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ON A CLASS OF BIVARIATE MEANS INCLUDING A LOT OF OLD AND NEW MEANS

Mustapha Raïssouli and Anis Rezgui

Abstract. In this paper we introduce a new formulation of symmetric homogeneous bivariate means that depends on the variation of a given continuous strictly increasing function on $(0, \infty)$. It turns out that this class of means includes a lot of known bivariate means among them the arithmetic mean, the harmonic mean, the geometric mean, the logarithmic mean as well as the first and second Seiffert means. Using this new formulation we introduce a lot of new bivariate means and derive some mean-inequalities.

1. Introduction

In [3], the authors introduced a new bivariate mean Z given by

$$\forall a, b > 0, Z(a, b) = \frac{2(a-b)}{\exp\big(1 - b/a\big) - \exp\big(1 - a/b\big)}, \text{ with } Z(a, a) = a, \tag{1.1}$$

where the notation $\exp(x) := e^x$ refers here to the standard exponential function of $x \in \mathbb{R}$. It is easy to see that the (symmetric homogeneous) mean Z can be included in the class of binary maps having the following form

$$m_f(a, b) = \frac{2(a-b)}{f(a/b) - f(b/a)}, \ a \neq b, \ \ m_f(a, a) = a, \tag{1.2}$$

where $f : (0, \infty) \longrightarrow \mathbb{R}$ is a real function. It is obvious that m_f is always symmetric (in a and b) and homogeneous. The following question arises: under which conditions on f such that $m_f(a, b)$ realizes a bivariate mean, i.e., satisfies $\min(a, b) \leq m_f(a, b) \leq \max(a, b)$ for all $a, b > 0$? As a first obvious necessary condition, f should be strictly increasing for $m_f(a, b)$ to be well-defined and to satisfy $m(a, b) > 0$ for all $a, b > 0$. The requirement $m_f(a, a) = a$ is ensured whenever the following condition

$$\lim_{x \to 1} \frac{2(x-1)}{f(x) - f(1/x)} = 1$$

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TABLE 1.1. Fitted standard means

The intrinsic function f	*The generated mean* $m_f := m_f(a,b),\ a \neq b$
$f(x) = x$ or $f(x) = -1/x$	Harmonic mean: $H = \frac{2ab}{a+b}$
$f(x) = -4/(x+1)$	Arithmetic mean: $A = \frac{a+b}{2}$
$f(x) = 2\sqrt{x}$	Geometric mean: $G = \sqrt{ab}$
$f(x) = \ln x$	Logarithmic mean: $L = \frac{a-b}{\ln a - \ln b}$
$f(x) = 4\arctan\sqrt{x}$	First Seiffert mean: $P = \frac{a-b}{2\arcsin\frac{a-b}{a+b}}$
$f(x) = 2\arctan x$	Second Seiffert mean: $T = \frac{a-b}{2\arctan\frac{a-b}{a+b}}$
$f(x) = 2\sinh^{-1}\frac{x-1}{x+1}$	Neuman-Sàndor mean: $NS = \frac{a-b}{2\sinh^{-1}\frac{a-b}{a+b}}$
$f(x) = -e^{1-x}$	$Z = \frac{2(a-b)}{\exp(1-b/a)-\exp(1-a/b)}$.

holds. If in addition the function f is assumed to be continuously differentiable, then this latter condition is equivalent (by using the standard mean-value theorem) to $f'(1) = 1$.

Actually, the answer to the previous question stems its importance in the fact that (1.2) includes a lot of known bivariate means. (See the table above.)

The remainder of this paper will be organized as follows: after this introduction, Section 2 displays the resolution of a functional equation that will be needed in the sequel. Section 3 is devoted to investigate, in a general context, the necessary and sufficient conditions on f which guarantee that (1.2) defines a bivariate mean. In Section 4 we have pointed out a relatively easy way to derive a lot of new means and their related mean-inequalities.

2. On a functional equation

In the aim to determine all $f : (0,\infty) \longrightarrow \mathbb{R}$ for which (1.2) defines a mean, we need to characterize the general solution of an involved functional equation.

Let $\mathbb{R}^{(0,\infty)}$ be the real vector space of all real functions defined from $(0,\infty)$ into $\mathbb{R}$.

We start this section by stating the following needed lemma.

