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ODES RELATED WITH SOME NONLOCAL SCHRÖDINGER EQUATIONS AND ITS APPLICATIONS

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Abstract. We obtain ordinary differential equations related with some nonlocal Schrödinger equations. As applications, we prove finite time blow-up or extinction of solutions.

1. Introduction

We are interested in the following systems of Schrödinger equations with nonlocal terms

$$
\begin{aligned}
&\mathrm{i}\partial_t u + \Delta u + V_1 u = \mathrm{i}\alpha u \left(\int_\Omega (|u|^2 + |v|^2)(x,t)dx \right)^p + f_1(|u|^2,\, |v|^2)u, \\
&\mathrm{i}\partial_t v + \Delta v + V_2 v = \mathrm{i}\beta v \left(\int_\Omega (|u|^2 + |v|^2)(x,t)dx \right)^p + f_2(|u|^2,\, |v|^2)v,
\end{aligned} \tag{1.1}
$$

and

$$
\begin{aligned}
&\mathrm{i}\partial_t u + \Delta u + V_1 u = \mathrm{i}\alpha u \left(\int_0^t \int_\Omega (|u|^2 + |v|^2)(x,s)dxds \right)^p + f_1(|u|^2, |v|^2)u, \\
&\mathrm{i}\partial_t v + \Delta v + V_2 v = \mathrm{i}\beta v \left(\int_0^t \int_\Omega (|u|^2 + |v|^2)(x,s)dxds \right)^p + f_2(|u|^2, |v|^2)v,
\end{aligned} \tag{1.2}
$$

with initial and boundary conditions

$$
\begin{aligned}
&u(x,\,0) = u_0(x), \quad v(x,\,0) = v_0(x) \quad x \in \Omega, \\
&u(x,\,t) = 0, \quad v(x,\,t) = 0 \quad t > 0, \quad x \in \partial\Omega.
\end{aligned}
$$

Here u, v are complex valued functions and V_j are real valued potential functions. The nonlinear terms f_j are real valued polynomials with respect to $|u|^2$ and $|v|^2$. α, β are real constants and p is an integer. Ω is a smoothly bounded

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domain of $\mathbb{R}^n$ or whole space $\mathbb{R}^n$. In the case of $\mathbb{R}^n$, the boundary condition is understood as $\lim_{|x|\to\infty}(u(x, t), v(x, t)) = (0, 0)$.

Some parabolic equations with nonlocal terms have been studied in [2]. The systems (1.1) and (1.2) are Schrödinger versions of them. We are also interested in the following system of Schrödinger equations

$$\begin{aligned} &\mathrm{i}\partial_t u + \Delta u + V_1 u = \mathrm{i}\alpha u \left(\int_{-\infty}^{x} (|u|^2 + |v|^2)(y, t) dy \right)^p + f_1(|u|^2, |v|^2)u, \\ &\mathrm{i}\partial_t v + \Delta v + V_2 v = \mathrm{i}\beta v \left(\int_{-\infty}^{x} (|u|^2 + |v|^2)(y, t) dy \right)^p + f_2(|u|^2, |v|^2)v, \end{aligned} \tag{1.3}$$

where u, v are complex valued functions defined on $\mathbb{R}^{1+1}$. While integrals in (1.1), (1.2) are functions of t, the integral in (1.3) is a function of x and t. The similar equations have been studied in [3].

We will derive some precise formulas for behaviors of L^2 norm of nonlinear nonlocal Schrödinger equations (1.1), (1.2) and (1.3) as long as solutions exist. From now on, we assume that $V_i = V_i(x)$ is a given smooth real-valued potential function satisfying

$$\sum_{j=0}^{m} \|\nabla^j V_i\|_{L^\infty} \le C_m < \infty \quad \text{for a positive integer } m > n/2.$$

Then it can be proved that the equations (1.1), (1.2) and (1.3) admit a unique local solution u, v in time interval $[0, T)$.

$$u,\, v \in C([0,T); W^{m,2}(\Omega)).$$

Therefore, the equations we will derive are valid in the time interval $[0,T)$ where solutions exist. Let us define functions

$$x(t) = \int_\Omega (|u|^2 + |v|^2)(x, t)dx \quad \text{and} \quad y(t) = \int_\Omega (|u|^2 - |v|^2)(x, t)dx.$$

Our first result is concerned with the system (1.1).

