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# SOME CONVERGENCE RESULTS FOR GENERALIZED NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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Abstract. The aim of this paper is to study convergence behaviour of Thakur iteration scheme in CAT(0) spaces for generalized nonexpansive mappings. In process, several relevant results of the existing literature are generalized and improved.

## 1. Introduction

Fixed point theory has been gaining much attention among the researchers as it provides useful tools to solve many problems that have application in different fields like engineering, economics, chemistry, game theory etc. Iteration process plays a crucial role in finding fixed points of a nonlinear mapping. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. Mann [24], Ishikawa [17], Halpern [16], Noor [26], Agarwal et al. [3] and Abbas and Nazir [2] are some of the well known and widely utilized iteration processes.

In 2008, Suzuki [32] introduced a new class of mappings which is larger than the class of nonexpansive mappings and named the defining condition as Condition (C) which is also referred as generalized nonexpansive mapping.

Following this, numerous results have been obtained for the class of generalized nonexpansive mappings in various spaces (e.g. [1, 4, 5, 9, 12, 13, 19, 20, 25, 28–30]).

Recently, Thakur et al. [33] introduced a new modified iteration process for finding fixed point of nonexpansive mappings. Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $X$, then the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$(1.1)\qquad \begin{cases} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)Tz_n + \alpha_n T y_n,\ n \in \mathbb{N}, \end{cases}$$

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where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0,1)$. Further, they showed that the new iteration process is faster than the above mentioned iteration processes.

The purpose of this paper is to study the convergence of Thakur iteration process (1.1) for generalized nonexpansive mappings in CAT(0) spaces thereby extending the classes of mappings as well as classes of spaces.

## 2. Preliminaries and lemmas

We begin by recalling some known facts in the existing literature of CAT(0) space.

In a metric space $(X,d)$, a geodesic path joining $x \in X$ and $y \in X$ is a map $c$ from a closed interval $[0,r] \subset \mathbb{R}$ to $X$ such that $c(0) = x,\ c(r) = y$ and $d(c(t),\ c(s)) = |s-t|$ for all $s,\ t \in [0,\ r]$. In particular, the mapping $c$ is an isometry and $d(x,\ y) = r$. The image of geodesic path (joining $x$ and $y$) under $c$ is called a geodesic segment joining $x$ and $y$ which is denoted by $[x,\ y]$ whenever such a segment exists uniquely. For any $x,\ y \in X$, we denote the point $z \in [x,\ y]$ by $z = (1-\alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$ if $d(x,\ z) = \alpha d(x,\ y)$ and $d(z,\ y) = (1-\alpha)d(x,\ y)$. The space $(X,\ d)$ is called a geodesic space if any two points of $X$ are joined by a geodesic, and $X$ is said to be uniquely geodesic if there is exactly one geodesic joining $x$ and $y$ for each $x,\ y \in X$. A subset $C$ of $X$ is called convex if $C$ contains every geodesic segment joining any two points in $C$.

A geodesic triangle $\triangle(x_1,\ x_2,\ x_3)$ in a geodesic metric space $(X,\ d)$ consists of three points of $X$ (as the vertices of $\triangle$) and a geodesic segment between each pair of points (as the edges of $\triangle$). A comparison triangle for $\triangle(x_1,\ x_2,\ x_3)$ in $(X,\ d)$ (denoted by $\overline{\triangle}$) is a triangle $\overline{\triangle}(x_1,\ x_2,\ x_3) := \triangle(\overline{x}_1,\ \overline{x}_2, \overline{x}_3)$ in Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i,\ \overline{x}_j) = d(x_i,\ x_j)$ for $i,\ j \in \{1,2,3\}$. A point $\overline{x} \in [\overline{x}_1,\ \overline{x}_2]$ is said to be comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x}_1, \overline{x})$. The comparison points on $[\overline{x}_2, \overline{x}_3]$ and $[\overline{x}_3, \overline{x}_1]$ are defined in same way.