Lemma 2.1. *Let* $g \in \mathbb{R}^{(0,\infty)}$. *Then the following two assertions are equivalent*:

(i) *For all* $x > 0$, *we have*

$$g(1/x) = -g(x) \quad \big(\textit{resp. } g(1/x) = g(x)\big). \tag{2.1}$$

(ii) *There exists an odd function* (*resp. even function*) $\mu : \mathbb{R} \longrightarrow \mathbb{R}$ *such that*

$$\forall x > 0 \quad g(x) = \mu \circ \ln x.$$

Proof. If there exists an odd function μ such that $g(x) = \mu \circ \ln x$ for all $x > 0$, obviously g satisfies (2.1). Suppose now that g satisfies (2.1). Setting $x = e^t > 0$ for $t \in \mathbb{R}$, (2.1) means that $g(e^{-t}) = -g(e^t)$ for any $t \in \mathbb{R}$. It follows that $\mu := g \circ \exp$ is an odd function, with $g(x) = \big(g \circ \ \exp\big) \circ \ln x := \mu \circ \ln x$ for every $x > 0$. The desired result is obtained, so finishing the proof. □

Now, we need more notations. We set

$$\mathcal{F}_- = \Big\{g = \mu \circ \ln;\ \ \mu : \mathbb{R} \longrightarrow \mathbb{R} \ \text{ is an odd function}\Big\},$$
$$\mathcal{F}_+ = \Big\{h = \nu \circ \ln;\ \ \nu : \mathbb{R} \longrightarrow \mathbb{R} \ \text{ is an even function}\Big\}.$$

With this, the previous lemma tells us that, $g \in \mathbb{R}^{(0,\infty)}$ satisfies $g(1/x) = -g(x)$ $\big($resp. $g(1/x) = g(x)\big)$ for all $x > 0$, if and only if $g \in \mathcal{F}_-$ $\big($resp. $g \in \mathcal{F}_+\big)$. The following proposition summarizes some needed properties of the two previous sets.

Proposition 2.2. *The following assertions are true*:
(i) $\mathcal{F}_-$ *and* $\mathcal{F}_+$ *are both subspaces of* $\mathbb{R}^{(0,\infty)}$.
(ii) $\mathcal{F}_-$ *and* $\mathcal{F}_+$ *are supplementary in* $\mathbb{R}^{(0,\infty)}$, *i.e.*,

$$\mathcal{F}_- \oplus \mathcal{F}_+ = \mathbb{R}^{(0,\infty)}.$$

Proof. It is straightforward. Details are simple and therefore omitted here for the reader. □

Of course, a large number of functions belonging to one of the previous subspaces can be immediately stated here. Some of the most interesting examples can be found in the literature. For instance, we cite the following.

Example 2.1. Let m be a symmetric homogeneous bivariate mean. It is easy to see that the map $g : x \longmapsto \frac{x-1}{m(x,1)}$ belongs to $\mathcal{F}_-$. If moreover $x \longmapsto m(x,1)$ is differentiable on $(0,\infty)$, it is easy to see that the map $h : x \longmapsto x\frac{d}{dx}\Big(\frac{x-1}{m(x,1)}\Big)$ belongs to $\mathcal{F}_+$. As an example, if we take $m = P$, the first Seiffert mean, a simple computation leads to (for all $x > 0$ and $t \in \mathbb{R}$)

$$g(x) = \frac{x-1}{P(x,1)} = 2\arcsin\frac{x-1}{x+1},$$
$$h(x) = xg'(x) = \frac{2\sqrt{x}}{x+1} = \nu \circ \ln(x), \text{ with } \nu(t) = \frac{1}{\cosh(t/2)}.$$

Example 2.2. Two special elements of $\mathcal{F}_+$ are the following celebrated functions

$$\mathcal{K}(x) = \frac{(x+1)^2}{4x}, \quad \mathcal{S}(x) = \frac{x^{\frac{1}{x-1}}}{e\ln\big(x^{\frac{1}{x-1}}\big)}, \text{ with } \mathcal{S}(1) = 1,$$

known in the literature as the Kantorovich constant and Specht ratio, respectively. It is not hard to check that the map

$$x \longmapsto \int_1^x \frac{\mathcal{K}(t)}{t}dt = \frac{1}{4}\frac{x^2-1}{x} + \frac{1}{2}\ln x$$

belongs to $\mathcal{F}_-$. By the same arguments, the function $x \longmapsto \int_1^x \frac{\mathcal{S}(t)}{t}dt$, which seems not be explicitly computable, defines an element of $\mathcal{F}_-$.

The previous functions $\mathcal{K}$ and $\mathcal{S}$ possess nice properties and appear as good tools in the refinement of a lot of mean-inequalities, see [1, 2, 4–6] for instance.

Example 2.3. Let $f : [-1,1] \longrightarrow [-1,1]$ be an even (resp. odd) function. Then the following map

$$x \longmapsto f\Big(\frac{x-1}{x+1}\Big)$$

belongs to $\mathcal{F}_+$ (resp. $\mathcal{F}_-$).