Theorem 1.1. *As long as the solutions u, v of* (1.1) *exist, we have the following ODEs.*

(i) *For the case $\beta = \alpha$, we have*

$$\frac{dx}{dt} = 2\alpha x^{p+1}.$$

(ii) *For the case $\beta = -\alpha$, we have*

$$\frac{dy}{dt} = 2\alpha(y^2 + c^2)^{\frac{1+p}{2}} \quad \textit{and} \quad x^2 = y^2 + c^2,$$

where $c^2 = 4\|u_0\|_{L^2}^2 \|v_0\|_{L^2}^2$.

As applications of the ODEs, we show finite time blow-up or extinction of solutions in Corollary 2.1. Our second result is concerned with the system (1.2). Let us define $X(t) = \int_0^t x(s)\, ds$.

Theorem 1.2. *Let p be a nonnegative integer. As long as the solutions u, v of* (1.2) *exist, we have the following equations.*

(i) *For the case $\beta = \alpha$, we have*

$$\frac{dX}{dt} = \frac{2\alpha}{p+1} X^{p+1}(t) + \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2.$$

(ii) *For the case $\beta = -\alpha$, we have*

$$y(t) = y(0) + \frac{2\alpha}{p+1}\left(\int_0^t \sqrt{y^2(s) + c^2}\, ds\right)^{p+1} \quad \text{and} \quad x^2 = y^2 + c^2,$$

where $c^2 = 4\|u_0\|_{L^2}^2 \|v_0\|_{L^2}^2$.

As mentioned above, the equation (1.3) is different from (1.1) in that the integral in (1.3) is a function of x and t.

Theorem 1.3. *As long as the solutions u, v of* (1.3) *exist, we have the following ODE.*

(i) *For the case $\beta = \alpha$, we have*

$$\frac{dx}{dt} = \frac{2\alpha}{p+1} x^{p+1}.$$

(ii) *For the case of $\alpha > 0$ and $p > 0$, we have*

$$\frac{d}{dt}\int_{\mathbb{R}} |u(x,t)|^2\, dx \geq \frac{2\alpha}{p+1}\left(\int_{\mathbb{R}} |u(x,t)|^2\, dx\right)^{p+1}.$$

As applications of Theorems 1.2 and 1.3, we can show several behaviors of solutions to (1.2) and (1.3). We prove Theorem 1.1 in Section 2. Theorems 1.2 and 1.3 are proved in Sections 3 and 4 respectively.

2. Proof of Theorem 1.1 and its applications

Multiplying (1.1) by $\bar{u}$, $\bar{v}$ respectively and taking imaginary parts of them, we obtain

$$\frac{\partial}{\partial t}|u(x,\, t)|^2 + \mathrm{i}(u\Delta\bar{u} - \bar{u}\Delta u) = 2\alpha|u(x,\, t)|^2 \left(\int_\Omega |u(y,\, t)|^2 + |v(y,\, t)|^2 dy\right)^p,$$

$$\frac{\partial}{\partial t}|v(x,\, t)|^2 + \mathrm{i}(v\Delta\bar{v} - \bar{v}\Delta v) = 2\beta|v(x,\, t)|^2 \left(\int_\Omega |u(y,\, t)|^2 + |v(y,\, t)|^2 dy\right)^p.$$

Integrating by parts and considering that $\int_\Omega |u(y,\, t)|^2 + |v(y,\, t)|^2 dy$ is a function of t, we can derive

$$\begin{aligned} \frac{d}{dt}\int_\Omega |u(x,\, t)|^2 dx &= 2\alpha \int_\Omega |u(x,\, t)|^2 dx \left(\int_\Omega |u(y,\, t)|^2 + |v(y,\, t)|^2 dy\right)^p, \\ \frac{d}{dt}\int_\Omega |v(x,\, t)|^2 dx &= 2\beta \int_\Omega |v(x,\, t)|^2 dx \left(\int_\Omega |u(y,\, t)|^2 + |v(y,\, t)|^2 dy\right)^p. \end{aligned} \tag{2.1}$$