A geodesic metric space $X$ is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom (CAT(0) inequality):

Let $\triangle$ be a geodesic triangle in $X$ and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,\ y \in \triangle$ and all comparison points $\overline{x},\ \overline{y}\ \in\ \overline{\triangle}$,

$$d(x,\ y) \leq d_{\mathbb{R}^2}(\overline{x},\ \overline{y}).$$

It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature and the complex Hilbert ball with a hyperbolic metric [14] is a CAT(0) space. Other examples include pre-Hilbert spaces, $\mathbb{R}$-trees [6] and Euclidean buildings [7]. For a thorough discussion of these spaces and of the fundamental role they play in geometry, see Bridson and Haefliger [6]. Also, one can refer Burago et al. [8] for more elementary information and Gromov [15] for comparatively deeper study about these spaces. It is worth mentioning that the results in CAT(0) space can be applied to any

CAT($k$) space with $k \leq 0$, since any CAT($k$) space is a CAT($k'$) space for every $k' \geq k$.

Next, we state the following lemmas to be used later on.

**Lemma 2.1** ([11])**.** *Let $(X, d)$ be a CAT(0) space. For $x,\ y \in X$ and $t\ \in\ [0,1]$, there exists a unique $z \in [x,\ y]$ such that*

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point $z$ of the above lemma.

**Lemma 2.2** ([11])**.** *For $x, y, z \in X$ and $t \in [0,1]$ we have*

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x,z) + td(y,z).$$

Now, we collect some basic geometric properties, which are instrumental throughout the discussions.

Let $\{x_n\}$ be a bounded sequence in a complete CAT(0) space $X$. For $x \in X$ write:

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x\ \in X\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

In 2006, Dhompongsa, Kirk and Sims proved that $A(\{x_n\})$ consists of exactly one point if $X$ is a CAT(0) space (Proposition 5 of [10]).

In 2008, Kirk and Panyanak [21] obtained an analogue result of weak convergence in Banach space and restriction of Lim's [23] concept of convergence to CAT(0) spaces which is known as $\Delta$-convergence.

**Definition 2.1** ([21])**.** A sequence $\{x_n\}$ in $X$ is said to be $\Delta$-convergent to $x \in X$ if $x$ is the unique asymptotic center of $u_n$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and read as $x$ is the $\Delta$-limit of $\{x_n\}$.

**Definition 2.2** ([27])**.** A Banach space $X$ is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in $X$ with $x_n \rightharpoonup x$ ($\rightharpoonup$ denotes weak convergence) implies that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in X$ with $y \neq x$.

From the definition of $\Delta$-convergence it can easily be seen that every CAT(0) space satisfies Opial's property.

Now, we list few results which will be frequently used throughout the text.

**Lemma 2.3.** *The following assertions hold in a CAT(0) space*:

(i) [21] *Every bounded sequence in a complete CAT(0) space admits a $\Delta$-convergent subsequence.*

(ii) [11] *If $C$ is a closed convex subset of a complete CAT(0) space $X$ and if $\{x_n\}$ is a bounded sequence in $C$, then the asymptotic center of $\{x_n\}$ is in $C$.*

**Lemma 2.4** ([11]). *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space with $A(\{x_n\}) = \{x\}$, $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

The following lemma is a consequence of Lemma 2.9 of [22] which will be used to prove our main result.

**Lemma 2.5** ([22]). *Let $(X, d)$ be a complete CAT(0) space and $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{u_n\}$, $\{v_n\}$ are sequences in $X$ such that $\limsup_{n\to\infty} d(u_n, x) \le r$, $\limsup_{n\to\infty} d(v_n, x) \le r$ and $\lim_{n\to\infty} d(t_n v_n \oplus (1-t_n)u_n, x) = r$ hold for some $r \ge 0$, then $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

**Definition 2.3** ([10]). A mapping $T$ defined on a subset $K$ of a Banach space $X$ is said to satisfy Condition (C) if

$$\frac{1}{2} d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le d(x, y)$$

for all $x, y \in K$.

Note that every nonexpansive mapping satisfies Condition (C), but the converse is not true (see [18]).

Next, we state the following result for generalized nonexpansive mappings in the setting of CAT(0) space which is very useful to prove our main results.