Finally, we end this section by stating another lemma that will be needed in the sequel.

Lemma 2.3. *Let $g \in \mathcal{F}_-$. Then the functional equation in the unknown $f \in \mathbb{R}^{(0,\infty)}$ generated by the following equality*

$$\forall x > 0 \qquad f(x) - f(1/x) = 2g(x) \tag{2.2}$$

has as solutions, functions of the form $f = g + h$ for $h \in \mathcal{F}_+$.

Proof. If $f = g + h$ for $h \in \mathcal{F}_+$ it is easy to check that f is a solution of the functional equation generated by (2.2).

Since the homogeneous functional equation associated to (2.2) is linear, then the set of solutions of (2.2) is expressed as follows: a particular solution of (2.2) $+h$, where h is the general solution of the associated homogeneous functional equation of (2.2). Following Lemma 2.1, the set of solutions of such homogeneous functional equation is $\mathcal{F}_+$. Now, it is easy to verify that if $g \in \mathcal{F}_-$, then g is a particular solution of (2.2). It follows that every solution of (2.2) can be written as $f = g + h$ with $h \in \mathcal{F}_+$. The proof is complete. □

3. On a class of bivariate means

We now are ready to answer the question asked in the introduction: Under which conditions the binary function defined in (1.2) defines a bivariate mean? The following theorem, which answers affirmatively this latter question, is the central result of this section.

Theorem 3.1. *The following assertions hold*:

(i) *Let f be a monotonic function on $(0, +\infty)$. If the binary function m_f defined by* (1.2) *realizes a bivariate mean, then there exists $(g, h) \in \mathcal{F}_- \times \mathcal{F}_+$ such that $f = g + h$, and satisfying, for all $x \geq 1$*

$$1 - \frac{1}{x} \leq g(x) \leq x - 1. \tag{3.1}$$

(ii) *Conversely, for any function* $g \in \mathcal{F}_-$ *that satisfies* (3.1) *and for any* $h \in \mathcal{F}_+$, *the binary function defined as in* (1.2) *with* $f = g + h$ *realizes a bivariate mean.*

Proof. First, if we put $2g(x) = f(x) - f(1/x)$ for $x > 0$, then g satisfies $g(x) = -g(1/x)$ for any $x > 0$. By virtue of the symmetric character of m_f defined by (1.2), we can assume without loss the generality that $0 < a < b$.

(i) Let f be a monotonic function on $(0, +\infty)$ and assume that the binary function m_f is a bivariate mean. Then for $0 < a < b$, the following two inequalities hold:

$$a \leq m_f(a, b) = \frac{2(b-a)}{f(b/a) - f(a/b)} \leq b. \tag{3.2}$$

If we put $x = b/a$, then for every $x > 1$, f satisfies

$$1 \leq \frac{2(x-1)}{f(x) - f(1/x)} \leq x, \tag{3.3}$$

or equivalently

$$\frac{1}{x} \leq \frac{g(x)}{x-1} \leq 1. \tag{3.4}$$

By using Lemma 2.1 the function g should be an element of $\mathcal{F}_-$ and (3.4) turns to

$$1 - \frac{1}{x} \leq g(x) \leq x - 1, \quad x \geq 1.$$

Since $f(x) - f(1/x) = 2g(x)$ for every $x > 0$ then by Lemma 2.3 there exists $h \in \mathcal{F}_+$ such that

$$f = g + h,$$

which finishes the proof of the necessary condition.

(ii) Now suppose that $f = g + h$ for a given $g \in \mathcal{F}_-$ that satisfies (3.1) and $h \in \mathcal{F}_+$. Then it is easy to see that

$$\frac{f(x) - f(1/x)}{2(x-1)} = \frac{g(x)}{x-1},$$

since $g(1/x) = -g(x)$ and $h(1/x) = h(x)$ for any $x > 0$. This finishes the proof of the sufficient condition. □

Denote by $C\big([0, \infty)\big)$ the space of all continuous real functions defined on $[0, \infty)$ and set

$$\mathcal{E} = \Big\{u \in C\big([0, \infty)\big), \quad \forall x \geq 0 \qquad e^{-x} \leq u(x) \leq e^{x}\Big\}.$$

The following result gives the way how to construct symmetric homogeneous bivariate means using the previous theorem.

