left(v\lambda + \sqrt{\lambda^2 u^2 - 2\lambda c_5}\right)}{\sqrt{\lambda}}, \end{cases} \tag{4.12}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$.

Theorem 4.2. *Suppose that* $\mathbf{M}^*$ *is a non harmonic dual surface which satisfies the condition* (3.1) *in* $\mathbb{I}_3^1$. *If the surface* $\mathbf{M}^*$ *satisfies the equation* $\Delta^{\mathbf{I}}\mathbf{x}_i^* = \lambda \mathbf{x}_i^*$, *where* $\lambda \in \mathbb{R}$, i=1, 2, 3, *then* $z(u, v)$ *can be written as* (4.11) *or* (4.12).

5. The dual surfaces defined by $z = f(u) + g(v)$ satisfying $\Delta^{\mathbf{II}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*$

In this section, we consider dual surfaces with non-degenerate II fundamental form in $\mathbb{I}_3^1$ under the condition

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*, \tag{5.1}$$

where $\lambda_i \in \mathbb{R}$, i=1, 2, 3 and

$$\Delta^{\mathbf{II}}\mathbf{x}^* = \left(\Delta^{\mathbf{II}}\mathbf{x}_1^*, \Delta^{\mathbf{II}}\mathbf{x}_2^*, \Delta^{\mathbf{II}}\mathbf{x}_3^*\right).$$

If dual surface $\mathbf{M}^*$ is constructed with component functions which are eigenfunctions of its Laplacian operator $\mathbf{\Delta^{II}}$, then we shall have

$$-\frac{f'''}{2f''} = \lambda_1 f' \tag{5.2}$$

$$-\frac{g'''}{2g''} = \lambda_2 g', \tag{5.3}$$

$$-2 - u\frac{f'''}{2f''} - v\frac{g'''}{2g''} = \lambda_3 \left(uf' + vg' - f - g\right), \tag{5.4}$$

where $\lambda_i \in \mathbb{R}$ and $\mathbf{M}^*$ is at least 3-type. From the equations (5.2), (5.3) and (5.4), we can write

$$\lambda_1 uf' - \lambda_3 uf' + \lambda_3 f - 2 = p = -\lambda_2 vg' + \lambda_3 vg' - \lambda_3 g. \tag{5.5}$$

We discuss eight cases according to constants $\lambda_1, \lambda_2, \lambda_3$. We have summarized the solutions of (5.5) of ordinary differential equation in the following table.

No	$\lambda_1, \lambda_2, \lambda_3$	$f(u)$	$g(v)$
1	$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0$	$f(u)$	$-$
2	$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 = 0$	$f(u)$	$c_1 - \frac{p \ln v}{\lambda_2}$
3	$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 \neq 0$	$c_1 u + \frac{2+p}{\lambda_3}$	$c_1 v - \frac{p}{\lambda_3}$
4	$\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	$c_1 u + \frac{2+p}{\lambda_3}$	B
5	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 = 0$	$c_1 + \frac{(2+p)\ln u}{\lambda_1}$	$-$
6	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 = 0$	$c_1 + \frac{(2+p)\ln u}{\lambda_1}$	$c_2 - \frac{p \ln v}{\lambda_2}$
7	$\lambda_1 \neq 0, \lambda_2 = 0, \lambda_3 \neq 0$	A	$c_1 v - \frac{p}{\lambda_3}$
8	$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$	A	B

where $p, c_i \in \mathbb{R}$ and

$$A = c_1 \left(u\lambda_1 - u\lambda_3\right)^{-\frac{\lambda_3}{\lambda_1 - \lambda_3}} + \frac{(2+p)\left(u\left(\lambda_1 - \lambda_3\right)\right)^{\frac{\lambda_3}{\lambda_1 - \lambda_3}} \left(u\lambda_1 - u\lambda_3\right)^{-\frac{\lambda_3}{\lambda_1 - \lambda_3}}}{\lambda_3},$$

$$B = c_2 \left(v\lambda_2 - v\lambda_3\right)^{-\frac{\lambda_3}{\lambda_2 - \lambda_3}} - \frac{p\left(v\left(\lambda_2 - \lambda_3\right)\right)^{\frac{\lambda_3}{\lambda_2 - \lambda_3}} \left(v\lambda_1 - v\lambda_3\right)^{-\frac{\lambda_3}{\lambda_2 - \lambda_3}}}{\lambda_3}.$$

In the first and the second rows of the table above, $f(u)$ can be any second order differentiable function. In the first and the fifth rows, we get contradictions for the function $g(v)$ $(p \neq 0)$. In the third, the fourth and the seventh rows, we have $L = 0$ or $N = 0$. So the second fundamental form in these cases are degenerate, that contradicts with the assumption. Substituting the eighth row into (5.2), (5.3) and (5.4), respectively, we can easily see that they do not satisfy these equations. Substituting the sixth row into (5.2), (5.3) and (5.4) yields $p = -1$ and $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$, respectively. They satisfy these equations. Similarly, if we choose $f(u) = c_1 + \ln u$, $g(v) = c_1 + \ln v$ yields $p = -1$ and $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 0)$ in the second row, then they also satisfy.