**Lemma 2.6** ([25]). *Let $C$ be a subset of a CAT(0) space and $T : C \to C$ be a generalized nonexpansive mapping. Then, for all $x, y \in C$ the following holds:*

$$d(x, Ty) \le 3d(x, Tx) + d(x, y).$$

We now modify (1.1) in a CAT(0) space as follows:

Let $C$ be a nonempty closed convex subset of a complete CAT(0) space $X$ and $T : C \to C$ be a mapping. Let $x_1 \in C$ be arbitrary, then the sequence $\{x_n\}$ is generated iteratively by:

$$\begin{cases} z_n &= (1-\gamma_n)x_n \oplus \gamma_n T x_n, \\ y_n &= (1-\beta_n)z_n \oplus \beta_n T z_n, \\ x_{n+1} &= (1-\alpha_n)T z_n \oplus \alpha_n T y_n, \ n \in \mathbb{N}, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $(0, 1)$.

In this paper, we prove $\Delta$ convergence and strong convergence of this iteration process. Our results generalize and extend the corresponding relevant results of Agarwal et al. [3], Khan and Abbas [18] and Thakur et al. [33].

## 3. Some Δ-convergence and strong convergence theorems

Let us begin with the following important lemma.

**Lemma 3.1.** *Let $T : C \to C$ be a generalized nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ such that $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by* (2.1)*, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$.*

*Proof.* For any $p \in F(T)$, we get

$$\frac{1}{2}d(p, Tp) = 0 \leq d(p, x_n).$$

Since $T$ is a generalized nonexpansive mapping, we obtain $d(Tx_n, Tp) \leq d(x_n, p)$. Similarly, we have $d(Ty_n, Tp) \leq d(y_n, p)$ and $d(Tz_n, Tp) \leq d(z_n, p)$.

Now, using Lemma 2.2, we have

$$\begin{aligned} d(z_n, p) &= d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(Tx_n, p) \\ &= (1-\gamma_n)d(x_n, p) + \gamma_n d(Tx_n, Tp) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} d(y_n, p) &= d((1-\beta_n)z_n \oplus \beta_n Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \\ &= (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, Tp) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &= d(x_n, p). \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2), we get

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Tz_n \oplus \alpha_n Ty_n, p) \\ &\leq (1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, p) \\ &= (1-\alpha_n)d(Tz_n, Tp) + \alpha_n d(Ty_n, Tp) \\ &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Thus, $\{d(x_n, p)\}$ is a non-increasing sequence of reals which is bounded below by zero and hence convergent. Therefore, $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. □

**Lemma 3.2.** *Let $T : C \to C$ be a generalized nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ such that $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by* (2.1)*, then $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.*

*Proof.* By Lemma 3.1, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists, say $\lim_{n\to\infty} d(x_n, p) = c$.

From (3.1) and (3.2) we have

$$\limsup_{n\to\infty} d(y_n, p) \le c \tag{3.3}$$

and

$$\limsup_{n\to\infty} d(z_n, p) \le c. \tag{3.4}$$

Since $T$ is a generalized nonexpansive mapping, we get

$$d(Tx_n, p) \le d(x_n, p),\ \ d(Ty_n, p) \le d(y_n, p) \ \text{ and } d(Tz_n, p) \le d(z_n, p)$$

which implies that

$$\limsup_{n\to\infty} d(Tx_n, p) \le c, \tag{3.5}$$

$$\limsup_{n\to\infty} d(Ty_n, p) \le c, \tag{3.6}$$

and

$$\limsup_{n\to\infty} d(Tz_n, p) \le c. \tag{3.7}$$

Now,

$$\begin{aligned} c &= \lim_{n\to\infty} d(x_n, p) \\ &= \lim_{n\to\infty} d(x_{n+1}, p) \\ &= d((1-\alpha_n)Tz_n \oplus \alpha_n Ty_n, p). \end{aligned}$$

So, by using Lemma 2.5, (3.6) and (3.7), we get

$$\lim_{n\to\infty} d(Tz_n, Ty_n) = 0. \tag{3.8}$$

Now,

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\alpha_n)Tz_n \oplus \alpha_n Ty_n, p) \\ &\le (1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, p) \\ &\le (1-\alpha_n)d(Tz_n, p) + \alpha_n d(Ty_n, Tz_n) + \alpha_n d(Tz_n, p) \\ &= d(Tz_n, p) + \alpha_n d(Ty_n, Tz_n) \end{aligned}$$