Corollary 3.2. *Let $u \in \mathcal{E}$ and let μ be the odd function defined on $\mathbb{R}$, with*

$$\forall x \geq 0 \qquad \mu(x) = \int_0^x u(t)dt.$$

Then the function

$$g(x) = \begin{cases} \mu\big(\ln x\big) & \text{for } x \geq 1, \\ -\mu\big(-\ln x\big) & \text{for } 0 < x < 1 \end{cases} \tag{3.5}$$

belongs to $\mathcal{F}_-$, satisfies (3.1) *and so realizes a* (*symmetric homogeneous*) *bivariate mean m_g defined, for $0 < a < b$, as follows*:

$$m_g(a,b) = \frac{2(b-a)}{g(b/a) - g(a/b)} = \frac{b-a}{g(b/a)}. \tag{3.6}$$

Proof. By its definition the function g belongs to $\mathcal{F}_-$. Now, since $u \in \mathcal{E}$ then, it satisfies by definition $e^{-1} \leq u(t) \leq e^t$ for all $t \geq 0$. By the definition of the function g we derive, for all $x \geq 1$

$$1 - \frac{1}{x} = \int_0^{\ln x} e^{-t}dt \leq g(x) = \int_0^{\ln x} u(t)dt \leq \int_0^{\ln x} e^t dt = x - 1,$$

with reversed inequalities if $0 < x \leq 1$. It follows that g satisfies (3.1). We finish the proof by using the previous theorem. □

Remark 3.1. Following the definition of u, μ and g, the symmetric mean m_g expressed in terms of g by (3.6) for $0 < a < b$, can be expressed in terms of μ and u, for all $a, b > 0,\ a \neq b$, by the following

$$m_g(a,b) = \frac{|a-b|}{\mu\big(|\ln a - \ln b|\big)} = \frac{|a-b|}{\displaystyle\int_0^{|\ln a/b|} u(t)dt} := M_u(a,b). \tag{3.7}$$

We now see how we can use the previous results for obtaining certain old bivariate means.

Example 3.1. (i) Consider first the trivial choice $u(x) = e^x$. Then it is easy to check that $g(x) = x - 1$ for $x \geq 1$, and so $g(x) = 1 - 1/x$ for $0 < x < 1$. For $0 < a < b$, we have

$$m_g(a,b) = \frac{b-a}{g(b/a)} = \frac{b-a}{b/a - 1} = a.$$

It follows that the associated (symmetric homogeneous) mean is $M_u(a,b) = \min(a,b)$. Note that if we choose $u(x) = e^{-x}$ we get $M_u(a,b) = \max(a,b)$.

(ii) Let $u(x) = 1$ for $x \geq 0$. It is easy to see that $g(x) = \ln x$ for every $x > 0$. It follows that, for $0 < a < b$, we have

$$m_g(a,b) = \frac{b-a}{g(b/a)} = \frac{b-a}{\ln b - \ln a}.$$

We then deduce $M_u(a,b) = L(a,b)$ the logarithmic mean.

(iii) Take $u(x) = \cosh x$. By the same arguments as previous we obtain $M_u(a,b) = H(a,b)$ the harmonic mean.

(iv) Let $u(x) = 1/\cosh x$. After a simple computation of integral we find $M_u(a,b) = T(a,b)$ the second Seiffert mean.

4. Derived new means and related mean-inequalities

We preserve the same notations as in the above. In the previous section we have pointed out some well known means using our formulation. In the ongoing section we will derive some new means and state some comparison results between them and known means.

We start by stating a result saying that simple comparison between two functions $u, v \in \mathcal{E}$ leads directly to comparison results between derived bivariate means M_u and M_v.

Proposition 4.1. *The following assertions hold*:

(i) *Let $u, v \in \mathcal{E}$ be such that $u(x) \le v(x)$ for all $x \ge 0$, (resp. $u(x) < v(x)$ for any $x > 0$). Then we have*:

$$\forall a,b>0 \quad M_u(a,b) \ge M_v(a,b), \ \ \textit{resp.}\ \forall a,b>0,\ a\neq b,\ M_u(a,b) > M_v(a,b).$$

(ii) *Let $u \in \mathcal{E}$ be such that $u(x) \ge 1$ for all $x \ge 0$, (resp. $u(x) > 1$ for any $x > 0$). Then one has*:

$$\forall a,b>0 \quad M_u(a,b) \le L(a,b) \le M_{1/u}(a,b),$$
$$\textit{resp.}\ \forall a,b>0,\ a\neq b,\ M_u(a,b) < L(a,b) < M_{1/u}(a,b).$$

Proof. (i) It is an immediate consequence of (3.7).

(ii) Follows from (i) with the help of Example 3.1(ii). □

The following result signifies in fact that M_u, given by (3.7), is characterized by $u \in \mathcal{E}$. However m_g, defined by (3.6), is characterized by $g \in \mathcal{F}_-$ modulo an element $h \in \mathcal{F}_+$.