Definition. A dual surface in $\mathbb{I}_3^1$ is define as **II**-harmonic under the condition that $\Delta^{\mathbf{II}}\mathbf{x}^*= \mathbf{0}$.

Corollary 5.1. *There is no* **II**-*harmonic dual surface satisfying the equation* $\Delta^{\mathbf{II}}\mathbf{x}^*= \mathbf{0}$ *in* $\mathbb{I}_3^1$.

Theorem 5.2. *Suppose that* $\mathbf{M}^*$ *is a non* **II**-*harmonic dual surface with non-degenerate second fundamental form given by* (3.3) *in* $\mathbb{I}_3^1$. *If* $\mathbf{M}^*$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x}_i^*=\lambda_i\mathbf{x}_i^*$ *for* $\lambda_i\in\mathbb{R}$, $i=1,2,3$, *then* $z(u,v)$ *can be written as*

$$z(u,v) = (c + \ln uv)\,,$$

where $c \in \mathbb{R}$.

6. The dual surfaces defined by $z = f(u) + g(v)$ satisfying $\Delta^{\mathbf{III}}\mathbf{x}_i^*= \lambda_i\mathbf{x}_i^*$

In this section, we consider dual surface with non-degenerate II fundamental form in $\mathbb{I}_3^1$ under the condition

$$\Delta^{\mathbf{III}}\mathbf{x}_i^*= \lambda_i\mathbf{x}_i^*, \tag{6.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{III}}\mathbf{x}^*= \left(\Delta^{\mathbf{III}}\mathbf{x}_1^*,\Delta^{\mathbf{III}}\mathbf{x}_2^*,\Delta^{\mathbf{III}}\mathbf{x}_3^*\right). \tag{6.2}$$

Using the equation (6.2), the Laplacian of $\mathbf{M}^*$ can be expressed as follows

$$\Delta^{\mathbf{III}}\mathbf{x}^* = (-f''', -g''', -f'' - g'' - uf''' - vg''')\,. \tag{6.3}$$

By using the equations (6.1) and (6.3), we have the following equations

$$-f''' = \lambda_1 f', \tag{6.4}$$

$$-g''' = \lambda_2 g', \tag{6.5}$$

$$-f'' - g'' - uf''' - vg''' = \lambda_3\,(uf' + vg' - f - g)\,, \tag{6.6}$$

where λ_1, λ_2 and $\lambda_3 \in \mathbb{R}$. Therefore, $\mathbf{M}^*$ is at least 3-type. Using the equations (6.4), (6.5) and (6.6), we have

$$-f'' + \lambda_1 uf' - \lambda_3 uf' + \lambda_3 f = g'' - \lambda_2 vg' + \lambda_3 vg' + \lambda_3 g, \tag{6.7}$$

where u, v are independent variables and both sides of the equation (6.7) are a constant. If we show this constant with p, we have

$$-f'' + \lambda_1 uf' - \lambda_3 uf' + \lambda_3 f = p = g'' - \lambda_2 vg' + \lambda_3 vg' - \lambda_3 g. \tag{6.8}$$

The differential equations (6.8) cannot be solved analytically, except in some special cases, i.e., $(\lambda_1, \lambda_2, \lambda_3) = (1, 1, 1)$ and $(\lambda_1, \lambda_2, \lambda_3) = (-1, -1, -1)$. Thus, the general solutions of (6.8) are given by

$$\begin{cases} f(u) = p + c_1 e^u + c_2 e^{-u}, \\ g(v) = -p + c_3 e^v + c_3 e^{-v}, \end{cases} \tag{6.9}$$

and

$$\left\{\begin{array}{l} f(u) = -p + c_1 \cos u + c_2 \sin u, \\ g(v) = p + c_1 \cos v + c_2 \sin v, \end{array}\right. \tag{6.10}$$

respectively. The remained cases with respect to λ_1, λ_2 and λ_3 are do not appear. Substituting the solutions (6.9) and (6.10) into (6.6), respectively, They don't satisfy this equation, respectively. Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (6.8), we obtain

$$-f'' = p = g''. \tag{6.11}$$

Hence, the general solutions of (6.8) are given by

$$\left\{\begin{array}{l} f(u) = c_1 + c_2 u - p\frac{u^2}{2}u, \\ g(v) = c_3 + c_4 u + p\frac{u^2}{2}u, \end{array}\right. \tag{6.12}$$

where $p, c_i \neq 0 \in \mathbb{R}$.

Definition. A dual surface in $\mathbb{I}_3^1$ is define as **III**-harmonic under the condition that $\Delta^{\mathbf{III}}\mathbf{x}^* = \mathbf{0}$.

Theorem 6.1. *Suppose that* $\mathbf{M}^*$ *is a dual surface with non-degenerate second fundamental in* $\mathbb{I}_3^1$*. If* $\mathbf{M}^*$ *is* **III***-harmonic, then* $z(u,v)$ *can be written as* (6.12)*.*

Theorem 6.2. *Suppose that* $\mathbf{M}^*$ *is a non* **III***-harmonic dual surface with non-degenerate second fundamental form in* $\mathbb{I}_3^1$*. Then, there is no dual surface* $\mathbf{M}^*$ *satisfying the condition* $\Delta^{\mathbf{III}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*$*, where* $\lambda_i \in \mathbb{R}$*.*

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