which on using (3.8) and taking infimum limit both sides yields that

$$c \le \liminf_{n\to\infty} d(Tz_n, p). \tag{3.9}$$

Owing to (3.7) and (3.9), we get

$$\lim_{n\to\infty} d(Tz_n, p) = c.$$

Also, we have

$$\begin{aligned} d(Tz_n, p) &\le d(Tz_n, Ty_n) + d(Ty_n, p) \\ &\le d(Tz_n, Ty_n) + d(y_n, p) \end{aligned}$$

which gives

$$c \leq \liminf_{n\to\infty} d(y_n, p). \tag{3.10}$$

Now, by using (3.3) and (3.10), we obtain

$$\lim_{n\to\infty} d(y_n, p) = c.$$

In view of Lemma 2.5, (3.4) and (3.7), we obtain

$$\lim_{n\to\infty} d(Tz_n, z_n) = 0.$$

Now, consider

$$\begin{aligned} d(y_n, p) &\leq (1-\beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \\ &\leq (1-\beta_n)d(z_n, p) + \beta_n[d(Tz_n, z_n) + d(z_n, p)] \\ &= d(z_n, p) + \beta_n d(Tz_n, z_n) \end{aligned}$$

so that

$$c \leq \limsup_{n\to\infty} d(z_n, p)$$

and hence using (3.4), we get

$$\begin{aligned} c &= \lim_{n\to\infty} d(z_n, p) \\ &= \lim_{n\to\infty} d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p). \end{aligned}$$

Then from Lemma 2.5 and (3.5), we obtain

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0,$$

which proves the result. □

**Theorem 3.1.** *Let $T : C \to C$ be a generalized nonexpansive mapping defined on a nonempty convex closed subset $C$ of a complete $CAT(0)$ space $X$ such that $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by iteration process* (2.1)*, then $\{x_n\}$ $\Delta$-converges to a fixed point of $T$*

*Proof.* From Lemmas 3.1 and 3.2, we have $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$ so that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Let $W_\omega(\{x_n\}) =: \cup A(\{u_n\})$, where union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. In order to show the $\Delta$-convergence of $\{x_n\}$ to a fixed point of $T$, firstly we will prove $W_\omega(\{x_n\}) \subset F(T)$ and thereafter argue that $W_\omega(\{x_n\})$ is a singleton set. To show $W_\omega(\{x_n\}) \subset F(T)$, let $u \in W_\omega(\{x_n\})$. Then, there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = u$. By Lemma 2.3, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v$ and $v \in C$. Since $\lim_{n\to\infty} d(Tx_n, x_n) = 0$ and $\{v_n\}$ is a subsequence of $\{x_n\}$, $\lim_{n\to\infty} d(v_n, Tv_n) = 0$. Since $T$ is a generalized nonexpansive mapping, by Lemma 2.6, we have

$$d(v_n, Tv) \leq 3d(v_n, Tv_n) + d(v_n, v).$$

By taking limsup of both the side, we get

$$\limsup_{n\to\infty} d(v_n, Tv) \le \limsup_{n\to\infty}\{3d(v_n, Tv_n) + d(v_n, v)\}$$
$$\le \limsup_{n\to\infty} d(v_n, v).$$

As $\Delta\text{-}\lim_n v_n = v$, by Opial property, we have

$$\limsup_{n\to\infty} d(v_n, v) \le \limsup_{n\to\infty} d(v_n, Tv).$$

Hence $Tv = v$, i.e., $v \in F(T)$.

Now, by Lemma 3.1, $\lim_{n\to\infty} d(x_n, v)$ exists. By Lemma 2.5, we obtain $u = v$ which shows that $W_\omega(\{x_n\}) \subset F(T)$. Now it is left to show that $W_\omega(\{x_n\})$ consists of single element only. For this, let $\{u_n\}$ be a subsequence of $\{x_n\}$. Again, by using Lemma 2.3, we can find a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_n v_n = v$ and $v \in C$. Let $A(\{u_n\}) = u$ and $A(\{x_n\}) = x$. We have already seen that $u = v$ and $v \in F(T)$. So, it is enough to show that $v = x$. Since $v \in F(T)$, by Lemma 3.1, $\{d(x_n, v)\}$ is convergent. Again, by Lemma 2.4, we have $v = x$ which proves that $W_\omega(\{x_n\})$ is a singleton set. Hence the conclusion follows. □

As a special case of Theorem 3.1, we obtain the following result which is an analogue of Theorem 4.3 of Thakur et al. [33].