Proposition 4.2. (i) *Let $u, v \in \mathcal{E}$ be such that $M_u = M_v$. Then we have $u = v$.*

(ii) *If $m_{g_1} = m_{g_2}$ for $g_1, g_2 \in \mathcal{F}_-$, then we have $g_1 = g_2 + h$ for some $h \in \mathcal{F}_+$.*

Proof. (i) By virtue of the symmetry and homogeneity of M_u we can assume that $x := a/b > 1$. If $M_u = M_v$, then by (3.7), we have

$$\int_0^{\ln x} u(t)dt = \int_0^{\ln x} v(t)dt$$

for all $x \ge 1$. Differentiating both sides of the above equation we obtain $u(\ln x) = v(\ln x)$ for any $x \ge 1$. Thus $u = v$ by a simple change of variables.

(ii) Assume that $m_{g_1} = m_{g_2}$ for $g_1, g_2 \in \mathcal{F}_-$. By (3.6) we have (with $x = a/b$)

$$g_1(x) - g_1(1/x) = g_2(x) - g_2(1/x) \ \text{ or } \ \big(g_1 - g_2\big)(x) = \big(g_1 - g_2\big)(1/x)$$

for all $x > 0$. According to Lemma 2.1 the desired result follows. □

Now, we state the following proposition which is of interest for deriving new examples of means.

Proposition 4.3. *Let $u \in \mathcal{E}$. Then the following assertions hold*:

(i) *If $u(x) \geq 1$ for all $x \geq 0$, then $1/u \in \mathcal{E}$ and*

$$\forall x \geq 0 \qquad e^{-x} \leq \frac{1}{u(x)} \leq 1 \leq u(x) \leq e^{x}.$$

(ii) *The map $x \longmapsto u(\alpha x) := (u.\alpha)(x)$ belongs to $\mathcal{E}$, for any $|\alpha| \leq 1$.*

(iii) *If u is strictly increasing (resp. decreasing), then we have $u.\alpha < (>) u.\beta$ whenever $\alpha < \beta$ and $|\alpha| \leq 1, |\beta| \leq 1$.*

Proof. It is immediate from the definition of $\mathcal{E}$, with convenient simple manipulations. Details are omitted here. □

The following two examples illustrate the previous results.

Example 4.1. Let $u(x) = \cosh(\alpha x)$, with $|\alpha| \leq 1$. Since $x \longmapsto \cosh x$ is an even function, we can restrict our situation to $0 \leq \alpha \leq 1$. After a simple computation of integral, by using (3.7), the associated symmetric homogeneous mean is given by

$$\forall a, b > 0,\ a \neq b \qquad M_u(a, b) = \frac{2\alpha a^{\alpha} b^{\alpha}(a-b)}{a^{2\alpha} - b^{2\alpha}} := L_{\alpha}(a, b).$$

This parameterized mean includes a lot of old means. In fact, if $\alpha = 1$, then $M_u(a, b) = H(a, b)$ the harmonic mean, and if $\alpha = 1/2$, then $M_u(a, b) = G(a, b)$ the geometric mean. Also, it is easy to see that

$$\lim_{\alpha \to 0} \frac{2\alpha a^{\alpha} b^{\alpha}(a-b)}{a^{2\alpha} - b^{2\alpha}} = \frac{a-b}{\ln a - \ln b} = L(a, b),$$

which corresponds to (ii) of the preceding example.

Example 4.2. Let $u(x) = 1/\cosh(\alpha x)$, with $0 \leq \alpha \leq 1$. By similar arguments as previous, our mean obtained here is given by

$$\forall a, b > 0,\ a \neq b \qquad M_u(a, b) = \frac{2\alpha(a-b)}{4\arctan(b/a)^{\alpha} - \pi} := S_{\alpha}(a, b).$$

Such weighted mean is an extension of certain known means. Indeed, the particular case $\alpha = 1$ corresponds to $M_u(a, b) = T(a, b)$ the second Seiffert mean whereas the case $\alpha = 1/2$ yields $M_u(a, b) = P(a, b)$ the first Seiffert mean. The limit case $\alpha \to 0$ also corresponds to $L(a, b)$.

Remark 4.1. Proposition 4.3 asserts that the map $\alpha \longmapsto L_{\alpha}(a, b)$ is strictly decreasing in $\alpha \in (0, 1],\ a \neq b$. This, when combined with Example 4.1, immediately yields $H(a, b) < G(a, b) < L(a, b)$ for all $a, b > 0,\ a \neq b$. Similar result for $\alpha \longmapsto S_{\alpha}(a, b)$ and, with Example 4.2, we then deduce $L(a, b) < P(a, b) < T(a, b)$. It follows that $H(a, b) < G(a, b) < L(a, b) < P(a, b) < T(a, b)$, which are well-known inequalities obtained here simultaneously and in a fast way. Otherwise, since $\cosh x > 1$ for all $x > 0$ we deduce, by Proposition

4.1, the following inequality $L_\alpha(a,b) < S_\alpha(a,b)$ for any $a,b>0,\ a\neq b$ and $0<\alpha\leq 1$.