When $\alpha = \beta$, we have from (2.1)

$$\frac{dx}{dt} = 2\alpha x^{1+p}. \tag{2.2}$$

When $p = 0$, we have $x(t) = x(0)e^{2\alpha t}$. For the case of $p \neq 0$, the ODE (2.2) leads us to

$$x^p(t) = \frac{x^p(0)}{1 - 2\alpha p x^p(0)t}. \tag{2.3}$$

When $q := -p > 0$, (2.3) can be rewritten as

$$x^q(t) = x^q(0) + 2\alpha q t.$$

As applications of (2.3), we can derive several behaviors of solutions like finite time blow-up or extinction.

Corollary 2.1. (1) *For $p > 0$ and $\alpha > 0$, we have a finite blow-up $x(t) \to \infty$ as $t \to 1/2\alpha p x^p(0)$.*

(2) *For $p > 0$ and $\alpha < 0$, we have $x(t) \to 0$ as $t \to \infty$.*

(3) *For $q := -p > 0$ and $\alpha > 0$, we have $x(t) \to \infty$ as $t \to \infty$.*

(4) *For $q := -p > 0$ and $\alpha < 0$, we have a finite time extinction* [1], *that is, the solution may become identically zero after some positive time. In fact, we can check $x(t) \to 0$ as $t \to -x^q(0)/2\alpha q$.*

When $\beta = -\alpha$, the equations (2.1) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= 2\alpha y x^p, \\ \frac{dy}{dt} &= 2\alpha x^{1+p}. \end{aligned} \tag{2.4}$$

From (2.4), we can derive

$$\frac{d}{dt}(x^2 - y^2)(t) = 0.$$

Considering $(x^2 - y^2)(t) = 4\|u(t)\|_{L^2}^2 \|v(t)\|_{L^2}^2$, we have

$$4\|u(t)\|_{L^2}^2 \|v(t)\|_{L^2}^2 = 4\|u_0\|_{L^2}^2 \|v_0\|_{L^2}^2 := c^2.$$

Then we obtain an ODE

$$\frac{dy}{dt} = 2\alpha(y^2 + c^2)^{\frac{1+p}{2}} \quad \text{or} \quad \frac{dx}{dt} = \pm 2\alpha\sqrt{x^2 - c^2}\,x^p.$$

When $p = 1$ in the above equation, we have an explicit solution

$$y(t) = c\,\frac{y(0) + c\tan(2\alpha c t)}{c - y(0)\tan(2\alpha c t)}.$$

3. Proof of Theorem 1.2 and its applications

In this section we assume that p is a nonnegative integer. For the equations (1.2), we can derive

(3.1)
$$\frac{d}{dt}\int_\Omega |u(x,\,t)|^2 dx = 2\alpha \int_\Omega |u(x,\,t)|^2 dx \left(\int_0^t \int_\Omega |u(y,\,s)|^2 + |u(y,\,s)|^2 dy\,ds\right)^p,$$
$$\frac{d}{dt}\int_\Omega |v(x,\,t)|^2 dx = 2\beta \int_\Omega |v(x,\,t)|^2 dx \left(\int_0^t \int_\Omega |u(y,\,s)|^2 + |u(y,\,s)|^2 dy\,ds\right)^p.$$

When $\alpha = \beta$, we have

$$x'(t) = 2\alpha x(t)\left(\int_0^t x(s)ds\right)^p. \tag{3.2}$$

Let $X(t) = \int_0^t x(s)ds$. Then (3.2) can be rewritten as

$$X''(t) = 2\alpha X^p(t)X'(t) = \left(\frac{2\alpha}{p+1}X^{p+1}\right)'.$$

Considering $X(0) = \int_0^0 x(s)ds = 0$ and $X'(0) = x(0) = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2$, we derive

$$X'(t) = \frac{2\alpha}{p+1}X^{p+1}(t) + x(0). \tag{3.3}$$

As applications of the ODE (3.3), we consider the following two cases.