**Corollary 3.1.** *Let $T : C \to C$ be a nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ with $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by iteration process* (2.1)*, then $\{x_n\}$ $\Delta$-converges to a fixed point of $T$.*

Now, we prove a strong convergence theorem under some suitable conditions.

**Theorem 3.2.** *Let $T : C \to C$ be a generalized nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ such that $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by* (2.1)*, then $\{x_n\}$ converges to a fixed point of $T$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

*Proof.* If the sequence $\{x_n\}$ converges to a point $p \in F(T)$, then

$$\liminf_{n\to\infty} d(x_n, p) = 0$$

so that

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, assume that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 3.1, we have

$$d(x_{n+1}, p) \le d(x_n, p) \text{ for any } p \in F(T)$$

so we have

$$d(x_{n+1}, F(T)) \le d(x_n, F(T)).$$

Thus, $d(x_n, F(T))$ forms a decreasing sequence which is bounded below by zero as well, so we get that $\lim_{n\to\infty} d(x_n, F(T))$ exists. As, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ so $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now, we prove that $\{x_n\}$ is a Cauchy sequence in $C$. Let $\epsilon$ >0 be arbitrarily chosen. Since $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, there exists $n_0$ such that for all $n \geq n_0$, we have

$$d(x_n, F(T)) < \frac{\epsilon}{4}.$$

In particular,

$$\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\epsilon}{4},$$

so there must exist a $p \in F(T)$ such that

$$d(x_{n_0}, p) < \frac{\epsilon}{2}.$$

Thus, for $m, n \geq n_0$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p) + d(x_n, p) \\ &< 2d(x_{n_0}, p) \\ &< 2\frac{\epsilon}{2} = \epsilon \end{aligned}$$

which shows that $\{x_n\}$ is a Cauchy sequence. Since $C$ is a closed subset of a complete metric space $X$, $C$ itself is a complete metric space and therefore $\{x_n\}$ must converge to some $x$ in $C$. As $\lim_{n\to\infty} d(x_n, F(T)) = 0$ which gives $d(x, F(T)) = 0$. In view of Lemma 2.7, $F(T)$ is closed so $x \in F(T)$. □

The following corollary of Theorem 3.2 provides an analogue of Theorem 4.4 of Thakur et al. [33].

**Corollary 3.2.** *Let $T : C \to C$ be a nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ such that $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by (2.1), then $\{x_n\}$ converges to a fixed point of $T$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.*

We recall (see [31]), mapping $T : C \to C$ is said to satisfy Condition (A) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$.

**Theorem 3.3.** *Let $T : C \to C$ be a generalized nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ with $F(T) \neq \phi$. If $\{x_n\}$ is a sequence defined by (2.1) and $T$ satisfies Condition (A), then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* By Lemma 3.1, $\lim_{n\to\infty} d(x_n, p)$ exists and $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in F(T)$.

We get

$$\inf_{p\in F(T)} d(x_{n+1},p) \leq \inf_{p\in F(T)} d(x_n,p),$$

which yields

$$d(x_{n+1},F(T)) \leq d(x_n,F(T)).$$

This shows that the sequence $\{d(x_n,F(T))\}$ is non-increasing and bounded below, so $\lim_{n\to\infty} d(x_n,F(T))$ exists.

Also, by Lemma 3.2 we have $\lim_{n\to\infty} d(x_n,Tx_n)=0$.

It follows from Condition (A) that

$$\lim_{n\to\infty} f(d(x_n,F(T))) \leq \lim_{n\to\infty} d(x_n,Tx_n)=0,$$

so that $\lim_{n\to\infty} f(d(x_n,F(T)))=0$.

Since $f$ is a non decreasing function satisfying $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$, $\lim_{n\to\infty} d(x_n,F(T))=0$.

By Theorem 3.2, the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. □

Note that Theorem 3.3 sets an analogue of Theorem 4.5 of Thakur et al. [33].

**Corollary 3.3.** *Let $T:C\to C$ be a nonexpansive mapping defined on a nonempty closed convex subset $C$ of a complete $CAT(0)$ space $X$ such that $F(T)\neq\phi$. If $\{x_n\}$ is a sequence defined by* (2.1) *and $T$ satisfies Condition* (A)*, then $\{x_n\}$ converges strongly to a fixed point of $T$.*

Now, we present a numerical example to illustrate the convergence of iteration (2.1) for a mapping which satisfies Condition (C) but is not a nonexpansive mapping.