Now, for $1\leq p\leq q,\ p,q$ integers, we set

$$\forall x\geq 0 \qquad r_{p,q}[\omega](x) := 1+\sum_{k=p}^{q}\omega_k\frac{x^k}{k!},$$

where the weight $\omega := (\omega_k)_{k=p}^{q}$ is such that $0\leq\omega_k\leq 1$ for each $k=p,p+1,\ldots,q$. For the sake of simplicity, we put

$$\Omega_{p,q}=\Big\{\omega := (\omega_k)_{k=p}^{q};\ \ 0\leq\omega_k\leq 1,\ \forall k=p,p+1,\ldots,q\Big\}.$$

With this, the following result may be stated.

Lemma 4.4. *With the previous notations, the following assertions hold*:

(i) $\Omega_{p,q}\subset\Omega_{p_1,q_1}$ *whenever* $1\leq p\leq p_1\leq q_1\leq q$.

(ii) *We have* $r_{p,q}[\omega]\in\mathcal{E}$ *for any* $1\leq p\leq q$ *and* $\omega\in\Omega_{p,q}$.

(iii) *Let* $x>0$. *Then, the map* $p\longmapsto r_{p,q}[\omega](x)$ *is strictly decreasing whereas* $q\longmapsto r_{p,q}[\omega](x)$ *is strictly increasing. That is,*

$$1\leq p_1<p_2\leq q_2 \Longrightarrow r_{p_1,q}[\omega](x) > r_{p_2,q}[\omega](x)\ \ \textit{for}\ \ \omega\in\Omega_{p_1,q},$$

and

$$1\leq p\leq q_1<q_2 \Longrightarrow r_{p,q_1}[\omega](x) < r_{p,q_2}[\omega](x)\ \ \textit{for}\ \ \omega\in\Omega_{p,q_2}.$$

Proof. (i) It is immediate from the definition of $\Omega_{p,q}$.

(ii) It is clear that, for all $x\geq 0$, we have

$$r_{p,q}[\omega](x)=1+\sum_{k=p}^{q}\omega_k\frac{x^k}{k!}\leq 1+\sum_{k=1}^{q}\frac{x^k}{k!}\leq 1+\sum_{k=1}^{\infty}\frac{x^k}{k!}=e^x.$$

This, with (i), immediately yields the desired result.

(iii) Let $1\leq p_1<p_2$ and $1\leq q_1<q_2$. Using the definition of $r_{p,q}[\omega](x)$, it is easy to see that the two following statements

$$r_{p_1,q}[\omega](x)-r_{p_2,q}[\omega](x)=\sum_{k=p_1}^{p_2-1}\omega_k\frac{x^k}{k!}>0\ \text{ and}$$

$$r_{p,q_2}[\omega](x)-r_{p,q_1}[\omega](x)=\sum_{k=q_1+1}^{q_2}\omega_k\frac{x^k}{k!}>0$$

hold for any $x>0$. The desired result is obtained. □

Now we can state the following results which give birth to a new family of symmetric homogeneous bivariate means.

Proposition 4.5. *Let* p *and* q *be integers such that* $1\leq p\leq q$ *and* $\omega\in\Omega_{p,q}$. *Then the following assertions hold*:

(i) *The following binary map*

$$\forall a,b>0,\ a\neq b, R_{p,q}[\omega](a,b)=\frac{|a-b|}{|\ln a-\ln b|+\displaystyle\sum_{k=p}^{q}\omega_k\frac{\big|\ln a-\ln b\big|^{k+1}}{(k+1)!}}, \tag{4.1}$$

with $R_{p,q}[\omega](a,a)=a$*, realizes a symmetric homogeneous bivariate mean.*

(ii) *For all* $a,b>0,\ a\neq b$*, the map* $p\longmapsto R_{p,q}[\omega](a,b)$ *is strictly increasing whereas* $q\longmapsto R_{p,q}[\omega](a,b)$ *is strictly decreasing.*

Proof. (i) Let $u\in\mathcal{E}$ with $u(x)=r_{p,q}[\omega](x)$ for $x\geq 0$. According to Corollary 3.2, with Remark 3.1, we obtain the desired result after an elementary computation. Details are simple and therefore omitted here.

(ii) It is sufficient to combine Proposition 4.1 with Lemma 4.4(ii). □

Before giving concrete examples we notice the following remarks.