(i) For $p > 0$ and $\alpha > 0$, we have finite time blow up.

(ii) For $p = 1$ and $\alpha = -\frac{p+1}{2}$, we have

$$X(t) = \sqrt{x(0)}\frac{1 - e^{1-2\sqrt{x(0)}\,t}}{1 + e^{1-2\sqrt{x(0)}\,t}}$$

which implies

$$x(t) = 4x(0)\frac{e^{1-2\sqrt{x(0)}\,t}}{(1 + e^{1-2\sqrt{x(0)}\,t})^2},$$

from which we have $x(t) \to 0$ as $t \to \infty$.

When $\beta = -\alpha$, we have from (3.1)

$$\begin{aligned} x'(t) &= 2\alpha y(t)\left(\int_0^t x(s)ds\right)^p, \\ y'(t) &= 2\alpha x(t)\left(\int_0^t x(s)ds\right)^p. \end{aligned} \tag{3.4}$$

From (3.4), we derive $\frac{d}{dt}(x^2 - y^2) = 0$ which implies $x^2 = y^2 + c^2$, where $c^2 = 4\|u_0\|_{L^2}^2\|v_0\|_{L^2}^2$. Plugging $x = \sqrt{y^2 + c^2}$ in the second equation of (3.4), we obtain an ODE

$$\frac{dy}{dt} = 2\alpha\sqrt{y^2(t)+c^2}\left(\int_0^t \sqrt{y^2(s)+c^2}\,ds\right)^p, \tag{3.5}$$

which can be rewritten as

$$\frac{d}{dt}\left(y - \frac{2\alpha}{p+1}\left(\int_0^t \sqrt{y^2(s)+c^2}\,ds\right)^{p+1}\right) = 0.$$

Therefore we have

$$y(t) = y(0) + \frac{2\alpha}{p+1}\left(\int_0^t \sqrt{y^2(s)+c^2}\,ds\right)^{p+1}. \tag{3.6}$$

Let us show another look at (3.5). Let $Y(t) = \int_0^t \sqrt{y^2(s)+c^2}\,ds$. Then (3.5) can be rewritten as

$$\frac{Y'Y''}{\pm\sqrt{(Y')^2-c^2}} = 2\alpha Y'Y^p, \tag{3.7}$$

where we use $y^2 = (Y')^2 - c^2$ and $\pm$ corresponds the sign of y. From now on, we consider the case $y(t) \geq 0$. The equation (3.6) implies $y(t) \geq 0$ for $y(0) \geq 0$, $\alpha \geq 0$. Then the equation (3.7) is equivalent to

$$\frac{d}{dt}\left(\sqrt{(Y')^2-c^2} - \frac{2\alpha}{p+1}Y^{p+1}\right) = 0.$$

Considering $Y(0) = 0$, $Y'(0) = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2$ and $c^2 = 4\|u_0\|_{L^2}^2\|v_0\|_{L^2}^2$, we obtain

$$\sqrt{(Y')^2-c^2} = \sqrt{y^2(0)} + \frac{2\alpha}{p+1}Y^{p+1},$$

where $y(0) = \|u_0\|_{L^2}^2 - \|v_0\|_{L^2}^2$. Then we obtain an ODE

$$\frac{dY}{dt} = \left[c^2 + \left(\sqrt{y^2(0)} + \frac{2\alpha}{p+1}Y^{p+1}\right)^2\right]^{\frac{1}{2}}.$$

In a similar way, we can derive an ODE for $X(t)$. Plugging $y = \pm\sqrt{x^2-c^2}$ in the first equation of (3.4), we obtain an ODE

$$\frac{dx}{dt} = \pm 2\alpha\sqrt{x^2(t)-c^2}\left(\int_0^t x(s)\,ds\right)^p. \tag{3.8}$$