**Example.** Define a mapping $T:[0,1]\to[0,1]$ by

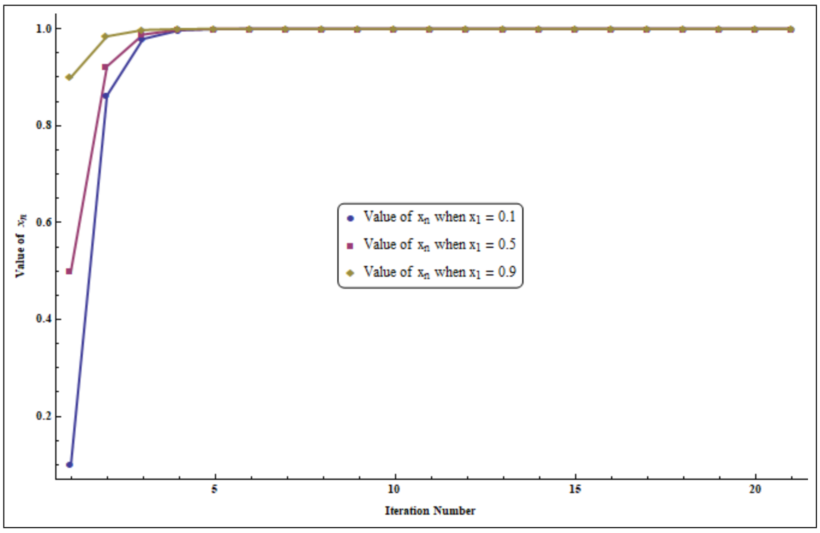
$$Tx=\begin{cases} 1-x, & x\in[0,\frac{1}{2}), \\ \frac{x+4}{5}, & x\in[\frac{1}{2},1]. \end{cases}$$

It was shown in [34] that $T$ is not a nonexpansive mapping but it satisfies Condition (C).

Let $\alpha_n=\sqrt{\frac{n+1}{5n+1}}$, $\beta_n=\frac{1}{\sqrt{2n+9}}$ and $\gamma_n=\frac{2n}{7n+9}$. Then, we obtain the following table of the iteration values for three different initial values:

| Step | When $x_1 = 0.1$ | When $x_1 = 0.5$ | When $x_1 = 0.9$ |
|---|---|---|---|
| 1 | 0.1 | 0.5 | 0.9 |
| 2 | 0.862281939962329 | 0.92253359122881 | 0.984506718245762 |
| 3 | 0.979036062576615 | 0.988207785199346 | 0.997641557039869 |
| 4 | 0.996841803065564 | 0.99822351422438 | 0.999644702844876 |
| 5 | 0.999527074385491 | 0.999733979341839 | 0.999946795868368 |
| 6 | 0.99992943516311 | 0.999960307279249 | 0.99999206145585 |
| 7 | 0.99998949397059 | 0.999994090358457 | 0.999998818071692 |
| 8 | 0.999998437867589 | 0.999999121300519 | 0.999999824260104 |
| 9 | 0.999999767906008 | 0.999999869447129 | 0.999999973889426 |
| 10 | 0.999999965530672 | 0.999999980611003 | 0.999999996122201 |
| 11 | 0.999999994881657 | 0.999999997120932 | 0.999999999424187 |
| 12 | 0.99999999923998 | 0.999999999572489 | 0.999999999914498 |
| 13 | 0.999999999887132 | 0.999999999936512 | 0.999999999987302 |
| 14 | 0.999999999983235 | 0.99999999999057 | 0.999999999998114 |
| 15 | 0.999999999997509 | 0.999999999998599 | 0.99999999999972 |
| 16 | 0.99999999999963 | 0.999999999999792 | 0.999999999999958 |
| 17 | 0.999999999999945 | 0.999999999999969 | 0.999999999999994 |
| 18 | 0.999999999999992 | 0.999999999999996 | 0.999999999999999 |
| 19 | 0.999999999999999 | 0.999999999999999 | 1.000000000000000 |
| 20 | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |

Next, the following graph shows the convergence behaviour of iteration (2.1) for the above example.

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