Remark 4.2. (i) If $\omega_k=1$ for any k, we drop ω in the two notations $r_{p,q}[\omega]$ and $R_{p,q}[\omega]$, for the sake of simplicity.

(ii) It is clear that if $\omega_k=1$ for any k, then we have $r_{1,\infty}(x):=\lim_{q\uparrow\infty}r_{1,q}(x)=e^x$ for all $x>0$.

(iii) If we choose $\omega_k=\alpha^k$ if k is even and $\omega_k=0$ if k is odd, for some $0\leq\alpha\leq 1$, then we have $r_{1,\infty}[\omega](x)=\cosh\alpha x$ for any $x>0$.

Remark 4.3. (i) By a simple manipulation, the bivariate mean given by (4.1) can be written as follows

$$R_{p,q}[\omega](a,b)=\frac{L(a,b)}{1+\sum_{k=p}^{q}\omega_k\frac{|\ln a-\ln b|^k}{(k+1)!}}.$$

(ii) If we choose $q=p$ and $\omega_p=1$ in the previous (i), we deduce that the following binary map

$$R_{p,p}(a,b)=\frac{L(a,b)}{1+\frac{|\ln a-\ln b|^p}{(p+1)!}},$$

with $R_{p,p}(a,a)=a$, realizes a symmetric homogeneous bivariate mean for each $p\geq 1$.

As a particular case of the previous proposition we can state the following example.

Example 4.3. Let $u(x)=r_{1,1}(x):=1+x\geq 1$ for $x\geq 0$. According to the previous corollary, the following

$$\forall a,b>0,\ a\neq b\qquad R_{1,1}(a,b)=\frac{2|a-b|}{\Big(|\ln a-\ln b|+1\Big)^2-1}=\frac{2L(a,b)}{2+|\ln a-\ln b|},$$

defines a symmetric homogeneous mean.

More concrete examples of interest may be given. In fact, if we take $u(x) = r_{p,q}[\omega](x) \geq 1$ for $x \geq 0$, Proposition 4.3 tells us that $1/u \in \mathcal{E}$ and so $M_{1/u} := \overline{R}_{p,q}[\omega]$ realizes a symmetric homogeneous mean. With this, it is immediate that for all $a, b > 0,\ a \neq b$, the map $p \longmapsto \overline{R}_{p,q}[\omega](a,b)$ is strictly decreasing while $q \longmapsto \overline{R}_{p,q}[\omega](a,b)$ is strictly increasing.

However, explicit computation of $\overline{R}_{p,q}[\omega](a,b)$ is not always possible, even if $\omega_k = 1$ for any k. It is worth mentioning that, for some particular choices of $u \in \mathcal{E}$, we can compute $\overline{R}_{p,q}$ as mentioned in the following example.

Example 4.4. (i) Let $u(x) = r_{1,1}(x) := 1 + x \geq 1$ for $x \geq 0$. We have $1/u \in \mathcal{E}$ and by simple computation we obtain

$$\forall a, b > 0,\ a \neq b,\ M_{1/u}(a,b) = \frac{|a-b|}{\ln\Big(1 + \big|\ln a - \ln b\big|\Big)} := \overline{R}_{1,1}(a,b).$$

(ii) Let $u(x) = r_{2,2}(x) := 1 + x^2/2 \geq 1$ for $x \geq 0$. By similar way as previous we have $1/u \in \mathcal{E}$ and simple computation leads to

$$\forall a, b > 0,\ a \neq b,\ M_{1/u}(a,b) = \frac{a-b}{\sqrt{2}\arctan\left(\dfrac{\ln a - \ln b}{\sqrt{2}}\right)} := \overline{R}_{2,2}(a,b).$$

(iii) Let $u(x) = r_{1,2}(x) := 1 + x + x^2/2 \geq 1$ for $x \geq 0$. By similar arguments, $1/u \in \mathcal{E}$ and we obtain (after a computation of integral)

$$\forall a, b > 0,\ a \neq b,\ M_{1/u}(a,b) = \frac{|a-b|}{2\arctan\Big(|\ln a - \ln b| + 1\Big) - \pi/2} := \overline{R}_{1,2}(a,b).$$

The previous bivariate means appear to us to be new. The following result concerns mean-inequalities involving these means.

Proposition 4.6. *With the above, the following assertions hold*:

(i) *For any* $1 \leq p \leq q$ *and all* $a, b > 0,\ a \neq b$ *we have*

$$R_{p,q}(a,b) < L(a,b) < \overline{R}_{p,q}(a,b).$$

(ii) *For all* $a, b > 0,\ a \neq b$ *one has*

$$R_{1,1}(a,b) < L(a,b) < \overline{R}_{1,1}(a,b) < \overline{R}_{1,2}(a,b),$$

$$L(a,b) < \overline{R}_{2,2}(a,b) < T(a,b) \ \text{ and } \ L(a,b) < \overline{R}_{2,2}(a,b) < \overline{R}_{1,2}(a,b).$$

Proof. (i) Since $r_{p,q}(x) > 1$ for any $x > 0$ then Proposition 4.1(ii) immediately yields the desired double inequality.