Let $X(t) = \int_0^t x(s)\,ds$. Then (3.8) can be rewritten as

$$X'' = \pm 2\alpha\sqrt{(X')^2-c^2}\,X^p.$$

Multiplying X' on both sides, we derive

$$\frac{X'X''}{\sqrt{(X')^2-c^2}} = \pm 2\alpha X'X^p$$

which is equivalent to $\frac{d}{dt}\left(\sqrt{(X')^2-c^2}\mp\frac{2\alpha}{p+1}X^{p+1}\right)=0$. Considering $X(0)=0$, $X'(0)=\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2$ and $c^2=4\|u_0\|_{L^2}^2\|v_0\|_{L^2}^2$, we obtain

$$\sqrt{(X')^2-c^2}=\sqrt{y^2(0)}\pm\frac{2\alpha}{p+1}X^{p+1}.$$

Then we obtain an ODE $\frac{dX}{dt}=\left[c^2+\left(\sqrt{y^2(0)}\pm\frac{2\alpha}{p+1}X^{p+1}\right)^2\right]^{\frac{1}{2}}$.

4. Proof of Theorem 1.3 and its applications

For the system (1.3), we can derive

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}|u(x,\,t)|^2dx=2\alpha\int_{\mathbb{R}}\left[|u(x,\,t)|^2\left(\int_{-\infty}^{x}(|u|^2+|v|^2)(y,\,t)dy\right)^p\right]dx,\\&\frac{d}{dt}\int_{\mathbb{R}}|v(x,\,t)|^2dx=2\beta\int_{\mathbb{R}}\left[|v(x,\,t)|^2\left(\int_{-\infty}^{x}(|u|^2+|v|^2)(y,\,t)dy\right)^p\right]dx.\end{aligned}\tag{4.1}$$

Let us define $f(x,t)=\int_{-\infty}^{x}|u(y,t)|^2\,dy$ and $g(x,t)=\int_{-\infty}^{x}|v(y,t)|^2\,dy$.

When $\alpha=\beta$, the system (4.1) implies

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(|u|^2+|v|^2)(x,\,t)dx\\&=2\alpha\int_{\mathbb{R}}\left[(|u|^2+|v|^2)(x,\,t)\left(\int_{-\infty}^{x}(|u|^2+|v|^2)(y,\,t)dy\right)^p\right]dx.\end{aligned}$$

Taking into account

$$\begin{aligned}&(|u|^2+|v|^2)(x,\,t)\left(\int_{-\infty}^{x}(|u|^2+|v|^2)(y,\,t)dy\right)^p\\&=\frac{d(f+g)}{dx}(f+g)^p=\frac{d}{dx}\left(\frac{1}{p+1}(f+g)^{p+1}\right),\end{aligned}$$

we obtain

$$\int_{-\infty}^{\infty}\frac{d}{dx}\left(\frac{1}{p+1}(f+g)^{p+1}\right)dx=\frac{1}{p+1}\left(\int_{-\infty}^{\infty}(|u|^2+|v|^2)(y,\,t)dy\right)^{p+1}.$$

Therefore we have

$$\frac{d}{dt}\int_{\mathbb{R}}(|u|^2+|v|^2)(x,\,t)dx=\frac{2\alpha}{p+1}\left(\int_{\mathbb{R}}(|u|^2+|v|^2)(x,\,t)dx\right)^{p+1},$$

which is the similar ODE to (2.2).

The system (4.1) can be rewritten as

$$\begin{aligned}&\frac{d}{dt}f(\infty,\,t)=2\alpha\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}(f+g)^p dx,\\&\frac{d}{dt}g(\infty,\,t)=2\beta\int_{-\infty}^{\infty}\frac{\partial g}{\partial x}(f+g)^p dx.\end{aligned}\tag{4.2}$$

As an application of (4.2), we consider the case of $\alpha > 0$, $p > 0$. Note that f, g are nonnegative functions. Moreover we have $\frac{\partial f}{\partial x} \geq 0$ and $\frac{\partial g}{\partial x} \geq 0$. Then the first equation of (4.2) implies

$$\frac{d}{dt} f(\infty, t) \geq 2\alpha \int_{-\infty}^{\infty} \frac{\partial f}{\partial x} f^p dx = \frac{2\alpha}{p+1} f^{p+1}(\infty, t),$$

where we note that $f(\infty, t) = \int_{\mathbb{R}} |u(x,t)|^2 \, dx$.

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