(ii) It can be proved by similar arguments as previous with the help of Example 4.2. Details are simple and therefore omitted. □

Remark 4.4. The two means $\overline{R}_{1,1}$ and $\overline{R}_{2,2}$ are not comparable. To show this, it is in fact sufficient to consider the following function

$$x \longmapsto F(x) = \ln\big(1 + \ln x\big) - \sqrt{2}\arctan\frac{\ln x}{\sqrt{2}}$$

and then study its sign for $x \geq 1$. It is not hard to see that

$$F(e^2) = \ln 3 - \sqrt{2}\arctan\sqrt{2} \approx -76.309... < 0 \text{ and } \lim_{x\to+\infty} F(x) = +\infty.$$

We can then conclude. We left to the reader the task for proving, in a similar way, that the two means T and $\overline{R}_{1,2}$ are not comparable.

In order to give more application and examples of our approach, we need to state the following result.

Proposition 4.7. *Let $u \in \mathcal{E}$. Then we have*:
(i) $u^\alpha \in \mathcal{E}$ *for each* $|\alpha| \leq 1$, *with* $u^\alpha(x) := \big(u(x)\big)^\alpha$ *for* $x \geq 0$.
(ii) $u_t \in \mathcal{E}$ *for every real number* $t > 0$, *where* $u_t(x) := \big(u(tx)\big)^{1/t}$ *for* $x \geq 0$.

Proof. It is straightforward. We left the detail for the reader. □

Now, we are in a position to add more new means itemized in the following examples.

Example 4.5. Taking $u(x) = r_{2,2}(x) := 1+x^2/2$ and $\alpha = -1/2$ in the previous proposition, then $x \longmapsto \big(1 + x^2/2\big)^{-1/2}$ belongs to $\mathcal{E}$. Simple computation of integral yields, for any $x \geq 0$,

$$\mu(x) = \int_0^x \frac{dt}{\sqrt{1+\frac{t^2}{2}}} = \sqrt{2}\sinh^{-1}\frac{x}{\sqrt{2}}.$$

By Corollary 3.2, with Remark 3.1, our mean obtained here is the following

$$(a,b) \longmapsto \frac{a-b}{\sqrt{2}\sinh^{-1}\frac{\ln a - \ln b}{\sqrt{2}}}, \quad a \neq b.$$

Example 4.6. Let $u(x) = r_{1,2}(x) := 1 + x + x^2/2$ and $\alpha = -1/2$. By similar way and arguments as in the previous example, we obtain here the following mean

$$(a,b) \longmapsto \frac{|a-b|}{\sqrt{2}\Big(\sinh^{-1}\big(1+\big|\ln a - \ln b\big|\big) - \sinh^{-1}1\Big)}, \quad a \neq b.$$

We left to the reader the routine task for comparing this mean with that obtained in the previous example.

Example 4.7. Following the previous proposition, for any $u \in \mathcal{E}$ the map $x \longmapsto u_r^\alpha(x) := \big(u(rx)\big)^{\alpha/r}$ belongs to $\mathcal{E}$, for each $r > 0$ and $|\alpha| \leq 1$. If we take $u(x) = r_{1,1}(x) = 1 + x$, the associated family of symmetric homogeneous means $M_{u_r^\alpha} := R_{1,1}^{(r,\alpha)}$ is given by (after an elementary computation of integral)

$$\forall a,b > 0,\ a \neq b \qquad R_{1,1}^{(r,\alpha)}(a,b) = \frac{(r+\alpha)|a-b|}{\Big(1 + r\big|\ln a - \ln b\big|\Big)^{\frac{r+\alpha}{r}} - 1}.$$

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Mustapha Raïssouli
Department of Mathematics
Science Faculty
Taibah University
Al Madinah Al Munawwarah, P.O.Box 30097, Zip Code 41477, Saudi Arabia
and
Department of Mathematics
Science Faculty
Moulay Ismail University
Meknes, Morocco
Email address: raissouli.mustapha@gmail.com

Anis Rezgui
Department of Mathematics
Science Faculty
Taibah University
Al Madinah Al Munawwarah, P.O.Box 30097, Zip Code 41477, Saudi Arabia
and
Mathematics Department
INSAT
Carthage University
Tunis, Tunisia
Email address: anis.rezguii@gmail.com